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A weakly chainable uniquely arcwise connected continuum without the fixed point property

by

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Abstract. A continuum is a metric compact connected space. A continuum is chainable if it is an inverse limit of arcs. A continuum is weakly chainable if it is a continuous image of a chainable continuum. A space X is uniquely arcwise connected if any two points in X are the endpoints of a unique arc in X. D. P. Bellamy asked whether if X is a weakly chainable uniquely arcwise connected continuum then every mapping $f : X \to X$ has a fixed point. We give a counterexample.

1. Introduction. A *continuum* is a metric compact connected space. A continuum is *chainable* if it is an inverse limit of arcs. A continuum is *weakly chainable* if it is a continuous image of a chainable continuum. A space X is *uniquely arcwise connected* if any two points in X are the endpoints of a unique arc in X. A space X has the *fixed point property* (briefly: f.p.p.) if every mapping $f : X \to X$ has a fixed point, i.e., a point $x \in X$ such that $f(x) = x$. Two papers [Mi1, Mi2] of P. Minc throw light on the relation of weak chainability and fixed points in continua. In the first he shows that a non-separating plane continuum X such that every indecomposable continuum in the boundary of X is contained in a weakly chainable continuum has the fixed point property. In particular a planar non-separating continuum with weakly chainable boundary has the f.p.p. In the second he constructs a treelike (i.e., an inverse limit of trees) weakly chainable continuum without the f.p.p. (the first treelike continuum without the f.p.p. was constructed by D. P. Bellamy [Be1]).

There are many papers about fixed points in uniquely connected continua. In particular K. Borsuk [Bo] proved that each dendroid (arcwise connected tree-like continuum) has the f.p.p. He presented there the *pursuit method*, which became a standard tool in the fixed point theory of uniquely arcwise connected spaces. C. Hagopian has shown that planar uniquely arc-

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wise connected continua have the f.p.p., and L. Mohler [Mo] has proved that any homeomorphism of a uniquely arcwise connected continuum has a fixed point. On the other hand G. S. Young [Yo] gave an example of a uniquely arcwise connected continuum without the f.p.p. consisting of a 'circle' made of two $\sin 1/x$-curves and a triod ending with a spiral approximating the 'circle'. Some vagueness in Young's description was removed by C. Hagopian and R. Mańka [HaMa]. They have shown that the fixed point property of the continuum depends on quite subtle details of the spiral.

In 1995 D. P. Bellamy [Be2] asked whether every weakly chainable uniquely arcwise connected continuum has the f.p.p. Here we will give a counterexample.

2. Description of the continuum. Let $S_0 \subset \mathbb{R}^2$ be the $\sin 1/x$-curve represented as the union $S_0 = L \cup I$ of the ray $L = \{(x, \sin\frac{3\pi}{2x}) : x \in (0,1]\}$ and the limit segment $I = \{(0,y) : y \in [-1,1]\}$. Let us denote by r the symmetry of $\mathbb{R}^2$ about the line $x = 1$. Set $R = r(L)$ and $I' = r(I)$ and let S denote the double $\sin 1/x$-curve $S_0 \cup R \cup I'$. We distinguish in S the sequence of right maxima $R^i = (2 - \frac{3}{4i+1}, 1)$ for $i = 1, 2, \ldots$, the sequence of left maxima $L^i = (\frac{3}{4i+1}, 1)$ for $i = 1, 2, \ldots$ and the central minimum $Q_0 = (1, -1)$. Moreover $B_0 = (2, -1)$ is the lower endpoint of I', and $D_0 = (0, -1)$ is the lower endpoint of I. Let W be the countable set with two limit points defined by

$$W = \{1 + 1/n : n = 1, 2, \ldots\} \cup \{1\} \cup \{-1 - 1/n : n = 1, 2, \ldots\} \cup \{-1\}.$$

Our example is a quotient space of $S \times W$. For clarity we proceed in two stages. First we clamp together all right limit segments $I' \times \{w\}$ for $w \in W$, all left limit segments $I \times \{w\}$ for $w \in W$, i.e., we consider the quotient space $\mathcal{C} = (S \times W)/\mathcal{E}_0$, where $\mathcal{E}_0$ is the equivalence relation generated by $(y, w) \sim (y, w')$ for $y \in I$ and $w, w' \in W$, and $(y, w) \sim (y, w')$ for $y \in I'$ and $w, w' \in W$. In the following we will use the same notation for points in a quotient space and for their representatives. The space $\mathcal{C}$ can be embedded in the plane as shown in Fig. 1.

