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Finitarily Bernoulli factors are dense

by

Stephen Shea (Manchester, NH)

Abstract. It is not known if every finitary factor of a Bernoulli scheme is finitarily isomorphic to a Bernoulli scheme (is finitarily Bernoulli). In this paper, for any Bernoulli scheme X, we define a metric on the finitary factor maps from X. We show that for any finitary map $f : X \to Y$, there exists a sequence of finitary maps $f_n : X \to Y(n)$ that converges to f, where each $Y(n)$ is finitarily Bernoulli. Thus, the maps to finitarily Bernoulli factors are dense. Let $(X(n))$ be a sequence of Bernoulli schemes such that each $Y(n)$ is finitarily isomorphic to $X(n)$. Let X' be a Bernoulli scheme with the same entropy as Y. Then we also show that $(X(n))$ can be chosen so that there exists a sequence of finitary maps to the $X(n)$ that converges to a finitary map to X'.

1. Introduction. For a complete introduction to ergodic theory, see [13]. When we refer to a process X, we are referring to $(X, \mathcal{U}, \mu, T)$ where $X = A^{\mathbb{Z}}$ for some (finite or countably-infinite) alphabet A, $\mathcal{U}$ is the σ-algebra generated by the coordinates, μ is a shift-invariant probability measure on $(X, \mathcal{U})$, and T is the shift map on $(X, \mathcal{U}, \mu)$. We say X is a *Bernoulli scheme* (BS) if $\mu = p^{\mathbb{Z}}$ for some probability vector p. Two processes X and Y are *isomorphic* if there exists an invertible, bimeasurable, equivariant map between subsets of full measure in X and Y that takes the measure in X to the measure in Y. Ornstein showed that entropy is a complete isomorphism invariant for BSs (see [4]). He later showed that every factor of a BS is isomorphic to a BS (see [5]).

For $x \in X$, we will use the notation $x[m, n]$ for the block $x_m, x_{m+1}, \dots, x_n$. An isomorphism ψ from X to Y is *finitary* if for almost every $x \in X$ there exist integers $m \le n$ such that the zero coordinates of $\psi(x)$ and $\psi(x')$ agree for almost all $x' \in X$ with $x[m, n] = x'[m, n]$, and similarly for ψ^{-1} (see [2]). If we drop the requirement that ψ be invertible, we say that ψ is a *finitary factor map*. Keane and Smorodinsky showed that entropy is a complete finitary isomorphism invariant for BSs (see [2]).

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A symbol a in the alphabet of a process X is a *renewal state* of X if $P[X_n = a] > 0$, and the σ-algebras $\mathcal{U}(X_{n+1}, X_{n+2}, \ldots)$ and $\mathcal{U}(\ldots, X_{n-2}, X_{n-1})$ are independent given the event $[X_n = a]$. A process is *Markov* if every state is a renewal state. Aperiodic finite-state Markov processes are finitarily isomorphic to BSs (*are fB*) (see [3]). In unpublished work, Smorodinsky showed that every finitary factor of a BS has exponentially decaying return times (see [9] for more details). Countable state mixing Markov processes with exponentially decaying return times are fB (see [7]). For an up-to-date survey of the literature on finitary maps, see [9].

In spite of all the progress with regard to finitary isomorphisms, there still does not exist a factor theorem in the finitary theory analogous to Ornstein's factor theorem in [5]. For a factor Y of a BS to be fB, Y must be a finitary factor of a BS (since, by definition, a finitary isomorphism is a finitary factor map). The question of whether all finitary factors are fB appears in the literature at least as early as [6]. In [12], it is conjectured that all finitary factors of BSs are fB.

Let X be a finite-state BS. We require X to be finite-state so that we can use the work in [1] and [2]. In Section 3, we define a metric d on the finitary factor maps of X. Let $f : X \to Y$ be a finitary factor map. Let X' be a BS such that $h(X') = h(Y)$, where h denotes the entropy. Here, we show that there exist sequences of processes $(Y(n))$ and $(X(n))$ where in d, $(Y(n)) \to Y$ and $(X(n)) \to X'$. Also, for each n, $X(n)$ is a BS, and $Y(n)$ is finitarily isomorphic to $X(n)$.

2. Finitary isomorphisms of r-processes. Following the terminology in [10], we say a renewal state a in X has *n-Bernoulli distribution* if for some nonnegative integer n, $P[X_{n'} = a \mid X_0 = a] = P[X_{n'} = a]$ for all $n' > n$. If a process has a renewal state with n-Bernoulli distribution for some $n \geq 0$, then we say that process is an *r-process.* A BS is an r-process where every state is a renewal state with 0-Bernoulli distribution. The proof of our first result will require the following theorem about r-processes.

