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Selivanovski hard sets are hard

by

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Abstract. Let $H \subseteq Z \subseteq 2^\omega$. For $n \geq 2$, we prove that if Selivanovski measurable functions from 2^ω to Z give as preimages of H all $\mathbf{\Sigma}^1_n$ subsets of 2^ω, then so do continuous injections.

Let $H \subseteq Z$ be subsets of the Cantor space $\mathcal{C} = 2^\omega$. Say that (H, Z) is $\mathbf{\Sigma}^1_n$-*hard* if for any $\mathbf{\Sigma}^1_n$ set $Q \subseteq \mathcal{C}$ there is a continuous function $f\colon \mathcal{C} \to Z$ with $Q = f^{-1}[H]$.

Kechris [1] proved ([1]) that using here Borel rather than continuous functions we get the same family of pairs. For $n \geq 2$ Sabok [4] improved this by replacing Borel functions with functions such that preimages of all sets from the canonical subbasis of $\mathcal{C}$ are in $\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1$.

We show for $n \geq 2$ that by changing in the definition of $\mathbf{\Sigma}^1_n$-hardness "continuous" to "Selivanovski measurable" we do not get more pairs, and by changing "continuous" to "continuous injective" we do not get fewer pairs.

Recall that a function is *Selivanovski measurable* if preimages of open sets belong to the σ-field of Selivanovski sets (also called C-sets), which is the least σ-field that contains all Borel sets and is closed under the Suslin operation.

Kechris and Sabok use effective descriptive set theory, and Kechris asked about a classical proof of his theorem. Our proof is classical and can be adapted to give Kechris's theorem (see [3] for a direct classical proof of Kechris's theorem).

Theorem. *Let $n \geq 2$ and $H \subseteq Z \subseteq \mathcal{C}$. If Selivanovski measurable functions from $\mathcal{C}$ to Z give as preimages of H all $\mathbf{\Sigma}^1_n$ subsets of 2^ω, then so do continuous injections.*

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([1]) Kechris formulated his result for $n = 1$, but his proof works for any $n \geq 1$.

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Note that since for any separable metrizable space S there exists a Borel injection $e\colon \mathcal{C} \to S$ whose inverse is continuous (e.g., $e(s)(i) = 1 \Leftrightarrow s \in O_i$, where $\{O_i\}_{i\in\omega}$ is a basis of S), and e can be chosen to be continuous if S is zero-dimensional, we can change in the Theorem the range space Z to any separable metrizable space and the domain space $\mathcal{C}$ to any zero-dimensional uncountable Polish space.

Note also that the Theorem cannot be extended to $n = 1$: pick distinct points z_0 and z_1 in $\mathcal{C}$ and let $Z = \{z_0, z_1\}$; if $Q \subseteq \mathcal{C}$ is $\mathbf{\Sigma}^1_1$, then the map sending Q to z_0 and $\mathcal{C} \setminus Q$ to z_1 is Selivanovski measurable; however, no non-clopen $Q \subseteq \mathcal{C}$ is a continuous preimage of $H = \{z_0\}$.

1. Spaces, pointclasses, functions. All our spaces are separable and metrizable; let X, Y, and Z range over such spaces. We identify the Baire space $\mathcal{N} = \omega^\omega$ with

$$\{x \in \mathcal{C}\colon \forall i\ \exists j > i\ x(j) = 1\}.$$

For $Q \subseteq X \times Y$, $f\colon X \times Y \to Z$, and $x \in X$, define the sections $Q_x \subseteq Y$ and $f_x\colon Y \to Z$ by $y \in Q_x \Leftrightarrow (x,y) \in Q$ and $f_x(y) = f(x,y)$.

A *pointclass* is a map Φ that assigns to any space X a family $\Phi_X = \Phi(X)$ of subsets of X; we often drop X if context permits. Let $\Phi_{XY} = \Phi(X,Y)$ be the family of all Φ *measurable functions* from X to Y, i.e., functions such that preimages of open subsets of Y are in $\Phi(X)$.

Let $\mathfrak{B}$ and $\mathfrak{S}$ be the pointclasses of Borel and Selivanovski sets. Selivanovski sets have the Baire property, and thus Selivanovski measurable functions are Baire measurable.

