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The tree property at the double successor of a measurable cardinal κ with 2^κ large

by

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Abstract. Assuming the existence of a λ^+-hypermeasurable cardinal κ, where λ is the first weakly compact cardinal above κ, we prove that, in some forcing extension, κ is still measurable, κ^{++} has the tree property and $2^\kappa = \kappa^{+++}$. If the assumption is strengthened to the existence of a θ-hypermeasurable cardinal (for an arbitrary cardinal $\theta > \lambda$ of cofinality greater than κ) then the proof can be generalized to get $2^\kappa = \theta$.

1. Introduction. For an infinite cardinal κ, a *κ-tree* is a tree T of height κ such that every level of T has size less than κ. A tree T is a *κ-Aronszajn tree* if T is a κ-tree which has no branches of length κ. We say that *the tree property holds at κ*, or $\mathrm{TP}(\kappa)$ *holds*, if every κ-tree has a branch of length κ. Thus, $\mathrm{TP}(\kappa)$ holds iff there is no κ-Aronszajn tree. $\mathrm{TP}(\aleph_0)$ holds in ZFC, and it is actually exactly the statement of the well-known König lemma. Aronszajn showed also in ZFC that there is an $\aleph_1$-Aronszajn tree. Hence, $\mathrm{TP}(\aleph_1)$ fails in ZFC.

Large cardinals are needed once we consider trees of height greater than $\aleph_1$. Silver proved that, for $\kappa > \aleph_1$, $\mathrm{TP}(\kappa)$ implies κ is weakly compact in L. Mitchell proved that given a weakly compact cardinal λ above a regular cardinal κ, one can make λ into κ^+ so that, in the extension, κ^+ has the tree property. Thus, $\mathrm{TP}(\aleph_2)$ is equiconsistent with the existence of a weakly compact cardinal.

For more of the relevant literature on the tree property we refer the reader to the following: Abraham [1], Cummings and Foreman [3], Foreman, Magidor and Schindler [5], and Neeman [10] have done work on the tree property at two or more successive cardinals; Magidor and Shelah [9], Neeman [10], and Sinapova [11], [12] have worked on the tree property at successors of singular cardinals.

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Natasha Dobrinen and the first author [4] used a generalization of Sacks forcing to reduce the large cardinal strength required to obtain the tree property at the double successor of a measurable cardinal κ from a supercompact to a weakly compact hypermeasurable cardinal. In their model $2^\kappa = \kappa^{++}$.

On the other hand, $\mathrm{TP}(\aleph_2)$ is consistent with large continuum (for a proof see [13]). In the present paper we prove the analogous result for $\mathrm{TP}(\kappa^{++})$ with κ measurable, using Mitchell's forcing together with a surgery argument (see [2]).

As in [4], the consistency of a cardinal κ of Mitchell order λ^+, where λ is weakly compact and greater than κ, is a lower bound on the consistency strength of $\mathrm{TP}(\kappa^{++})$ with κ measurable and $2^\kappa = \kappa^{+++}$. Therefore our result is in fact almost an equiconsistency result.

2. The theorem. We say that a cardinal κ is *γ-hypermeasurable* if there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ such that $H(\gamma)^V = H(\gamma)^M$.

THEOREM. *Assume that V is a model of ZFC and κ is λ^+-hypermeasurable in V, where λ is the least weakly compact cardinal greater than κ. Then there exists a forcing extension of V in which κ is still measurable, κ^{++} has the tree property and $2^\kappa = \kappa^{+++}$.*

Proof. Let κ be λ^+-hypermeasurable. Let $j : V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $H(\lambda^+)^V = H(\lambda^+)^M$. We may assume that M is of the form $M = \{j(f)(\alpha) \mid \alpha < \lambda^+,\ f : \kappa \to V,\ f \in V\}$. We first define some forcing notions in order to describe the intended model.

For a regular cardinal α and an arbitrary cardinal β let $\mathrm{Add}(\alpha, \beta)$ denote the forcing for adding β many α-Cohens. The conditions are partial functions from $\alpha \times \beta$ into $\{0, 1\}$ of size $< \alpha$.

