FUNDAMENTA
MATHEMATICAE
223 (2013)

Dimension-raising maps in a large scale

by

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Abstract. Hurewicz's dimension-raising theorem states that $\dim Y \le \dim X + n$ for every n-to-1 map $f : X \to Y$. In this paper we introduce a new notion of finite-to-one like map in a large scale setting. Using this notion we formulate a dimension-raising type theorem for asymptotic dimension and asymptotic Assouad–Nagata dimension. It is also well-known (Hurewicz's finite-to-one mapping theorem) that $\dim X \le n$ if and only if there exists an $(n+1)$-to-1 map from a 0-dimensional space onto X. We formulate a finite-to-one mapping type theorem for asymptotic dimension and asymptotic Assouad–Nagata dimension.

1. Introduction. Let us recall the classical Hurewicz dimension theorems for maps.

THEOREM 1.1 (Dimension-lowering theorem). *Let $f : X \to Y$ be a closed surjective map between metrizable spaces. Then $\dim X \le \dim Y + \dim f$, where $\dim f = \sup\{\dim f^{-1}(y) : y \in Y\}$.*

THEOREM 1.2 (Dimension-raising theorem). *Let $f : X \to Y$ be a closed surjective map between metrizable spaces such that $|f^{-1}(y)| \le n+1$ for each $y \in Y$. Then $\dim Y \le \dim X + n$.*

THEOREM 1.3 (Finite-to-one mapping theorem). *Let X be a metrizable space. Then $\dim X \le n$ if and only if there exists a zero-dimensional metric space Y and a closed surjective map $f : Y \to X$ such that $|f^{-1}(x)| \le n+1$ for each $x \in X$.*

G. Bell and A. Dranishnikov [1] proved the dimension-lowering theorem for asymptotic dimension, and N. Brodskiy, J. Dydak, M. Levin and A. Mitra [3] generalized it to Assouad–Nagata dimension and asymptotic Assouad–Nagata dimension. However, there is no simple translation of the

2010 *Mathematics Subject Classification*: Primary 54F45; Secondary 54E35.
Key words and phrases: asymptotic dimension, asymptotic Assouad–Nagata dimension, dimension-raising map, finite-to-one map, coarse category.

DOI: 10.4064/fm223-1-6

dimension-raising theorem in a large scale setting since there are simple one-to-one dimension-raising coarse maps.

In this paper we introduce conditions, called $(\mathrm{B})_n$ and $(\mathrm{C})_n$, respectively, which correspond to the condition that a map is n-to-1. Using those conditions we formulate a dimension-raising type theorem and a finite-to-one mapping type theorem for asymptotic dimension and asymptotic Assouad–Nagata dimension.

Our main theorems for asymptotic dimension (asdim) are:

THEOREM 1.4. *Let X and Y be metric spaces, and let $f : X \to Y$ be a coarse map with the following property:*

$(\mathrm{B})_n$ *For each $r < \infty$, there exists $d < \infty$ such that for each subset B of Y with $\operatorname{diam}(B) \le r$, $f^{-1}(B) = \bigcup_{i=1}^{n} A_i$ for some subsets A_i of X with $\operatorname{diam}(A_i) \le d$ for $i = 1, \dots, n$.*

Then

$$\operatorname{asdim} Y \le (\operatorname{asdim} X + 1)n - 1.$$

THEOREM 1.5. *Let X be a metric space. Then $\operatorname{asdim} X \le n$ if and only if there exist a metric space Y with $\operatorname{asdim} Y = 0$ and a coarse map $f : Y \to X$ with property $(\mathrm{B})_{n+1}$.*

Our main theorems for asymptotic Assouad–Nagata dimension ($\operatorname{asdim}_{\mathrm{AN}}$) state:

THEOREM 1.6. *Let X and Y be metric spaces, and let $f : X \to Y$ be an asymptotically Lipschitz map with the following property:*