We shall also use the pointclases $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, and $\mathbf{\Delta}^1_n$, $n \geq 1$. For an arbitrary space X, the families $\mathbf{\Sigma}^1_n(X)$, $\mathbf{\Pi}^1_n(X)$, and $\mathbf{\Delta}^1_n(X)$ are defined in the same way as for a Polish space (see [2, 25.A]): the $\mathbf{\Pi}^1_n(X)$ sets are the complements of $\mathbf{\Sigma}^1_n(X)$ sets, and the $\mathbf{\Sigma}^1_n(X)$ sets are the projections of $\mathbf{\Pi}^1_{n-1}(X \times \mathcal{N})$ sets, if $n > 1$, and of closed subsets of $X \times \mathcal{N}$, if $n = 1$; also, $\mathbf{\Delta}^1_n(X) = \mathbf{\Sigma}^1_n(X) \cap \mathbf{\Pi}^1_n(X)$.

We have

$$\mathfrak{B}(X) \subseteq \mathbf{\Delta}^1_1(X) \subseteq \mathfrak{S}(X) \subseteq \mathbf{\Delta}^1_2(X);$$

if X is an uncountable Polish space, then the first inclusion is improper, and the next two are proper (see [2]; for $\mathfrak{S} \neq \mathbf{\Delta}^1_2$ see Section 4).

LEMMA 1. *Let* $\Phi \in \{\mathfrak{B}, \mathfrak{S}, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n, \mathbf{\Delta}^1_n\}$.

(1) *If* $X \subseteq X'$, *then:*

- (a) $Q' \in \Phi_{X'} \Rightarrow X \cap Q' \in \Phi_X$,
- (b) $Q \in \Phi_X \Rightarrow \exists Q' \in \Phi_{X'}\ Q = X \cap Q'$, *if* $\Phi \neq \mathbf{\Delta}^1_n$,
- (c) $Q \in \Phi_X \wedge X \in \Phi_{X'} \Rightarrow Q \in \Phi_{X'}$.

(2) *If Y is $\mathbf{\Sigma}^1_n$ in a Polish space, then projections along Y of $\mathbf{\Sigma}^1_n$ subsets of $X \times Y$ are $\mathbf{\Sigma}^1_n(X)$.*
(3) *Φ is closed under countable unions, countable intersections, and sections. The class of Φ measurable functions is closed under sections.*
(4) *If $f_0 \colon X_0 \to Y_0$ and $f_1 \colon X_1 \to Y_1$ are Φ measurable, then the Cartesian product function $(x_0, x_1) \mapsto (f_0(x_0), f_1(x_1))$ is Φ measurable.*
(5) *A function is Φ measurable iff preimages of closed sets are Φ sets. For any function, the notions of $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, and $\mathbf{\Delta}^1_n$ measurability coincide.*
(6) *The graph of a Φ measurable function is a Φ set.*
(7) *Preimages of Φ sets under Φ measurable functions are Φ sets.*
(8) *If the domain of a $\mathbf{\Delta}^1_n$ measurable function is $\mathbf{\Sigma}^1_n$ in a Polish space, then images of $\mathbf{\Sigma}^1_n$ sets are $\mathbf{\Sigma}^1_n$ sets.*
(9) *If Y is $\mathbf{\Sigma}^1_n$ in a Polish space and the graph of $f \colon X \to Y$ is $\mathbf{\Sigma}^1_n$, then $f \in \mathbf{\Delta}^1_n(X, Y)$.*

Proof. (4) The open subsets of $Y_0 \times Y_1$ are the countable unions of products $V_0 \times V_1$, with V_0 and V_1 open; the preimage of $V_0 \times V_1$ is $f_0^{-1}(V_0) \times f_1^{-1}(V_1) \in \Phi_{X_0 \times X_1}$.

(5) Closed sets are G_δ, and open sets are F_σ.

(6) If $f \in \Phi_{XY}$, then $\mathsf{graph}\, f$ is the preimage of the diagonal of Y^2 under the Φ measurable function $(x, y) \mapsto (f(x), y)$.

(7) We give a proof for $\Phi = \mathbf{\Sigma}^1_n$. Embed Y into a Polish space Y'; given any $Q \in \mathbf{\Sigma}^1_n(Y)$, get $Q' \in \mathbf{\Sigma}^1_n(Y')$ with $Q' \cap Y = Q$; then

$$f^{-1}(Q) = \{x \in X \colon \exists y \in Y' \ y \in Q' \wedge f(x) = y\}$$

is the projection along Y' of the intersection of $\mathbf{\Sigma}^1_n(X \times Y')$ sets: $X \times Q'$ and $\mathsf{graph}\, f$.

(8) For $Q \subseteq X$, $f(Q)$ is the projection of $(Q \times Y) \cap \mathsf{graph}\, f$ along X.