Define a forcing notion P_κ as follows. Let ρ_0 be the first inaccessible cardinal and let λ_0 be the least weakly compact cardinal above ρ_0. For $k < \kappa$, given λ_k, let ρ_{k+1} be the least inaccessible cardinal above λ_k and let λ_{k+1} be the least weakly compact cardinal above ρ_{k+1}. For limit ordinals $k \le \kappa$, let ρ_k be the least inaccessible cardinal greater than or equal to $\sup_{l<k} \lambda_l$ and let λ_k be the least weakly compact cardinal above ρ_k. Note that $\rho_\kappa = \kappa$ and λ_κ is the least weakly compact cardinal above κ.

Let P_0 be the trivial forcing. For $i < \kappa$, if $i = \rho_k$ for some $k < \kappa$, let $\dot{Q}_i$ be a P_i-name for the forcing $\mathrm{Add}(\rho_k, \lambda_k^+)$. Otherwise let $\dot{Q}_i$ be a P_i-name for the trivial forcing. Let $P_{i+1} = P_i * \dot{Q}_i$. Let P_κ be the iteration $\langle\langle P_i, \dot{Q}_i\rangle : i < \kappa\rangle$ with Easton support.

We define the *Mitchell forcing* $M(\kappa, \beta)$ as the iteration $\mathrm{Add}(\kappa, \beta) * Q$, where

$$Q = \{q \mid q \text{ is a partial function of cardinality } \leq \kappa \text{ on the regular cardinals below } \beta \text{ such that for each } \gamma \text{ in } \mathrm{Dom}(q),\ \emptyset \Vdash^{\mathrm{Add}(\kappa,\gamma)} \text{“}q(\gamma) \in \mathrm{Add}(\kappa^+,1)\text{”}\}.$$

Since $M(\kappa,\lambda)$ is known to preserve the tree property at λ while making λ into the κ^{++} of the extension (see [1]), the idea is simply to force with $\mathrm{Add}(\kappa,\lambda^+)$ over $V^{M(\kappa,\lambda)}$. However, in order to preserve the measurability of κ, our intended model will be a little different:

Let $j_0 : V \to M_0$ be the measure ultrapower embedding via the normal measure $U_0 = \{X \subseteq \kappa \mid \kappa \in j(X)\}$ derived from j with critical point κ such that ${}^{\kappa}M_0 \subseteq M_0$ and let λ_0 be the first weakly compact cardinal of M_0 above κ. To prove the theorem we force over V with

$$P_\kappa * \mathrm{Add}(\kappa,(\lambda_0^+)^{M_0}) * M(\kappa,\lambda) * \mathrm{Add}(\kappa,\lambda^+) * R,$$

where P_κ is the ‘preparatory’ forcing defined above, and R is the forcing notion defined in the following paragraph:

Let G, g_0 be generic filters on $P_\kappa, \mathrm{Add}(\kappa,(\lambda_0^+)^{M_0})$, respectively. We lift the embedding $j_0 : V \to M_0$ to an embedding of $V[G]$ as follows. The forcing $j_0(P_\kappa)$ can be factored into the three obvious parts $j_0(P_\kappa)_{|\kappa} * j_0(P_\kappa)_\kappa * j_0(P_\kappa)_{\kappa+1,j_0(\kappa)}$, but since V and M_0 have the same H_{κ^+}, we have $j_0(P_\kappa)_{|\kappa} = P_\kappa$. By elementarity, $j_0(P_\kappa)_\kappa$ is the forcing $\mathrm{Add}(\kappa,(\lambda_0^+)^{M_0})$. Therefore, $G * g_0$ is generic for $j_0(P_\kappa)_{|\kappa} * j_0(P_\kappa)_\kappa$ over M_0. We can easily construct in $V[G][g_0]$ a generic filter H_0 over $M_0[G][g_0]$ for the remaining forcing $j_0(P_\kappa)_{\kappa+1,j_0(\kappa)}$, using the facts that $j_0(P_\kappa)_{\kappa+1,j_0(\kappa)}$ is κ^+-closed in $M_0[G][g_0]$, $V[G][g_0] \cap {}^{\kappa}M_0[G][g_0] \subseteq M_0[G][g_0]$, and each dense subset of $j_0(P_\kappa)_{\kappa+1,j_0(\kappa)}$ in $M_0[G][g_0]$ has an $\mathrm{Add}(\kappa,(\lambda_0^+)^{M_0})$-name in $M_0[G]$ of the form $j_0(f)(\kappa)$ for some function $f \in V[G], f : \kappa \to H(\kappa^+)$. Therefore, j_0 lifts in $V[G][g_0]$ to an elementary embedding $j_0 : V[G] \to M_0[G][g_0][H_0]$ because j_0 is the identity on the conditions in G, and hence obviously $j_0[G] \subseteq G * g_0 * H_0$. The forcing R is defined as $\mathrm{Add}(j_0(\kappa),\lambda^+)$ of $M_0[G][g_0][H_0]$. We note here that R is an element of $V[G][g_0]$. Since $j_0(\lambda) = \lambda$, R is actually the image of $\mathrm{Add}(\kappa,\lambda^+)$ under j_0.