Next, for $i, j = 1, 2, \ldots$ let

$$\begin{aligned} R^i_j &= (R^i, 1 + 1/j) \in S \times W, \\ \tilde{R}^i_j &= (R^i, -1 - 1/j) \in S \times W, \\ L^i_j &= (L^i, 1 + 1/j) \in S \times W, \\ \tilde{L}^i_j &= (L^i, -1 - 1/j) \in S \times W. \end{aligned}$$

We consider in $\mathcal{C}$ the equivalence relation $\mathcal{E}$ generated by $R^i_i \sim \tilde{R}^i_i$ and $\tilde{L}^{i+1}_i \sim L^{i+1}_{i+1}$ for $i = 1, 2, \ldots$. The relations $\mathcal{E}_0$ and $\mathcal{E}$ induce upper semicontinuous decompositions, hence the quotient space $\mathcal{C}/\mathcal{E}$ is metric and compact (in fact a

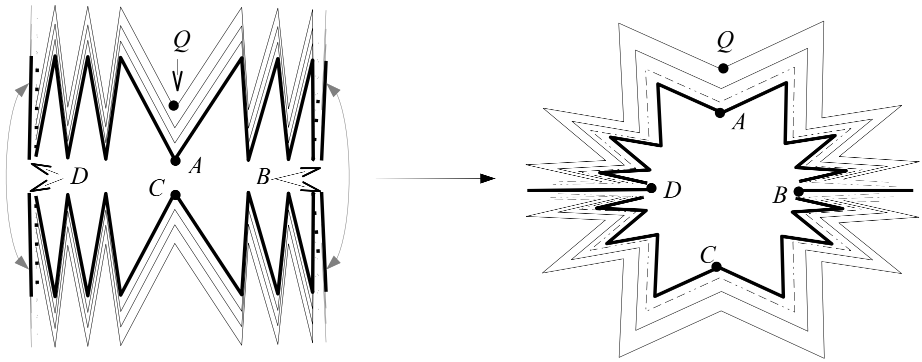

Fig. 1. $\mathcal{C} = (S \times W)/\mathcal{E}_0$. Left: $S \times W$, consisting of two symmetric halves, the upper one with $A = (Q_0, 1)$ and the lower one with $C = (Q_0, -1)$.

continuum). We complete $\mathcal{C}/\mathcal{E}$ to an arcwise connected continuum by adding a simple 5-od $\mathcal{P}$, being the union of five arcs J_A, J_B, J_C, J_D, J_Q with common endpoint O and otherwise disjoint. The endpoints of J_A, J_B, J_C, J_D, J_Q opposite to O are respectively $A = (Q_0, 1), B = (B_0, w), C = (Q_0, -1), D = (D_0, w), Q = (Q_0, 2) \in \mathcal{C}/\mathcal{E}$. Moreover $\mathcal{P} \cap \mathcal{C}/\mathcal{E} = \{A, B, C, D, Q\}$. Finally our example is $X = \mathcal{C}/\mathcal{E} \cup \mathcal{P}$. One can easily check that X is uniquely arcwise connected.

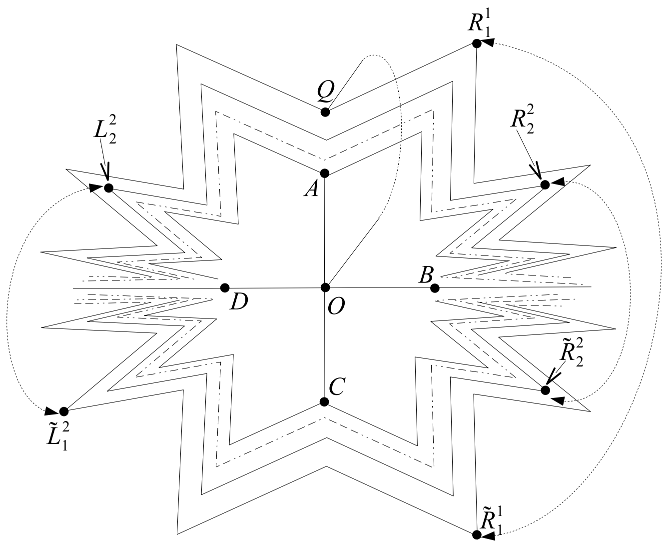

Fig. 2. The example X. Two-sided arrows indicate pairs of glued points.

3. X is weakly chainable. First we define on $S \times W$ an equivalence relation $\mathcal{R}$ generated by the following identifications:

- $(y,w)\sim(y,-w)$ for $y\in I', i=1,2,\dots$ and $w\in W$,
- $\big(y,1+\frac{1}{2i-1}\big)\sim\big(y,1+\frac{1}{2i}\big)$ for $y\in I$ and $i=1,2,\dots$,
- $\big(y,-1-\frac{1}{2i}\big)\sim\big(y,-1-\frac{1}{2i+1}\big)$ for $y\in I$ and $i=1,2,\dots$.