$(\mathrm{C})_n$ *There exist $c, d > 0$ such that for each $r < \infty$ and for each subset B of Y with $\operatorname{diam}(B) \le r$, $f^{-1}(B) = \bigcup_{i=1}^{n} A_i$ for some subsets A_i of X with $\operatorname{diam}(A_i) \le cr + d$ for $i = 1, \dots, n$.*

Then

$$\operatorname{asdim}_{\mathrm{AN}} Y \le (\operatorname{asdim}_{\mathrm{AN}} X + 1)n - 1.$$

THEOREM 1.7. *Let X be a metric space. Then $\operatorname{asdim}_{\mathrm{AN}} X \le n$ if and only if there exist a metric space Y with $\operatorname{asdim}_{\mathrm{AN}} Y = 0$ and an asymptotic Lipschitz map $f : Y \to X$ with property $(\mathrm{C})_{n+1}$.*

The “if” parts of Theorems 1.5 and 1.7 immediately follow from Theorems 1.4 and 1.6, respectively. For the “only if” parts of Theorems 1.5 and 1.7, we introduce the notion of n-precode structure, which is a sequence of covers with some conditions determining a map with property $(\mathrm{B})_{n+1}$ or $(\mathrm{C})_{n+1}$ from an ultrametric space to the given space.

We give various examples of dimension-raising maps. In particular, we present a simple example of 1-precode structure for $(\mathbb{Z}, d_\varepsilon)$ with the Euclidean metric d_ε.

A finite-to-one mapping theorem for Asouad–Nagata dimension was obtained in [6], where a condition called (B) (see Section 3) was introduced. Using condition (B), we show a dimension-raising type theorem for Assouad–Nagata dimension as well.

Throughout the paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_+$ denote the set of nonnegative integers, the set of integers, the set of real numbers, and the set of positive real numbers, respectively. For any set X, let id_X denote the identity map on X.

2. Asymptotic dimension, Assouad–Nagata dimension, and asymptotic Assouad–Nagata dimension. In this section, we recall the definitions and properties of asymptotic dimension, Assouad–Nagata dimension, and asymptotic Assouad–Nagata dimension. For more details, the reader is referred to [1], [2], [5], and [4].

Let (X,d) be a metric space. For each $x \in X$ and $r > 0$, let $\mathrm{B}(x,r) = \{y \in X : d(x,y) < r\}$, and $\overline{\mathrm{B}}(x,r) = \{y \in X : d(x,y) \leq r\}$. For each subset A of X, let $\mathrm{diam}(A)$ denote the diameter of A.

Let $\mathcal{U}$ be a cover of X. The *multiplicity* of $\mathcal{U}$, denoted $\mathrm{mult}(\mathcal{U})$, is defined as the largest integer n such that no point of X is contained in more than n elements of $\mathcal{U}$, and the *r-multiplicity* of $\mathcal{U}$, denoted $r\text{-mult}(\mathcal{U})$, is the largest integer n such that no subset of diameter at most r meets more than n elements of $\mathcal{U}$. The *Lebesgue number* of $\mathcal{U}$, denoted $\mathrm{Leb}(\mathcal{U})$, is defined as the supremum of positive numbers r such that for every subset A with $\mathrm{diam}(A) \leq r$, there exists $U \in \mathcal{U}$ with $A \subset U$. The *mesh* of $\mathcal{U}$, denoted $\mathrm{mesh}(\mathcal{U})$, is $\sup\{\mathrm{diam}(U) : U \in \mathcal{U}\}$, and $\mathcal{U}$ is said to be *uniformly bounded* if $\mathrm{mesh}(\mathcal{U}) < \infty$. A family $\mathcal{U}$ of subsets of X is said to be *r-disjoint* if $d(x,x') > r$ for any x and x' that belong to different elements of $\mathcal{U}$.