For technical reasons, we rewrite our forcing as

$$P_\kappa * \mathrm{Add}(\kappa,\lambda^+) * Q * R,$$

where Q is this time defined only using the even components i of $\mathrm{Add}(\kappa,\lambda^+)$ with $(\lambda_0^+)^{M_0} \leq i < \lambda$. More precisely, for an interval I of ordinals let $\mathrm{Add}(\kappa,I)_{|\mathrm{even}}$ be the forcing whose conditions are partial functions from $\kappa \times \{\text{even ordinals in } I\}$ into $\{0,1\}$ of size $< \kappa$. Then, for $q \in Q$ and $\gamma \in \mathrm{Dom}(q)$, $q(\gamma)$ is an $\mathrm{Add}(\kappa,[(\lambda_0^+)^{M_0},\gamma))_{|\mathrm{even}}$-name for a condition in $\mathrm{Add}(\kappa^+,1)$.

We denote the final model $V^{P_\kappa * \mathrm{Add}(\kappa,\lambda^+) * Q * R}$ as W.

Definition. Let A and B be two partial orderings. A function $\pi : B \to A$ is called a *projection* if the following hold:

- π is order-preserving and $\pi(B)$ is dense in A.
- If $\pi(b) = a$ and $a' < a$, then there is $b' \leq b$ such that $\pi(b') \leq a'$.

Fact. *If $\pi : B \to A$ is a projection, then the forcing B is forcing-equivalent to $A * B/A$ for some quotient B/A (see* [1] *for details).*

Since both Q and $\mathrm{Add}(\kappa, [(\lambda_0^+)^{M_0}, \lambda))_{|\mathrm{even}}$ exist in the model $V[G][g_0]$, we can also consider the forcing $\mathrm{Add}(\kappa, [(\lambda_0^+)^{M_0}, \lambda))_{|\mathrm{even}} \times Q$. In order not to confuse it with $\mathrm{Add}(\kappa, [(\lambda_0^+)^{M_0}, \lambda))_{|\mathrm{even}} * Q$, which has a different ordering, we will write $\mathrm{Add}(\kappa, [(\lambda_0^+)^{M_0}, \lambda))_{|\mathrm{even}} \times Q'$. For the same reason, the conditions (p, q) in the product will be denoted as $(p, (0, q))$.

It can be shown that Q is κ^+-distributive, and Q' is obviously κ^+-closed in $V[G][g_0]$. See [1] for a proof of the following lemma.

Lemma 1. *The map π given by $\pi(p, (0, q)) = (p, q)$ is a projection from*

$$\mathrm{Add}(\kappa, [(\lambda_0^+)^{M_0}, \lambda))_{|\mathrm{even}} \times Q' \quad \textit{onto} \quad \mathrm{Add}(\kappa, [(\lambda_0^+)^{M_0}, \lambda))_{|\mathrm{even}} * Q.$$

This projection can be naturally extended to a projection from

$$\mathrm{Add}(\kappa, [(\lambda_0^+)^{M_0}, \lambda^+)) \times Q' \times R \quad \text{onto} \quad \mathrm{Add}(\kappa, [(\lambda_0^+)^{M_0}, \lambda^+)) * Q * R.$$

Lemma 2. *R is κ^+-closed and λ-Knaster in $V[G][g_0]$.*

Proof. The closure follows easily because R is κ^+-closed in $M_0[G][g_0][H_0]$ and $M_0[G][g_0][H_0]$ is closed under κ-sequences in $V[G][g_0]$. Let $\langle p_\alpha : \alpha < \lambda \rangle$ be a sequence of conditions in R, and let p_α be of the form $j_0(f_\alpha)(\kappa)$ for some function $f_\alpha : \kappa \to \mathrm{Add}(\kappa, \lambda^+)$, $f_\alpha \in V[G]$. A Δ-system argument shows that λ many of the functions f_α are pointwise compatible. It follows that λ many of the conditions p_α are compatible. ∎