(9) For $Q \subseteq Y$, $f^{-1}(Q)$ is the projection of $(X \times Q) \cap \mathsf{graph}\, f$ along Y. ∎

Denote by $\mathcal{P}(X)$ the family of all Cantor (i.e., homeomorphic to $\mathcal{C}$) subsets of X endowed with the Vietoris topology. Note that if G is G_δ in X then $\mathcal{P}(G)$ is G_δ in $\mathcal{P}(X)$. Also, if X is a perfect Polish space, then so is $\mathcal{P}(X)$, and if G is comeager in such an X, then $\mathcal{P}(G)$ is comeager in $\mathcal{P}(X)$.

Recall that if $g \colon X \to Y$ is Baire measurable, then there is a comeager set $G \subseteq X$ such that $g|G$ is continuous. So, if X is a perfect Polish space, then g is continuous on comeagerly many $p \in \mathcal{P}(X)$ (on any $p \in \mathcal{P}(G)$). Equivalently, if sets $Q^n \subseteq X$, $n \in \omega$, have the Baire property, then there is a comeager set $G \subseteq X$ such that the sets $G \cap Q^n$, $n \in \omega$, are clopen in G. So, if X is a perfect Polish space, then for comeagerly many $p \in \mathcal{P}(X)$, the sets $p \cap Q^n$, $n \in \omega$, are clopen in p.

Let $\mathcal{P} = \mathcal{P}(\mathcal{C})$, and let $\pi\colon \mathcal{P} \times \mathcal{C} \to \mathcal{C}$ be a continuous function such that each section π_p, $p \in \mathcal{P}$, is a homeomorphism from $\mathcal{C}$ onto p (e.g., let π_p be induced by the unique bijection from $2^{<\omega}$ onto the split nodes of the tree $\{s|l\colon s \in p,\ l \in \omega\}$ which preserves the lexicographic ordering).

For $z \in \mathcal{C}$, define $z^* \in \mathcal{C}$ by $z^*(i) = z(2i)$, and write

$$\mathcal{C}_x = \{z \in \mathcal{C}\colon z^* = x\},\ \ \mathcal{P}_x = \mathcal{P}(\mathcal{C}_x), \quad x \in \mathcal{C}.$$

Fix also a list $\{I_n\}_{n\in\omega}$ all of clopen subsets of $\mathcal{C}$, with $I_0 = \varnothing$.

Finally, the main notion: if $n \geq 1$ and $H \subseteq Z \subseteq \mathcal{C}$, we say that (H, Z) is

- $\mathbf{\Sigma}^1_n$*-hard* if $\forall Q \in \mathbf{\Sigma}^1_n(\mathcal{C})\ \exists$ continuous $f\colon \mathcal{C} \to Z$ with $Q = f^{-1}[H]$,
- $\mathfrak{S}\mathbf{\Sigma}^1_n$*-hard* if $\forall Q \in \mathbf{\Sigma}^1_n(\mathcal{C})\ \exists$ Selivanovski measurable $f\colon \mathcal{C} \to Z$ with $Q = f^{-1}[H]$.

2. Injections. We first show how hardness can be realized via injections.

LEMMA 2. *Suppose that (H, Z) is $\mathbf{\Sigma}^1_n$-hard for some $n \geq 1$. Then any $\mathbf{\Sigma}^1_n$ subset of $\mathcal{C}$ can be obtained as the preimage of H under a continuous injection from $\mathcal{C}$ into Z.*

Proof. Define $c\colon \mathcal{N} \times \mathcal{C} \to \mathcal{C}$ by

$$c(s, y)(i) = 1 \ \Leftrightarrow\ y \in I_{s(i)}.$$

Then c is continuous, and $\{c_s\}_{s\in\mathcal{N}}$ is the family of all continuous functions from $\mathcal{C}$ to $\mathcal{C}$.

CLAIM. $\exists\, \mathfrak{p} \in \mathfrak{B}_{\mathcal{N}\mathcal{P}}\ \forall s \in \mathcal{N}\ \mathfrak{p}(s) \subseteq \mathcal{C}_s \wedge c_s|\mathfrak{p}(s)$ *is injective or constant.*

Proof of Claim. Let

$$Q = \{(s, p) \in \mathcal{N} \times \mathcal{P}\colon p \subseteq \mathcal{C}_s \wedge c_s|p \text{ is injective or constant}\}.$$

We claim that (1) Q is G_δ, and (2) $\forall s \in \mathcal{N}\ Q_s$ is nonmeager in $\mathcal{P}_s$. Once this is established, we can use the uniformization theorem for Borel sets with "large sections" [2, 18.6] to get the desired $\mathfrak{p}$.