A metric space X is said to have *asymptotic dimension at most n*, written $\mathrm{asdim}\, X \leq n$, if there exists a function $D_X : \mathbb{R}_+ \to \mathbb{R}_+$ (called an *n-dimensional control function* for X) such that for every $r < \infty$ there exist r-disjoint families $\mathcal{U}^0, \dots, \mathcal{U}^n$ of subsets of X such that $\bigcup_{i=0}^n \mathcal{U}^i$ is a cover of X and $\mathrm{mesh}(\mathcal{U}) \leq D_X(r)$.

A metric space X is said to have *Assouad–Nagata dimension at most n*, written $\dim_{\mathrm{AN}} X \leq n$, if there exists an n-dimensional control function D_X such that $D_X(r) = cr$ for some $c \geq 0$.

A metric space X is said to have *asymptotic Assouad–Nagata dimension at most n*, written $\mathrm{asdim}_{\mathrm{AN}}\, X \leq n$, if there exists an n-dimensional control function D_X such that $D_X(r) = cr + d$ for some $c, d \geq 0$.

We write $\mathrm{asdim}\, X = n$ if $\mathrm{asdim}\, X \leq n$ and $\mathrm{asdim}\, X \not\leq n-1$, and write $\mathrm{asdim}\, X = \infty$ if $\mathrm{asdim}\, X \not\leq n$ for any nonnegative integer n. Similarly, we define $\dim_{\mathrm{AN}} X = n$ and $\mathrm{asdim}_{\mathrm{AN}}\, X = n$.

The following characterizations of asymptotic dimension, Assouad–Nagata dimension, and asymptotic Assouad–Nagata dimension are well-known (see [2], [5], and [4]).

Proposition 2.1. *Let X be a metric space. Then the following conditions are equivalent:*

(1) $\operatorname{asdim} X \leq n$.
(2) *For every uniformly bounded cover $\mathcal{V}$ of X, there exists a uniformly bounded cover $\mathcal{U}$ of X such that* $\operatorname{mult}(\mathcal{U}) \leq n+1$ *and* $\mathcal{V} < \mathcal{U}$.
(3) *For every $s < \infty$, there exists a uniformly bounded cover $\mathcal{V}$ of X such that* $s\text{-}\operatorname{mult}(\mathcal{V}) \leq n+1$.
(4) *For every $t < \infty$, there exists a uniformly bounded cover $\mathcal{W}$ of X such that* $\operatorname{Leb}(\mathcal{W}) \geq t$ *and* $\operatorname{mult}(\mathcal{W}) \leq n+1$.

Proposition 2.2. *Let X be a metric space. Then the following conditions are equivalent:*

(1) $\operatorname{asdim}_{\mathrm{AN}} X \leq n$ (*resp.*, $\dim_{\mathrm{AN}} X \leq n$).
(2) *There exists $c > 0$ (resp., there exist $c, s_0 > 0$) such that for every $s < \infty$ (resp., $s \geq s_0$), there exists a cover $\mathcal{V}$ of X with* $\operatorname{mesh}(\mathcal{V}) \leq cs$ *and* $s\text{-}\operatorname{mult}(\mathcal{V}) \leq n+1$.
(3) *There exists $c > 0$ (resp., there exist $c, t_0 > 0$) such that for every $t < \infty$ (resp., $t \geq t_0$), there exists a cover $\mathcal{W}$ of X with* $\operatorname{mesh}(\mathcal{W}) \leq ct$, $\operatorname{Leb}(\mathcal{W}) \geq t$, *and* $\operatorname{mult}(\mathcal{W}) \leq n+1$.

The following characterization of the asymptotic dimension will be used in Section 6.

Proposition 2.3. *Let X be a metric space. Then the following conditions are equivalent:*

(1) $\operatorname{asdim} X \leq n$.
(2) *For all $s, t < \infty$, there exists a uniformly bounded cover $\mathcal{U}$ of X such that* $s\text{-}\operatorname{mult}(\mathcal{U}) \leq n+1$ *and* $\operatorname{Leb}(\mathcal{U}) \geq t$.