Lemma 3. *The forcing $Q * R$ is κ^+-distributive in $V^{P_\kappa * \mathrm{Add}(\kappa, \lambda^+)}$.*

Proof. The forcings Q', R are closed in the model $V^{P_\kappa * \mathrm{Add}(\kappa, (\lambda_0^+)^{M_0})}$ in which they are defined, therefore their product $Q' \times R$ is closed there as well. By Easton's lemma, after forcing with the κ^+-c.c. forcing $\mathrm{Add}(\kappa, [(\lambda_0^+)^{M_0}, \lambda^+))$, the product $Q' \times R$ will remain κ^+-distributive. Since κ^+-distributivity is equivalent to not adding new κ-sequences of ordinals, it follows from the above facts about projections that $Q * R$ is distributive in $V^{P_\kappa * \mathrm{Add}(\kappa, \lambda^+)}$ as well. ∎

Lemma 4. *In W, $\kappa^+ = (\kappa^+)^V$, $\kappa^{++} = \lambda$, and $\kappa^{+++} = (\lambda^+)^V$. In particular, $2^\kappa = \kappa^{+++}$.*

Proof. $\kappa^+ = (\kappa^+)^V$: This follows from the facts that $P_\kappa * \mathrm{Add}(\kappa, \lambda^+)$ is κ^+-c.c in V, and $Q * R$ is κ^+-distributive in $V^{P_\kappa * \mathrm{Add}(\kappa, \lambda^+)}$.

$\kappa^{++} = \lambda, \kappa^{+++} = (\lambda^+)^V$: The Mitchell forcing $M(\kappa, \lambda)$ collapses precisely the cardinals between κ^+ and λ (see [1, Lemma 2.4] for a proof). On the other hand, in the model $V^{P_\kappa * \mathrm{Add}(\kappa,(\lambda_0^+)^{M_0})}$, in which all cardinals are preserved, R has the λ-Knaster property and $M(\kappa, \lambda) * \mathrm{Add}(\kappa, \lambda^+)$ satisfies the λ-c.c. It follows that their product also satisfies the λ-c.c., which means that all cardinals above λ are preserved. ∎

REMARK. In the general case where κ is θ-hypermeasurable we can first force to add a function $f : \kappa \to \kappa$ with $j(f)(\kappa) = \theta$. Then θ_0, M_0's version of θ, is less than κ^{++}, because $\theta_0 = j_0(f)(\kappa) < j_0(\kappa) < \kappa^{++}$. It follows that the forcing R still has the λ-Knaster property in $V^{P_\kappa * \mathrm{Add}(\kappa,\theta_0)}$.

To complete our proof we need to show that, in the extension, κ is still measurable and $\lambda = \kappa^{++}$ still has the tree property.

LEMMA 5. *κ remains measurable in W.*

Proof. In order to prove that κ remains measurable in W we intend to extend the elementary embedding $j : V \to M$ to an embedding of W. We have already picked generics G, g_0 for the forcings $P_\kappa, \mathrm{Add}(\kappa, (\lambda_0^+)^{M_0})$, respectively. Let g be an $\mathrm{Add}(\kappa, [(\lambda_0^+)^{M_0}, \lambda^+))$-generic filter over $V[G][g_0]$. We first use a 'surgery' argument to lift j to an embedding of $V[G][g_0][g]$. For completeness we give the full proof.

The embedding j can be factored as $k \circ j_0$, where $k : M_0 \to M$ is defined by $k([F]_U) := j(F)(\kappa)$. The embedding k is also elementary and its critical point is $(\kappa^{++})^{M_0}$. By elementarity and GCH, $(\kappa^{++})^{M_0} < j_0(\kappa) < \kappa^{++}$. Note also that $k(\lambda_0) = \lambda$.

On page 57 we have lifted in $V[G][g_0]$ the embedding $j_0 : V \to M_0$ to an embedding $j_0 : V[G] \to M_0[G][g_0][H_0]$.