THEOREM 2.1. *Let X be a finitary factor of a BS and a finite or countably-infinite state r-process. Then X is fB.*

Proof. In [10], we provide a proof of Theorem 2.1 for finite-state r-processes using the marker and filler methods of [2]. We now extend to the countable state case by using the result in [7]. We will show there exists a finitary isomorphism ϕ from X to a mixing countable state Markov process Y with exponentially decaying return times. Since the construction mimics that in the proof of Theorem 15 in [11], we will not provide the easily verifiable details. Let X have renewal state a. Let $(\phi(x))_n = a$ if $x_n = a$, and otherwise let $(\phi(x))_n = x[n-j, n]$ where $j \geq 0$, $x_{n-j} = a$ and $x_{n-i} \neq a$ for i

in $[0, j-1]$. In other words, the image under ϕ is a record of x back to the last occurrence of a. Since a is a renewal state in X, every state in Y is a renewal state. So, Y is a Markov process. Since ϕ is a finitary isomorphism, and X is a finitary factor of a BS, Y is a finitary factor of a BS. Therefore, Y is mixing and has exponentially decaying return times. By [7], there exists a finitary isomorphism ψ from Y to a BS. The composition of the finitary isomorphisms ϕ and ψ provides a finitary isomorphism from our r-process to a BS. ∎

3. Main results. Let X be a finite-state BS. Let $f : X \to Y$ and $g : X \to Y'$ be finitary factor maps. Define a map $\phi_{f,g} : X \to Z$ where $(\phi_{f,g}(x))_i = 0$ if $(f(x))_i = (g(x))_i$, and $(\phi_{f,g}(x))_i = 1$ otherwise. Since f and g are finitary factor maps, ϕ is a finitary factor map. Let Z have measure ν. Now define $d(f, g) = \nu(1)$. Let F_X be the set of finitary factor maps from X. It is easy to verify that (F_X, d) is a metric space.

Let $h(X)$ denote the entropy of X. Let $f_n : X \to Y(n)$ be a sequence of finitary factor maps such that $f_n \to f$ in d. For each n, $d(f_n, f) \geq \bar{d}(Y(n), Y)$ where $\bar{d}$ is Ornstein's d-bar distance. See [8] for details on the d-bar distance. So, $\lim_{n\to\infty} \bar{d}(Y(n), Y) = 0$. Entropy is continuous in $\bar{d}$ (see [8] again). This implies that $\lim_{n\to\infty} h(Y(n)) = h(Y)$. We will use this fact in the proof of Theorem 3.2.

Keane and Smorodinsky showed that for any two BSs X and X' where $h(X) > h(X')$, there exists a finitary factor map from X to X' (see [1]). So, if Y is a finitary factor of a BS X, and Y' is a finitary factor of a BS X', and $h(X) > h(X')$, then both Y and Y' are finitary factors of X. If $h(X) = h(X')$, both Y and Y' are finitary factors of X and X' by [2]. So, given any two finitary factors Y and Y' of BSs, there exists a BS X and finitary factor maps $f : X \to Y$ and $g : X \to Y'$. Thus, (f, Y) and (g, Y') can be compared using the metric d.

We will need the following two definitions. The first can be found in [12], for example. The second is new to this article. The process $X^{(k)}$ called the *k-stringing* (or *k-block presentation*) of X is defined as follows. The state space of $X^{(k)}$ is all allowable sequences of length k in X, and $X_n^{(k)} = (X_n, X_{n+1}, \ldots, X_{n+k-1})$. Let $f : X \to Y$ be finitary, and let $x \in X$. If $(f(x'))_k = (f(x))_k$ for almost all $x' \in X$ with $x'[i, j] = x[i, j]$, then we say $x[i, j]$ is *nice* with respect to f and k. We are now ready to state and prove the first of our main results.

THEOREM 3.1. *Let X be a finite-state BS, and let $f : X \to Y$ be a finitary factor map. There exists a sequence of finitary factor maps $f_n : X \to Y(n)$ such that each $Y(n)$ is fB, and* $\lim_{n\to\infty} d(f_n, f) = 0$.

Proof. Suppose X has alphabet A with $a \in A$. Let α be a symbol not in the alphabet of X or Y. For each n, we define $f_n : X \to Y(n)$ as follows.

First, $f_n(x)[i,i+n-1] = \alpha^n$ if $x[i,i+n-1] = a^n$. Suppose $x[i,i+n-1] = a^n$, $x[j,j+n-1] = a^n$ where j is some integer greater than $i+n$, and $x[i',i'+n-1] \neq a^n$ for any i' where $i < i' < j$. Then for all k such that $i+n-1 < k < j$, let $(f_n(x))_k = (f(x))_k$ if $x[i,j+n-1]$ is nice with respect to f and k. If $x[i,j+n-1]$ is not nice, let $(f_n(x))_k = x_k$.