(1) Consider in $\mathcal{N} \times \mathcal{C}^2$ the open set ∇ and the closed set Δ defined by

$$\nabla = \{(s, y_0, y_1) \in \mathcal{N} \times \mathcal{C}^2\colon c_s(y_0) \neq c_s(y_1)\},$$
$$\Delta = \{(s, y_0, y_1) \in \mathcal{N} \times \mathcal{C}^2\colon y_0 = y_1\}.$$

Note that $(s, p) \in Q$ iff $p \subseteq \mathcal{C}_s$ and

$$\{s\} \times p^2 \subseteq \nabla \cup \Delta \ \vee\ \{s\} \times p^2 \subseteq (\mathcal{N} \times \mathcal{C}^2) \setminus \nabla.$$

Now, "$p \subseteq \mathcal{C}_s$" defines a closed set in $\mathcal{N} \times \mathcal{P}$. The displayed line defines, in turn, a G_δ set: the map $(s, p) \mapsto \{s\} \times p^2$ is continuous, and the set

$$\mathcal{P}(\nabla \cup \Delta) \cup \mathcal{P}((\mathcal{N} \times \mathcal{C}^2) \setminus \nabla)$$

is G_δ in $\mathcal{P}(\mathcal{N} \times \mathcal{C}^2)$ because the sets $\nabla \cup \Delta$ and $(\mathcal{N} \times \mathcal{C}^2) \setminus \nabla$ are G_δ in $\mathcal{N} \times \mathcal{C}^2$.

(2) Fix $s \in \mathcal{N}$. Either c_s is constant on a nonempty open set $U \subseteq \mathcal{C}_s$ — then $\mathcal{P}(U)$ is nonempty open in $\mathcal{P}_s$, and $p^2 \subseteq \mathcal{C}^2 \smallsetminus \nabla_s$ for $p \in \mathcal{P}(U)$; or else $\mathcal{C}_s^2 \cap \nabla_s$ is dense open in $\mathcal{C}_s^2$ —then there are comeagerly many $p \in \mathcal{P}_s$ with $p^2 \subseteq \nabla_s \cup \Delta_s$ by the Kuratowski–Mycielski theorem [2, 19.1]. ■Claim

Now, consider the following Borel injection from $\mathcal{N} \times \mathcal{C}$ into $\mathcal{C}$:

$$h(s, y) = \pi(\mathfrak{p}(s), y).$$

If $Q \in \mathbf{\Sigma}^1_n(\mathcal{C})$, then $h[\mathcal{N} \times Q] \in \mathbf{\Sigma}^1_n(\mathcal{C})$. As (H, Z) is $\mathbf{\Sigma}^1_n$-hard, for some continuous $f\colon \mathcal{C} \to Z$,

$$h[\mathcal{C} \times Q] = f^{-1}[H].$$

Hence, since h is injective,

$$\mathcal{C} \times Q = h^{-1}[f^{-1}[H]].$$

Pick s with $f = c_s$. Then

$$Q = h_s^{-1}[c_s^{-1}[H]] = (c_s h_s)^{-1}[H].$$

But $c_s h_s$ is injective or constant, as h_s is a bijection onto $\mathfrak{p}(s)$, and $c_s|\mathfrak{p}(s)$ is injective or constant.

If $c_s h_s$ is injective, we are done. Otherwise, it must be the case that $Q \in \{\mathcal{C}, \varnothing\}$. Then there is also a continuous injective $g\colon \mathcal{C} \to Z$ with $Q = g^{-1}[H]$ since both H and $Z \smallsetminus H$ contain copies of $\mathcal{C}$ ([2]). ■

3. Suslin operation. For any set Λ, a set $T \subseteq \Lambda^{<\omega}$ is a tree if it is closed under initial segments. A tree T is *well-founded* if $\neg\exists t \in \Lambda^\omega\ \forall l \in \omega\ t|l \in T$.

Henceforth let $A = \omega^{<\omega}$, and let $\mathcal{E}$ be the set of all nonempty well-founded subtrees of $A^{<\omega}$. Identifying $\mathsf{Pow}(A^{<\omega})$ with $\mathcal{C}$, we view $\mathcal{E}$ as a $\mathbf{\Pi}^1_1$ subset of $\mathcal{C}$.