Next we lift the embedding $k : M_0 \to M$ to $M_0[G][g_0][H_0]$. It lifts trivially to $k : M_0[G] \to M[G]$. Note that $k(\mathrm{Add}(\kappa, (\lambda_0^+)^{M_0})) = \mathrm{Add}(\kappa, \lambda^+)$, and that k is almost the identity on the conditions in g_0, namely, it only shifts the ith component to the $k(i)$th component. Therefore, we can rearrange the generic filter $g_0 \times g$ into some $(g_0 \times g)'$ such that the ith component of $g_0 \times g$ is the same as the $k(i)$th component of $(g_0 \times g)'$. Then $k[g_0] \subseteq (g_0 \times g)'$ and the embedding k lifts in $V[G][g_0][g]$ to $k : M_0[G][g_0] \to M[G][(g_0 \times g)']$ (note that k restricted to λ_0^+ belongs to $M[G]$). And finally, we 'transfer' H_0 along k to build a filter H generic for $k(j_0(P_\kappa)_{\kappa+1,j_0(\kappa)}) = j(P_\kappa)_{\kappa+1,j(\kappa)}$. Namely, let $H = \{p \in j(P_\kappa)_{\kappa+1,j(\kappa)} \mid k(p_0) \le p \text{ for some } p_0 \in H_0\}$; then H is generic for $j(P_\kappa)_{\kappa+1,j(\kappa)}$. To see this, note that each open dense set $D \subseteq j(P_\kappa)_{\kappa+1,j(\kappa)}$ in $M[G][(g_0 * g)']$ is of the form $k(f)(a)$ for some $f \in M_0[G][g_0]$ with domain of size $(\lambda_0^+)^{M_0}$, because every element of M is of the form $j(f')(\alpha) = (k(j_0(f')))_{\restriction \lambda^+}(\alpha) = k(j_0(f')_{\restriction (\lambda_0^+)^{M_0}})(\alpha)$ for some $f' \in V$, $\alpha < \lambda^+$. We may assume that $f(x)$ is an open dense subset of $j_0(P_\kappa)_{\kappa+1,j_0(\kappa)}$ for each

$x \in \mathrm{Dom}(f)$, and since $j_0(P_\kappa)_{\kappa+1,j_0(\kappa)}$ is $(\lambda_0^{++})^{M_0}$-closed in $M_0[G][g_0]$, we may choose $r_0 \in H_0$ belonging to each $f(x)$, $x \in \mathrm{Dom}(f)$. It follows that $k(r_0) \in D \cap H$.

Therefore, we can lift k to $k : M_0[G][g_0][H_0] \to M[G][(g_0 \times g)'][H]$ in $V[G][g_0][g]$, getting the commutative diagram

$$\begin{tikzcd} V[G] \arrow[rr, "j"] \arrow[dr, "j_0"'] & & M[G][(g_0 \times g)'][H] \\ & M_0[G][g_0][H_0] \arrow[ur, "k"'] & \end{tikzcd}$$

We now plan to lift $j : V[G] \to M[G][(g_0 \times g)'][H]$ to an embedding of $V[G][g_0][g]$. Let $G_Q \times h$ be a filter on $Q \times R$ which is generic over $V[G][g_0][g]$. We transfer h along k, just as we did with H_0, in order to get a generic h^* for $j(\mathrm{Add}(\kappa, \lambda^+))$ so that we could lift j to $j : V[G][g_0][g] \to M_0[G][g_0][H_0][h^*]$. The fact that h can be transferred to create a generic for $j(\mathrm{Add}(\kappa, \lambda^+))$, and the fact that $R = j_0(\mathrm{Add}(\kappa, \lambda^+))$ is not a harmful forcing in $V[G][g_0]$, i.e. has κ^+-closure and λ-Knaster property, are the main advantages of factoring j as $k \circ j_0$.

This lifting argument is called surgery, because we still have to make sure that $j[g_0 \times g] \subseteq h^*$, and that is done by altering the generic h^* on a small part, as follows. Let $F = \bigcup g_0 \times g : \kappa \times \lambda^+ \to 2$ be the function corresponding to the generic $g_0 \times g$. Then $\bigcup j[g_0 \times g]$ is the function $F^* : \kappa \times j[\lambda^+] \to 2$ defined by $F^*(\gamma, j(\delta)) = F(\gamma, \delta)$. We have to modify h^* to h^{**} so that each q^* in h^{**} is compatible with F^* so that $j[g_0 \times g] \subseteq h^{**}$. Finally we show that h^{**} is also a generic filter.