We first show that each $Y(n)$ is finitarily isomorphic to an r-process. Then we use Theorem 2.1 to show that each $Y(n)$ is fB.

Let X have σ-algebra $\mathcal{U}$. Let $Y(n)$ have σ-algebra $\mathcal{V}$. Since X is a BS, $\mathcal{U}(\ldots,X_{i-2},X_{i-1})$ and $\mathcal{U}(X_{i+n},X_{i+n+1},\ldots)$ are independent given that $X[i,i+n-1] = a^n$. We have defined f_n so that what occurs between two occurrences of α^n in $Y(n)$ is determined by what occurs between the two occurrences of a^n in X. So, $\mathcal{V}(\ldots,Y(n)_{i-2},Y(n)_{i-1})$ and $\mathcal{V}(Y(n)_{i+n}, Y(n)_{i+n+1},\ldots)$ are independent given $Y(n)[i,i+n-1] = \alpha^n$. Therefore, α^n is a renewal state in $Y(n)^{(n)}$ (the n-stringing of $Y(n)$).

The word a^n is a block of length n in the BS X. So, a^n has $(n-1)$-Bernoulli distribution in $X^{(n)}$. Then α^n has $(n-1)$-Bernoulli distribution in $Y(n)^{(n)}$. Therefore, $Y(n)^{(n)}$ is an r-process. Theorem 2.1 implies $Y(n)^{(n)}$ is fB. Since k-stringings are finitary isomorphisms, and finitary isomorphism is transitive, $Y(n)$ is fB.

We now show that the sequence (f_n) converges to f in d. Since f is a finitary factor map, for any integer i and any $x \in X$, we can, with full probability, find integers m and m' (where $m' \geq m$) such that $x[m,m']$ is nice with respect to f and i. Then, with full probability, there exist integers j, j' and k such that $j+k-1 < m$, $j' > m'$, $x[j,j+k-1] = a^k$, $x[j',j'+k-1] = a^k$ and $x[i',i'+k-1] \neq a^k$ for any i' where $j < i' < j'$. Then for each i, there exists a positive integer k such that $(f(x))_i = (f_{k'}(x))_i$ for all $k' > k$. Therefore, $\lim_{n\to\infty} d(f_n,f) = 0$. ■

A corollary to Theorem 3.1 is that the maps to fB factors are dense among the finitary factor maps.

With our next theorem, we will show the following. For any finitary factor Y of a BS and any BS X' such that $h(Y) = h(X')$, there exist a process Z arbitrarily close to Y and a BS Z' arbitrarily close to X' such that Z and Z' are finitarily isomorphic.

Recall that for two BSs X and X' where $h(X) \geq h(X')$, there exists a finitary factor map from X to X' (see [1], [2]). Given a finitary factor map $g: X \to X'$, we define $\hat{g} : X \times X \to X' \times \{*\}$ so that $(\hat{g}(x^1,x^2))_i = ((g(x^1))_i, *)$ where $x^1, x^2 \in X$, and $*$ is a constant symbol. We define $\bar{g} : X \times X \to X'$ so that $(\bar{g}(x^1,x^2))_i = (g(x^1))_i$. We introduce $\hat{g}$ and $\bar{g}$ for technical reasons that will become apparent in the proof of our next theorem.

THEOREM 3.2. *Let* X, Y, $(Y(n))$, (f_n), *and* f *be as in Theorem* 3.1. *Let* X' *be a BS such that* $h(X') = h(Y)$. *Let* g *be a finitary factor map from* X

to X'. Then there exists a sequence of finitary factor maps $g_n : X \times X \to X(n)$ such that the following hold:

(i) *$(g_n) \to \hat{g}$ or $\bar{g}$ in d;*
(ii) *for each n, $X(n)$ is a BS;*
(iii) *for each n, $X(n)$ is finitarily isomorphic to $Y(n)$.*

Proof. Since $\lim_{n\to\infty} h(Y(n)) = h(Y)$, there exists a monotonic subsequence $(h(Y(n_k)))$ of entropies of the $Y(n)$. We consider three cases. First, there exists a subsequence $(Y(n_k))$ where $h(Y(n_k)) = h(X')$ for all k. Next, we consider cases of subsequences $(Y(n_k))$ where $h(Y(n_k))$ is strictly increasing or strictly decreasing. We verify (i) and (ii) for each of the three cases separately. Then, at the end of the proof, we establish (iii) for all three cases together.

