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Erratum to "Fields of surreal numbers and exponentiation"

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by

Lou van den Dries (Urbana, IL) and **Philip Ehrlich** (Athens, OH)

Lemma 4.5 in [2] is false. The correct result is the Lemma below. We use the following conventions and notations: Γ is an ordered abelian group, $S \subseteq \Gamma$; we let $[S] := \{s_1 + \ldots + s_k : k \in \mathbb{N}, s_1, \ldots, s_k \in S\}$ be the additive monoid generated by S in Γ; for $a \in \Gamma$, put $S^{<a} := \{s \in S : s < a\}$ and define $S^{\leq a}$ and $S^{\geq a}$ similarly; if S is well-ordered, we let $o(S)$ be its ordinal. Also α, λ, μ are ordinals, and sums and products of ordinals are their natural sums and natural products.

LEMMA. *Suppose $S \subseteq \Gamma^{\geq 0}$ is well-ordered with $o(S) \leq \mu$. Then $[S]$ is well-ordered with $o([S]) \leq \omega^{\omega\mu}$.*

Lemma 4.5 in [2] claims the sharper bound $o([S]) \leq \omega^{\mu}$. We will see below that this is correct if $\mu < \varepsilon_0$, but incorrect for $\mu = \varepsilon_0$.

Replacing Lemma 4.5 in [2] by the lemma above does not affect any of the main results of [2] but leads to minor changes in some proofs:

(1) In the proof of Lemma 4.6, replace "ω^{α}" by "$\omega^{\omega\alpha}$" and "ω^{σ}" by "$\omega^{\omega\sigma}$".

(2) Lemma 4.10: in its statement and proof, replace "$\omega^{(\omega+n)\mu}$" by "$\omega^{\omega(n+1)\mu}$", and in its proof replace "$\omega^{n\mu}$" by "$\omega^{\omega n\mu}$".

(3) In the proofs of Proposition 4.11, Lemma 5.2 and Lemma 5.4, replace "$\omega + 1$" (occurring as a factor in some exponents) by "$\omega 2$", and "$2\omega + 2$" by "$\omega 4$".

Proof of Lemma. We proceed by induction on μ. The lemma holds trivially for $\mu = 0$ ($S = \emptyset$) and $\mu = 1$, so let $\mu > 1$, and assume inductively that the desired result holds for smaller values.

Case 1: *μ is not additive.* This means that $\mu = \mu_1 + \mu_2$ for ordinals $\mu_1, \mu_2 < \mu$. Then $S = S_1 \cup S_2$ with $o(S_1) \le \mu_1$ and $o(S_2) \le \mu_2$. Hence $[S] = [S_1] + [S_2]$, so

$$o([S]) \le o([S_1]) \cdot o([S_2]) \le \omega^{\omega\mu_1} \cdot \omega^{\omega\mu_2} = \omega^{\omega\mu}.$$

Case 2: *μ is additive.* Then $\mu = \omega^\lambda$, $\lambda > 0$. Let $0 < a \in S$, $0 < n \in \mathbb{N}$. It suffices to show that then $[S]^{\le na} < \omega^{\omega\mu}$, since the elements na are cofinal in $[S]$. Note that $[S]^{\le na} \subseteq [S^{\le a}] + (S \cup \{0\}) + \ldots + (S \cup \{0\})$ where there are n terms $S \cup \{0\}$. Hence, with $o(S^{\le a}) = \alpha < \mu$, and using the fact that $o(S \cup \{0\}) \le \mu = \omega^\lambda$, we obtain

$$o([S]^{\le na}) \le \omega^{\omega\alpha}\omega^\lambda \ldots \omega^\lambda = \omega^{\omega\alpha + n\lambda}.$$

Thus it remains to show that $\omega\alpha + n\lambda < \omega\mu$. To this end we write α in Cantor normal form as $\alpha = \omega^{\alpha_1}n_1 + \ldots + \omega^{\alpha_k}n_k$ with $\lambda > \alpha_1 > \ldots > \alpha_k$ and positive integers $n_1, \ldots, n_k$. Then the Cantor normal form of $\omega\alpha$ has leading term $\omega^{\alpha_1+1}n_1$, so $\omega\alpha \le \omega^\lambda(n_1+1) = (n_1+1)\mu$. Hence $\omega\alpha + n\lambda \le (n_1+1)\mu + n\mu = (n_1+1+n)\mu < \omega\mu$.

In trying to carry out a similar inductive proof with the bound ω^μ (instead of $\omega^{\omega\mu}$), case 1 presents no problem, but case 2 leads to the inequality $\alpha + n\lambda < \mu$ (instead of $\omega\alpha + n\lambda < \omega\mu$). This inequality holds for $\lambda < \mu$, since μ is additive, but it fails when $\lambda = \mu$, that is, when μ is an ε-number. We conclude that the original Lemma 4.5 in [2] holds for $\mu < \varepsilon_0$.

Lemma 4.5 fails for $\mu = \varepsilon_0$: Let $\Gamma = \mathbb{R}$, the additive ordered group of real numbers, and take for S a well-ordered subset of the open interval $(0,1)$ with $o(S) = \varepsilon_0$. Then $S \subseteq [S]$ and $[S]$ has elements ≥ 1, so $\varepsilon_0 < o([S])$. Thus $o([S]) > \omega^{\varepsilon_0} = \varepsilon_0$.

The *Remark* following Lemma 4.5 is also incorrect. (It did not play any further role in [2].) First, the assumption "$S \subseteq K^{>0}$" in this *Remark* should be replaced by "$S \subseteq K^{\ge 1}$". Then a correct bound follows by noting that the semiring generated by S equals the additive monoid generated by the multiplicative monoid generated by S. This multiplicative monoid has ordinal at most $\omega^{\omega\mu}$ by our corrected lemma, and thus the semiring generated by S has ordinal at most $\omega^{\omega\omega^{\omega\mu}}$, which equals $\omega^{\omega^{1+\omega\mu}}$. The *Remark* gives instead the bound ω^{ω^μ}. This last bound (with $S \subseteq K^{\ge 1}$) is correct for $\mu < \varepsilon_0$ (by the valid part of Lemma 4.5), but incorrect for $\mu = \varepsilon_0$ (by the counterexample in the last paragraph).

Earlier results on $o([S])$ are by Carruth [1] and by Gonshor and Harkleroad [4].

We take this opportunity to point out that part (3) of Lemma 4.2 in [2] is immediate from Theorem 5.12 of [3].

References

[1] P. Carruth, *Arithmetic of ordinals with applications to ordered Abelian groups*, Bull. Amer. Math. Soc. 48 (1942), 262–271.

[2] L. van den Dries and P. Ehrlich, *Fields of surreal numbers and exponentiation*, Fund. Math. 167 (2001), 173–188.

[3] H. Gonshor, *An Introduction to the Theory of Surreal Numbers*, London Math. Soc. Lecture Note Ser. 110, Cambridge Univ. Press, 1986.

[4] L. Harkleroad and H. Gonshor, *The ordinality of additively generated sets*, Algebra Universalis 27 (1990), 507–510.

Department of Mathematics
University of Illinois
Urbana, IL 61801, U.S.A.
E-mail: vddries@math.uiuc.edu

Department of Philosophy
Ohio University
Athens, OH 45701, U.S.A.
E-mail: ehrlich@ohiou.edu

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