For any $q \in h^*$ let q^* be defined by altering q on $\mathrm{Dom}(q) \cap (\kappa \times j[\lambda^+])$ to agree with F^*. We claim that q^* belongs to $M[G][(g_0 \times g)'][H]$, and therefore is a condition in $j(\mathrm{Add}(\kappa, \lambda^+))$. We can write q as $j(f)(\alpha)$ for some $\alpha < \lambda$ and some function $f : \kappa \to \mathrm{Add}(\kappa, \lambda^+)$, $f \in V[G]$. If $(\gamma, j(\delta))$ belongs to $\mathrm{Dom}(q)$, then (γ, δ) belongs to $\mathrm{Dom}(f(\beta))$ for some $\beta < \kappa$, so $\{(\gamma, \delta) \mid (\gamma, j(\delta)) \in \mathrm{Dom}(q)\}$ is contained in $Z_0 = \bigcup_\beta \mathrm{Dom}(f(\beta)) \in V[G]$. As Z_0 has size at most κ and P_κ is κ-c.c., there is $Z \in V$ with $Z_0 \subseteq Z \subseteq \kappa \times \lambda^+$ of size at most κ. Then Z belongs to M and $j{\restriction}Z$ also belongs to M. Using $q, g_0, g, j{\restriction}Z$ we can define q^*, and therefore q^* belongs to $M[G][(g_0 \times g)'][H]$.

Claim. *$h^{**} := \{q^* \mid q \in h^*\}$ is $j(\mathrm{Add}(\kappa, \lambda^+))$-generic over the model $M[G][(g_0 \times g)'][H]$.*

Proof. Suppose that D is an open dense subset of $j(\mathrm{Add}(\kappa, \lambda^+))$, $D \in M[G][(g_0 \times g)'][H]$. For any $q \in j(\mathrm{Add}(\kappa, \lambda^+))$ define $N(q)$ to be the set of conditions r with the same domain as q which disagree with q on a set of size at most κ. Then $E = \{q \mid N(q) \subseteq D\}$ is a dense subset of $j(\mathrm{Add}(\kappa, \lambda^+))$

as well, by the $j(\kappa)$-closure of $j(\mathrm{Add}(\kappa, \lambda^+))$. Choose q in $E \cap h^*$. Then q^* belongs to $N(q)$, and therefore to D. It follows that h^{**} intersects D. This shows the Claim. ∎

So far we have proven that in $V[G][g_0][g][h]$ there is a definable elementary embedding $j : V[G][g_0][g] \to M[G][(g_0 \times g)'][H][h^{**}]$.

We now need to find a generic filter $G_{j(Q)} \times h_{j(R)}$ for $j(Q \times R)$ such that $j[G_Q \times h] \subseteq G_{j(Q)} \times h_{j(R)}$, in order to define our final lifting

$$j : V[G][g_0][g][G_Q][h] \to M[G][(g_0 \times g)'][H][h^{**}][G_{j(Q)}][h_{j(R)}].$$

This last step is, however, just another transferring argument since, by Lemma 3, $Q \times R$ is κ^+-distributive over $V[G][g_0][g]$, that is, $G_{j(Q)} \times h_{j(R)} := \{(q, r) \mid j(q_0, r_0) \le (q, r) \text{ for some } (q_0, r_0) \in G_Q \times h\}$ is an appropriate generic. This completes the proof of Lemma 5. ∎

Lemma 6. *κ^{++} has the tree property in W.*

Proof. In order to get a contradiction suppose that there is a κ^{++}-Aronszajn tree in W.