In the following:

- $\langle \dagger \rangle$ is the one-term sequence consisting of $\dagger$;
- $i \in \omega$;
- $\sigma, \varsigma, \tau \in A$; $\theta, \vartheta \in A^{<\omega}$;
  $\varnothing$, resp. $\emptyset$, is the empty sequence in A, resp. $A^{<\omega}$;
- for $\theta \neq \emptyset$, $\mathsf{last}\,\theta =$ the last term of θ;
- $\varsigma\hat{\ }\sigma$ and $\vartheta\hat{\ }\theta$ denote the concatenations of the respective sequences; but $\sigma\hat{\ }i = \sigma\hat{\ }\langle i\rangle$ and $\vartheta\hat{\ }\sigma = \vartheta\hat{\ }\langle\sigma\rangle$; so $\mathsf{last}\,\vartheta\hat{\ }\emptyset = \mathsf{last}\,\vartheta$ and $\mathsf{last}\,\vartheta\hat{\ }\varnothing = \varnothing$;
- $\varepsilon \in \mathcal{E}$; $\theta\hat{\ }\varepsilon = \{\theta\hat{\ }\vartheta \colon \vartheta \in \varepsilon\}$; $\varepsilon_\theta = \{\vartheta \colon \theta\hat{\ }\vartheta \in \varepsilon\}$;
- $s, t \in \mathcal{N}$; $s \leq t$ iff $\forall l\ s(l) \leq t(l)$.

([2]) Fix $G \in G_\delta(\mathcal{C}) \smallsetminus F_\sigma(\mathcal{C})$. Let $g\colon \mathcal{C} \to Z$ be continuous with $G = g^{-1}[H]$. Then $g[G]$ is uncountable, as otherwise $G = g^{-1}[g[G]]$ would be F_σ. Being an uncountable $\mathbf{\Sigma}^1_1$ set, $g[G]$ contains a copy of $\mathcal{C}$. The same argument works for g with $Q = g^{-1}[Z \smallsetminus H]$.

We use the symbol (3) $\overline{\bigwedge}$ for the Suslin operation: given sets $\{Q^\sigma\}_{\sigma\in A}$,

$$\overline{\bigwedge_\sigma} Q^\sigma = \bigcup_s \bigcap_{\sigma\subseteq s} Q^\sigma.$$

Note that if a family $\mathcal{F} \subseteq \mathsf{Pow}(X)$ is closed under the operation $X \setminus \overline{\bigwedge}_\sigma Q^\sigma$, then $\mathcal{F}$ is a σ-field closed under the Suslin operation; so, if $\mathcal{F}$ also contains a basis of X, then $\mathcal{F} \supseteq \mathbf{S}_X$.

Lemma 3. *Suppose that X is compact and $\{Q^\sigma\}_{\sigma\in A} \subseteq \mathsf{Pow}(X)$. If each $\overline{\bigwedge}_\tau Q^{\sigma\hat{}\tau}$, $\sigma \in A$, is clopen, then there exists $t \in \mathcal{N}$ such that*

$$\overline{\bigwedge_\sigma} Q^\sigma = \bigcup_{s\le t} \bigcap_{\sigma\subseteq s} Q^\sigma.$$

Proof. Let $\tilde{Q}^\sigma = \overline{\bigwedge}_\tau Q^{\sigma\hat{}\tau}$. Note that $\tilde{Q}^\varnothing = \overline{\bigwedge}_\sigma Q^\sigma$, and for each σ,

$$\tilde{Q}^\sigma = \bigcup_{i\in\omega} \tilde{Q}^{\sigma\hat{}i}.$$

Since the tilded sets above are compact and clopen, there exist $k_\sigma \in \omega$ such that if $k \ge k_\sigma$ and if "$i \in \omega$" is changed to "$i \le k$", then the equality is preserved. It follows that $t \in \mathcal{N}$ given by

$$t(\ell) = \max\{k_\sigma : |\sigma| = \ell \wedge \forall\, l < \ell\ \sigma(l) \le t(l)\}$$

works. ∎

4. Coding. We construct a $\mathbf{\Delta}^1_2$ measurable function that is universal for $\mathbf{S}_{\mathcal{CC}}$. Define $U^\theta_\varepsilon \subseteq \mathcal{C}$ by

$$U^\theta_\varepsilon = \begin{cases} I_{|\mathsf{last}\,\theta|}, & \theta \notin \varepsilon, \\ \mathcal{C} \setminus \overline{\bigwedge}_\sigma U^{\theta\hat{}\sigma}_\varepsilon, & \theta \in \varepsilon, \end{cases}$$

and then define $u : \mathcal{E} \times \mathcal{C} \to \mathcal{C}$ by

$$u(\varepsilon, x)(i) = 1 \ \Leftrightarrow\ x \in U^{\langle\langle i\rangle\rangle}_\varepsilon.$$