Suppose there exists a subsequence $(Y(n_k))$ where $h(Y(n_k)) = h(X')$ for all k. Then define $g_{n_k} : X \times X \to X(n_k)$ to be $\bar{g}$ (and thus $X(n_k) = X'$).

Suppose that $(h(Y(n)))$ has a strictly decreasing subsequence. Then we have $h(Y(n_k)) > h(X')$ for all n_k. To simplify notation, we will now drop the subscript k. In other words, $Y(n)$ will refer to our subsequence with strictly decreasing entropies. Let $X(n) = X' \times W(n)$ where each $W(n)$ is a BS such that $h(X(n)) = h(Y(n))$. Since $\lim_{n\to\infty} h(Y(n)) = h(Y)$, $\lim_{n\to\infty} h(X(n)) = h(X')$. So, for some positive integer N, if $n > N$, then $h(X(n)) - h(X') < \log 2$. For $n > N$, we can take $W(n)$ to be a two-state BS. Let $W(n)$ have alphabet $\{b, *\}$ and measure p_n where $p_n(*) > p_n(b)$. Define $(W(n))_{n>N}$ so that $p_{n+1}(*) \geq p_n(*)$. Since $\lim_{n\to\infty} h(X(n)) = h(X')$, $\lim_{n\to\infty} p_n(*) = 1$.

For each n, $W(n)$ is a BS such that $h(W(n)) < h(X)$. By [1], there exists a finitary factor map $\psi_n : X \to W(n)$. Define $g_n : X \times X \to X(n)$ where $(g_n(x^1, x^2))_i = ((g(x^1))_i, (\psi_n(x^2))_i)$. We know $(g_n(x^1, x^2))_i = ((g(x^1))_i, *)$ if $(\psi_n(x^2))_i = *$. Since $p_n(*) \to 1$ as $n \to \infty$, $(g_n) \to \hat{g}$ in d.

Now suppose that $(h(Y(n)))$ has a strictly increasing subsequence. Then $h(Y(n_k)) < h(X')$ for each n_k. Again, we will drop the subscript k. Suppose X' has measure $q^{\mathbb{Z}}$ where $q = (q_1, \ldots, q_m)$, and $q_1 \geq \cdots \geq q_m$. Let $\hat{X}'$ be a BS with probability vector $\hat{q} = (q_1, \ldots, q_{m-1} + C, q_m - C)$ where C is a positive constant less than q_m, and $C/q_m \leq q_m - C$. Since $\lim_{n\to\infty} h(Y(n)) = h(Y)$, there exists a positive integer N such that for all $n > N$, $h(\hat{X}') \leq h(Y(n)) < h(X')$. So, for each $n > N$, there exists a positive constant c_n such that $0 \leq c_n \leq C$, and $h(Y(n)) = h(X(n))$ if we define $X(n)$ to have probability vector $(q_1, \ldots, q_{m-1} + c_n, q_m - c_n)$. Let $X(n)$ have corresponding symbol space $\{1, \ldots, m\}$.

Let $Z(n)$ be a BS with measure $(c_n/q_m, 1 - c_n/q_m)^{\mathbb{Z}}$ and corresponding symbol space $\{a, b\}$. Since $c_n/q_m \leq q_m$ and $h(X') \leq h(X)$, we have $h(X) \geq h(Z(n))$. So by [1], there exists a finitary factor map $\phi_n : X \to Z(n)$.

Let $g_n : X \times X \to X(n)$ be defined as follows. Let $(g_n(x^1, x^2))_i = ((g(x^1))_i$ if $(g(x^1))_i \neq m$ or $(\phi_n(x^2))_i \neq a$. If $(g(x^1))_i = m$ and $(\phi_n(x^2))_i = a$, then let $(g_n(x^1, x^2))_i = m-1$. The image is $X(n)$ since $P[(x_i^2 = a) \cap (x_i^1 = m)] = (c_n/q_m) \cdot q_m = c_n$. Since $\lim_{n\to\infty} h(X(n)) = h(X')$, $\lim_{n\to\infty} c_n = 0$. Thus, $g_n \to \bar{g}$ in d.

Now we establish (iii). By Theorem 3.1, each $Y(n)$ is fB. By [2], for any sequence $(X(n))$ of BSs where for each n, $h(X(n)) = h(Y(n))$, $Y(n)$ will be finitarily isomorphic to $X(n)$. ■

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Stephen Shea
Department of Mathematics
St. Anselm College
100 St. Anselm Drive #1792
Manchester, NH 03102, U.S.A.
E-mail: sshea@anselm.edu

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