Recall that W can be written as $V^{P_\kappa * \mathrm{Add}(\kappa, (\lambda_0^+)^{M_0}) * M(\kappa,\lambda) * \mathrm{Add}(\kappa,\lambda^+) * R}$. Let V_1 denote the model $V^{P_\kappa * \mathrm{Add}(\kappa, (\lambda_0^+)^{M_0}) * M(\kappa,\lambda)}$ and let $R' = R_{|\lambda}$ be the forcing $\mathrm{Add}(j_0(\kappa), \lambda)$ of $M_0[G][g_0][H_0]$. We first notice that there must be a κ^{++}-Aronszajn tree already in $V_1^{\mathrm{Add}(\kappa,\lambda) \times R'}$. Indeed, note that $\mathrm{Add}(\kappa, \lambda^+) \times R$ has the λ-c.c. in V_1 (see the proof of Lemma 4) and let π be an $\mathrm{Add}(\kappa, \lambda^+) \times R$-name in V_1 for a subset of $\lambda = \kappa^{++}$ which codes a κ^{++}-Aronszajn tree. Then for every $\alpha < \kappa^{++}$ there is a maximal antichain A_α of size less than λ such that each $q \in A_\alpha$ decides the statement $\check{\alpha} \in \pi$. Let $B = \bigcup\{\mathrm{Dom}(q) \mid q \in A_\alpha \text{ for some } \alpha\}$. Then $|B| = \lambda$, and $(\mathrm{Add}(\kappa, \lambda^+) \times R)_{|B}$ is isomorphic to $\mathrm{Add}(\kappa, \lambda) \times R'$. Thus, we can replace $\mathrm{Add}(\kappa, \lambda^+) \times R$ with its isomorphic copy such that π is an $\mathrm{Add}(\kappa, \lambda) \times R'$-name, which means that there is a κ^{++}-Aronszajn tree in $V_1^{\mathrm{Add}(\kappa,\lambda) \times R'}$.

Just as before, rewrite $P_\kappa * \mathrm{Add}(\kappa, (\lambda_0^+)^{M_0}) * M(\kappa, \lambda) * \mathrm{Add}(\kappa, \lambda) \times R'$ as $P_\kappa * \mathrm{Add}(\kappa, (\lambda_0^+)^{M_0}) * \mathrm{Add}(\kappa, \lambda) * Q \times R'$, where Q is defined only using the even components of $\mathrm{Add}(\kappa, \lambda)$. Hence, in terms of our chosen generics, the above means that there is a κ^{++}-Aronszajn tree T in $V[G][g_0][g_{|\lambda}][G_Q][h_{|\lambda}]$. Let $\dot{T}$ be an $\mathrm{Add}(\kappa, \lambda) * Q \times R'$-name in $V[G][g_0]$ for T.

Recall that λ is a weakly compact cardinal in $V[G][g_0]$. Therefore, there exist in $V[G][g_0]$ transitive ZF^--models N_0, N_1 of size λ and an elementary embedding $k : N_0 \to N_1$ with critical point λ, such that $N_0 \supseteq H(\lambda)^{V[G][g_0]}$ and $G, g_0, \dot{T} \in N_0$.

Note that $g_{|\lambda} * G_Q * h_{|\lambda}$ is also $\mathrm{Add}(\kappa, \lambda) * Q \times R'$-generic over N_0. Since λ is the critical point of k, we can factor $k(\mathrm{Add}(\kappa, \lambda) * Q \times R')$ as

$$\mathrm{Add}(\kappa, \lambda) * \mathrm{Add}(\kappa, [\lambda, k(\lambda))) * Q * Q^* * R' * R^*$$

where Q^* and R^* denote the tail forcings $k(Q)/Q$ and $k(R')/R'$, respectively, with components indexed from the interval $[\lambda, k(\lambda))$. Since k is the identity on $g_{|\lambda} * G_Q * h_{|\lambda}$ we can extend the embedding $k : N_0 \to N_1$ in some large universe U to an embedding

$$k : N_0[g_{|\lambda}][G_Q][h_{|\lambda}] \to N_1[g_{|\lambda}][g^*][G_Q][G_{Q^*}][h_{|\lambda}][h^*]$$

where g^*, G_{Q^*}, h^* are arbitrary generics for $\mathrm{Add}(\kappa, [\lambda, k(\lambda)))$, Q^*, R^*, respectively, picked in U. (We can assume that G_Q is generic over $N_1[g_{|\lambda}][g^*]$ and that $h_{|\lambda}$ is generic over $N_1[g_{|\lambda}][g^*][G_Q][G_{Q^*}]$, for one can start the argument by first picking an $\mathrm{Add}(\kappa, \lambda) * \mathrm{Add}(\kappa, [\lambda, k(\lambda))) * Q * Q^* * R' * R^*$-generic filter $g_{|\lambda} * g^* * G_Q * G_{Q^*} * h_{|\lambda} * h^*$ over V in some large universe U, and then restricting it to $\mathrm{Add}(\kappa, \lambda) * Q \times R'$ to get $g_{|\lambda} * G_Q * h_{|\lambda}$.)