Lemma 4. *$u \in \mathbf{\Delta}^1_2(\mathcal{E} \times \mathcal{C}, \mathcal{C})$ and $\{u_\varepsilon\}_{\varepsilon\in\mathcal{E}} = \mathbf{S}_{\mathcal{CC}}$.*

Proof. For the first part it is enough to see that $x \in U^\theta_\varepsilon$ is $\mathbf{\Delta}^1_2$. We have

$$x \in U^\theta_\varepsilon \ \Leftrightarrow\ \exists d \subseteq \varepsilon\ \varphi \wedge \theta \in d \ \Leftrightarrow\ \forall d \subseteq \varepsilon\ \varphi \Rightarrow \theta \in d,$$

where φ is

$$\forall\theta\ \big((\theta \notin \varepsilon \Rightarrow x \in I_{|\mathsf{last}\,\theta|}) \wedge (\theta \in \varepsilon \Rightarrow \neg\exists s\ \forall\sigma \subseteq s\ \theta\hat{}\sigma \in d)\big).$$

For the second part it is enough to see that $\{U^\theta_\varepsilon\}_{\varepsilon\in\mathcal{E}} = \mathbf{S}_{\mathcal{C}}$ whenever $\theta = \langle\langle i\rangle\rangle$. In fact, this is true for any θ.

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(3) `\DeclareMathOperator*{\suslin}{\vphantom{\bigwedge}\mathpalette\souslin\bigwedge}`
`\def\souslin#1#2{#1\overline{\smash{#2}}}`
`\suslin_{\sigma}Q^{\sigma}` ⇝ $\overline{\bigwedge}_\sigma Q^\sigma$, $\overline{\bigwedge_\sigma} Q^\sigma$.

The $\subseteq$ inclusion is clear. To see the $\supseteq$ inclusion note first that $U_\varepsilon^{\theta\hat{\ }\vartheta} = U_{\varepsilon_\theta}^{\vartheta}$ for all θ and ϑ. Also, for any θ and any $\{\varepsilon^\sigma\}_{\sigma\in A}$, if

$$\varepsilon = \{\theta|l\colon l \le |\theta|\} \cup \bigcup_\sigma (\theta\hat{\ }\sigma)\hat{\ }\varepsilon^\sigma,$$

then

$$U_\varepsilon^\theta = \mathcal{C} \setminus \mathop{\mathsf{A}}_\sigma U_{\varepsilon^\sigma}^{\theta}.$$

It follows that $\{U_\varepsilon^\theta\}_{\varepsilon\in\mathcal{E}}$ is a σ-field closed under the Suslin operation.

We still need to see that $\{U_\varepsilon^\theta\}_{\varepsilon\in\mathcal{E}}$ contains a basis of $\mathcal{C}$. First, if θ is terminal in ε, then $U_\varepsilon^{\theta\hat{\ }\varnothing} = I_{|\varnothing|} = \varnothing$, so $\mathop{\mathsf{A}}_\sigma U_\varepsilon^{\theta\hat{\ }\sigma} = \varnothing$, hence $U_\varepsilon^\theta = \mathcal{C}$. Next, given any $n \ne 0$, let

$$\varepsilon = \{\theta|l\colon l \le |\theta|\} \cup \{\theta\hat{\ }\sigma\colon |\sigma| \ne n\}.$$

Now, if $|\sigma| \ne n$ then $\theta\hat{\ }\sigma$ is terminal in ε, so $U_\varepsilon^{\theta\hat{\ }\sigma} = \mathcal{C}$, and if $|\sigma| = n$ then $\theta\hat{\ }\sigma \notin \varepsilon$, so $U_\varepsilon^{\theta\hat{\ }\sigma} = I_{|\sigma|} = I_n$. Altogether this gives $\mathop{\mathsf{A}}_\sigma U_\varepsilon^{\theta\hat{\ }\sigma} = I_n$, hence $U_\varepsilon^\theta = \mathcal{C} \setminus I_n$. ■

5. Uniformization

LEMMA 5. *There is $\mathfrak{p} \in \mathbf{\Delta}_2^1(\mathcal{E},\mathcal{P})$ such that for each ε,*

$$\mathfrak{p}(\varepsilon) \in \mathcal{P}_\varepsilon \quad \text{and} \quad u_\varepsilon|\mathfrak{p}(\varepsilon) \text{ is continuous.}$$