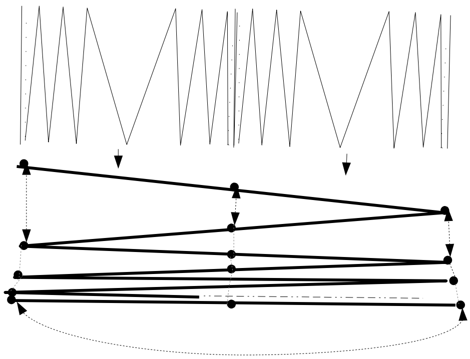

Fig. 3. Chainable continuous model $\mathcal{Z}$ of X. Each interval between two consecutive marked points in the infinite polygonal curve represents a double $\sin 1/x$-curve.

Denote $\mathcal{Z}=(S\times W)/\mathcal{R}$. The space $\mathcal{Z}$ is an infinite accordion-like booklet whose pages are unions of two double $\sin 1/x$-curves with a common limit segment, and it can be represented as an inverse limit of analogous finite booklets. But each of these booklets is chainable, hence $\mathcal{Z}$ is also chainable. Let us remark that layers of $\mathcal{R}$ are finer than those of $\mathcal{E}_0$, hence $\mathcal{C}$ is a continuous image of $\mathcal{Z}$. This means that $\mathcal{C}/\mathcal{E}$ is weakly chainable. Finally we can map $\mathcal{C}/\mathcal{E}$ onto X by forming the 5-od $\mathcal{P}$ from an arc containing the point Q. Thus X is weakly chainable.

4. A fixed point free mapping on X. If there exists in a space only one arc with endpoints A,B we will denote it by $\widehat{AB}$. Now we define a homeomorphism $f:S\to S\subset\mathbb{R}^2$ by the conditions:

- $f(R^1)=R^2$ and f maps $\widehat{Q_0R^1}$ onto $\widehat{Q_0R^2}$,
- $f(R^i)=R^{i+1}$ and f maps $\widehat{R^iR^{i+1}}$ onto $\widehat{R^{i+1}R^{i+2}}$ for $i=2,3,\dots$ in such a way that the second coordinate of points remains unchanged,
- f restricted to $S_0\cup I'$ is the identity.

Define $s:W\to W$ by $s(-1-1/n)=-1-1/(n+1)$ for $n=1,2,\dots$ and $s(w)=w$ for $w\neq -1-1/n$ for all $n=1,2,\dots$. Let $\tau:W\to W$ denote the symmetry $\tau(w)=-w$. Set $\sigma=r\times\tau:S\times W\to S\times W$ (it acts on $\mathcal{C}$ like central symmetry, see Fig. 1).

Now we consider the composition $\phi=\sigma\circ(f\times s):S\times W\to S\times W$. It transforms equivalence classes of $\mathcal{E}$ into equivalence classes of $\mathcal{E}$, hence it induces a continuous mapping $\tilde{\phi}$ on the quotient space $\mathcal{C}/\mathcal{E}$. Its action is

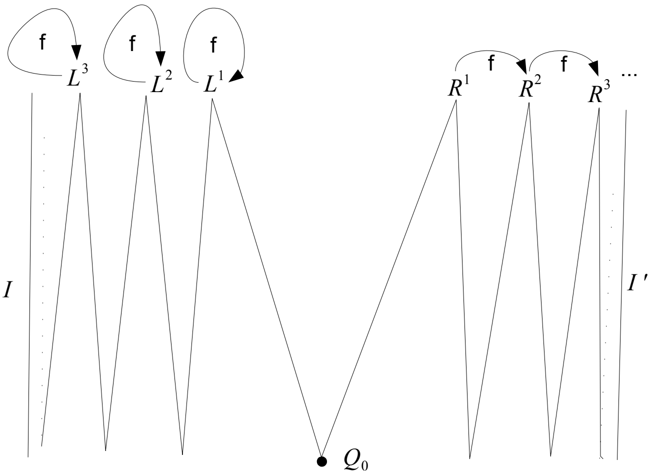

Fig. 4. Action of f on S

fixed point free. We can extend it to a fixed point free mapping $g : X \to X$ in the following way:

- g restricted to $\mathcal{C}/\mathcal{E}$ is $\tilde{\phi}$,
- $g(O) = Q$ and g maps J_Q homeomorphically onto the arc $\widehat{Q\tilde{\phi}(Q)}$ in $\mathcal{C}/\mathcal{E}$,
- $g(A) = C$, $g(B) = D$, $g(C) = A$, $g(D) = B$ and g stretches the arcs J_A, J_B, J_C, J_D homeomorphically onto $J_C \cup J_Q$, $J_D \cup J_Q$, $J_A \cup J_Q$, $J_B \cup J_Q$ respectively.

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