Since $T \in N_0[g_{|\lambda}][G_Q][h_{|\lambda}]$ is a λ-Aronszajn tree, by elementarity $k(T)$ is a $k(\lambda)$-Aronszajn tree in $N_1[g_{|\lambda}][g^*][G_Q][G_{Q^*}][h_{|\lambda}][h^*]$ which coincides with T up to level λ. Hence T has a cofinal branch b in $N_1[g_{|\lambda}][g^*][G_Q][G_{Q^*}][h_{|\lambda}][h^*]$. We will show that b must actually belong to $N_1[g_{|\lambda}][G_Q][h_{|\lambda}]$ (i.e. the tail generics g^*, G_{Q^*}, h^* cannot add a new branch), and thereby reach the desired contradiction to the assumption that T has no cofinal branches in $V[G][g_0][g][G_Q][h]$!

Similarly to the discussion following Lemma 1, in N_1 there is a projection from the product

$$\mathrm{Add}(\kappa, \lambda) \times \mathrm{Add}(\kappa, [\lambda, k(\lambda))) \times Q' \times Q^{*\prime} \times R' \times R^*$$

onto

$$\mathrm{Add}(\kappa, \lambda) * \mathrm{Add}(\kappa, [\lambda, k(\lambda))) * Q * Q^* * R' * R^*,$$

where Q' and $Q^{*\prime}$ are κ^+-closed forcings defined in N_1. Let $G_{Q'} \times G_{Q^{*\prime}}$ be $Q' \times Q^{*\prime}$-generic over $N_1[g_{|\lambda}][g^*]$. (Again we can assume that $h_{|\lambda}$ is generic over the bigger model $N_1[g_{|\lambda}][g^*][G_{Q'}][G_{Q^{*\prime}}]$.)

If we can show that the bigger generic $g^* * G_{Q^{*\prime}} * h^*$ does not add the branch b through T over the bigger model $N_1[g_{|\lambda}][G_{Q'}][h_{|\lambda}]$, then in particular the smaller generic $g^* * G_{Q^*} * h^*$ does not add b over the smaller model $N_1[g_{|\lambda}][G_Q][h_{|\lambda}]$, and we are done.

Since all the forcings are defined in N_1, we can 'reorder the generics' in $N_1[g_{|\lambda}][g^*][G_{Q'}][G_{Q^{*\prime}}][h_{|\lambda}][h^*]$ as we want. Write

$$N_1[g_{|\lambda}][g^*][G_{Q'}][G_{Q^{*\prime}}][h_{|\lambda}][h^*] \quad \text{as} \quad N_1[G_{Q'}][h_{|\lambda}][g_{|\lambda}][g^*][G_{Q^{*\prime}}][h^*].$$

Note that in $N_1[G_{Q'}][h_{|\lambda}]$, $Q^{*\prime} \times R^*$ is a κ^+-closed forcing and $\mathrm{Add}(\kappa, k(\lambda))$ is κ^+-c.c. Therefore, it can be shown that $Q^{*\prime} \times R^*$ does not add any branches to T over the model $N_1[G_{Q'}][h_{|\lambda}][g_{|\lambda}][g^*]$ (for a detailed proof of this lemma see [13]).

Finally, $\mathrm{Add}(\kappa, [\lambda, k(\lambda)))$ has the κ^{++}-Knaster property, which means that it could not have added the branch b over the model $N_1[G_{Q'}][h_{|\lambda}][g_{|\lambda}]$ either.

This proves Lemma 6 and hence the proof of the Theorem is complete. ∎

3. Open questions

1. Is it possible to singularize the cardinal κ of the above model preserving the tree property of κ^{++}?
2. Does the consistency of the existence of a measurable cardinal κ, such that $\mathrm{TP}(\kappa^{++})$ and $2^\kappa = \kappa^{+++}$, follow from the consistency of the existence of a cardinal κ of Mitchell order λ^+ where λ is weakly compact (yielding an equiconsistency result)?
3. Is it consistent to have $\mathrm{TP}(\aleph_{\omega+2})$, $\aleph_\omega$ strong limit, and $2^{\aleph_\omega}$ large?
4. Is it consistent to have $\mathrm{TP}(\lambda^+)$, λ singular strong limit, and 2^λ large?

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