Proof. The desired $\mathfrak{p}$ is obtained by the $\mathbf{\Sigma}_2^1$ uniformization theorem applied to the set Q of all $(\varepsilon, p) \in \mathcal{E}\times\mathcal{P}$ with $p \in \mathcal{P}_\varepsilon$ for which there exist $\bar{n} \in \omega^{A^{<\omega}}$ and $\bar{s} \in \mathcal{N}^{A^{<\omega}}$ such that $\forall\theta \notin \varepsilon\ |\mathsf{last}\,\theta| = \bar{n}(\theta)$ and

$$\forall\theta\in\varepsilon\ \Big(\mathop{\mathsf{A}}_\sigma p\cap I_{\bar{n}(\theta\hat{\ }\sigma)} \subseteq p\setminus I_{\bar{n}(\theta)} \subseteq \bigcup_{s\le\bar{s}(\theta)}\ \bigcap_{\sigma\subseteq s} p\cap I_{\bar{n}(\theta\hat{\ }\sigma)}\Big).$$

Note that u_ε is continuous on any $p \in Q_\varepsilon$ since $\forall\theta\ p\cap U_\varepsilon^\theta = p\cap I_{\bar{n}(\theta)}$.

We will show: (1) $Q \in \mathbf{\Sigma}_2^1(\mathcal{E}\times\mathcal{P})$, and (2) $\forall\varepsilon\ Q_\varepsilon \ne \varnothing$.

(1) The conditions "$p\in\mathcal{P}_\varepsilon$" and "$\forall\theta\notin\varepsilon\ |\mathsf{last}\,\theta| = \bar{n}(\theta)$" define closed sets in $\mathcal{E}\times\mathcal{P}$ and $\mathcal{E}\times\omega^{A^{<\omega}}$. The displayed condition, in turn, defines a $\mathbf{\Pi}_1^1$ set in

$$\mathcal{E}\times\mathcal{P}\times\omega^{A^{<\omega}}\times\mathcal{N}^{A^{<\omega}}.$$

Its first inclusion gives clearly a $\mathbf{\Pi}_1^1$ set in $\mathcal{P}\times\omega^{A^{<\omega}}$. Its second inclusion gives a closed set in $\mathcal{P}\times\omega^{A^{<\omega}}\times\mathcal{N}^{A^{<\omega}}$, as it says that the compact set $p\setminus I_{\bar{n}(\theta)}$ is contained in the projection of the compact set

$$\{(x,s)\in\mathcal{C}\times\mathcal{N}\colon s\le\bar{s}(\theta)\wedge\forall\sigma\subseteq s\ x\in p\cap I_{\bar{n}(\theta\hat{\ }\sigma)}\}.$$

(2) Since for any θ and ς, the set $\mathcal{C}_\varepsilon \cap \mathop{\mathsf{A}}_\sigma U_\varepsilon^{\theta\hat{\ }\langle\varsigma\hat{\ }\sigma\rangle}$ has the Baire property in $\mathcal{C}_\varepsilon$, we can choose $p\in\mathcal{P}_\varepsilon$ such that for any θ and ς, the set $p\cap\mathop{\mathsf{A}}_\sigma U_\varepsilon^{\theta\hat{\ }\langle\varsigma\hat{\ }\sigma\rangle}$ is clopen in p. In particular, for any $\theta\in\varepsilon$, the set $p\cap U_\varepsilon^\theta = p\setminus\mathop{\mathsf{A}}_\sigma U_\varepsilon^{\theta\hat{\ }\langle\varnothing\hat{\ }\sigma\rangle}$ is clopen in p.

To get $\bar{n} \in \omega^{A^{<\omega}}$, if $\theta \notin \varepsilon$ then let $\bar{n}(\theta) = |\mathsf{last}\,\theta|$, and if $\theta \in \varepsilon$ then let $\bar{n}(\theta)$ be any n such that

$$p \cap U_\varepsilon^\theta = p \cap I_n.$$

To get $\bar{s} \in \mathcal{N}^{A^{<\omega}}$, if $\theta \notin \varepsilon$ then let $\bar{s}(\theta)$ be any element of $\mathcal{N}$, and if $\theta \in \varepsilon$ then let $\bar{s}(\theta)$ be the t of Lemma 3 applied to p and $\{p \cap I_{\bar{n}(\theta\hat{}\sigma)}\}_{\sigma\in A}$ so that

$$\bigwedge_{\sigma} p \cap I_{\bar{n}(\theta\hat{}\sigma)} = \bigcup_{s\le\bar{s}(\theta)} \bigcap_{\sigma\subseteq s} p \cap I_{\bar{n}(\theta\hat{}\sigma)}. \ \blacksquare$$

6. Proof of the Theorem. In view of Lemma 2, we just need to get $\mathbf{\Sigma}_n^1$-hardness from $\mathbf{S\Sigma}_n^1$-hardness. Consider the following $\mathbf{\Delta}_2^1$ measurable injection from $\mathcal{E} \times \mathcal{C}$ to $\mathcal{C}$:

$$g(\varepsilon, x) = \pi(\mathfrak{p}(\varepsilon), x),$$

where $\mathfrak{p}$ is from Lemma 5. If $Q \in \mathbf{\Sigma}_n^1(\mathcal{C})$, then $g[\mathcal{E} \times Q] \in \mathbf{\Sigma}_n^1(\mathcal{C})$ by Lemma 1(8). So, if (H, Z) is $\mathbf{S\Sigma}_n^1$-hard, then for some $f \in \mathbf{S}_{\mathcal{C}Z}$,

$$g[\mathcal{E} \times Q] = f^{-1}[H].$$

Hence, since g is injective,

$$\mathcal{E} \times Q = g^{-1}[f^{-1}[H]].$$

Pick ε with $f = u_\varepsilon$. Then

$$Q = g_\varepsilon^{-1}[u_\varepsilon^{-1}[H]] = (u_\varepsilon g_\varepsilon)^{-1}[H].$$

But $u_\varepsilon g_\varepsilon$ is continuous because g_ε is a homeomorphism onto $\mathfrak{p}(\varepsilon)$ and $u_\varepsilon|\mathfrak{p}(\varepsilon)$ is continuous.

7. Kechris's Theorem. Change A to ω, $A^{<\omega}$ to $\omega^{<\omega}$, $\mathbf{S}$ to $\mathbf{B}$, $\mathbf{\Sigma}_2^1$ to $\mathbf{\Sigma}_1^1$, $\mathbf{\Delta}_2^1$ to $\mathbf{\Delta}_1^1$, and $\bigwedge$ to $\bigcup$. So, $\mathcal{E}$ is now the set of all nonempty well-founded subtrees of $\omega^{<\omega}$, and u is $\mathbf{\Delta}_1^1$ measurable.

In Lemma 5, let Q_ε consist of all $p \in \mathcal{P}_\varepsilon$ on which u_ε is continuous. Then Q_ε is comeager in $\mathcal{P}_\varepsilon$. Also, Q is $\mathbf{\Pi}_1^1$, since $u_\varepsilon|p$ is continuous iff

$$\forall n\ \exists m\ \forall x \in p\ x \in I_m \Leftrightarrow u(\varepsilon, x) \in I_n,$$

and "$u(\varepsilon, x) \in I_n$" gives a $\mathbf{\Delta}_1^1(\mathcal{E}\times\mathcal{C}\times\omega)$ set. To get $\mathfrak{p}$, use the uniformization theorem for $\mathbf{\Pi}_1^1$ sets with "large sections" [1, 36.23] that provides here a $\mathbf{\Delta}_1^1$ measurable uniformization. (The $\mathbf{\Pi}_1^1$ uniformization theorem may fail to give a $\mathbf{\Delta}_1^1$ measurable function.)

Now, if g is as in Section 6 and $Q \in \mathbf{\Sigma}_1^1(\mathcal{C})$, then

$$g[\mathcal{E} \times Q] = \{z \in \mathcal{C} \colon \exists y \in Q\ g(z^*, y) = z\} \in \mathbf{\Sigma}_1^1(g[\mathcal{E} \times \mathcal{C}]),$$

as we have here the projection along $Q \in \mathbf{\Sigma}_1^1(\mathcal{C})$ of the $\mathbf{\Delta}_1^1(g[\mathcal{E} \times \mathcal{C}] \times \mathcal{C})$ set given by the preimage of $\{(z, z) \colon z \in \mathcal{C}\}$ via the $\mathbf{\Delta}_1^1$ measurable function

$$g[\mathcal{E} \times \mathcal{C}] \times \mathcal{C} \ni (z, y) \mapsto (g(z^*, y), z).$$

So, for some ε, $g[\mathcal{E} \times Q] = g[\mathcal{E} \times \mathcal{C}] \cap u_\varepsilon^{-1}[H]$, hence $\mathcal{E} \times Q = g^{-1}[u_\varepsilon^{-1}[H]]$, and, as before, $Q = (u_\varepsilon g_\varepsilon)^{-1}[H]$.

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