Proof. The implication (2)⇒(1) is obvious by Proposition 2.1. To show (1)⇒(2), suppose $\operatorname{asdim} X \leq n$. Let $s, t < \infty$ and $r \geq s + 4t$. Then by definition there exist uniformly bounded r-disjoint families $\mathcal{U}^0, \dots, \mathcal{U}^n$ of subsets of X such that $\mathcal{U}' = \bigcup_{i=0}^n \mathcal{U}^i$ is a cover of X. Consider the cover $\mathcal{U} = \{\mathrm{B}(U, 2t) : U \in \mathcal{U}'\}$. Then $s\text{-}\operatorname{mult}(\mathcal{U}) \leq n+1$. Indeed, let A be a subset of X with $\operatorname{diam}(A) \leq s$ such that $A \cap \mathrm{B}(U, 2t) \neq \emptyset$ and $A \cap \mathrm{B}(U', 2t) \neq \emptyset$ for some $U, U' \in \mathcal{U}'$. Then $d(U, U') \leq s + 4t \leq r$, which implies $U \in \mathcal{U}^i$ and $U' \in \mathcal{U}^{i'}$ for some i, i' with $i \neq i'$. Thus A intersects at most $n+1$ elements of $\mathcal{U}$, proving that $s\text{-}\operatorname{mult}(\mathcal{U}) \leq n+1$. To show $\operatorname{Leb}(\mathcal{U}) \geq t$, let A be a subset of X such that $\operatorname{diam}(A) \leq t$. Then $A \cap U \neq \emptyset$ for some $U \in \mathcal{U}'$, and hence $A \subset \mathrm{B}(U, 2t)$. This shows (2). ∎

The following characterization of asymptotic Assouad–Nagata dimension will be used in Section 7.

PROPOSITION 2.4. *Let X be a metric space. Then the following conditions are equivalent:*

(1) $\operatorname{asdim}_{\mathrm{AN}} X \leq n$.
(2) *There exist $c, d > 0$ such that for all $s, t < \infty$, there exists a cover $\mathcal{U}$ of X such that $\mathcal{U}$ is $(c(s+4t)+d)$-bounded, s-$\mathrm{mult}(\mathcal{U}) \leq n+1$, and $\mathrm{Leb}(\mathcal{U}) \geq t$.*

Proof. (2)⇒(1) is obvious by Proposition 2.2. (1)⇒(2) can be proved by the same argument as in the proof of Proposition 2.3. Indeed, let $s, t < \infty$, and let $c, d > 0$ be the constants as in the definition of $\operatorname{asdim}_{\mathrm{AN}} X \leq n$. Without loss of generality, we can assume $c \geq 2$. Put $r = s + 4t$, and let $\mathcal{U}^0, \ldots, \mathcal{U}^n$ be $(cr/2 + d)$-bounded r-disjoint families of subsets of X such that $\mathcal{U}' = \bigcup_{i=0}^n \mathcal{U}^i$ is a cover of X. Then the cover $\mathcal{U} = \{\mathrm{B}(U, 2t) : U \in \mathcal{U}'\}$ satisfies the required conditions. Note that $(c(s+4t)+d)$-boundedness of $\mathcal{U}$ follows from the following inequalities:

$$\mathrm{mesh}(\mathcal{U}) \leq \mathrm{mesh}(\mathcal{U}') + 4t \leq c(s+4t)/2 + d + 4t \leq c(s+4t) + d. \blacksquare$$

3. Dimension-raising maps: properties $(\mathrm{B})_n$ **and** $(\mathrm{C})_n$**.** In this section, we prove dimension-raising type theorems for Assouad–Nagata dimension, asymptotic dimension, and asymptotic Assouad–Nagata dimension.

A map $f : (X, d_X) \to (Y, d_Y)$ is said to be *bornologous* if there exists a function $\delta_f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $d_Y(f(x), f(x')) \leq \delta_f(d_X(x, x'))$ for all $x, x' \in X$, and it is *coarse* if it is bornologous and proper. It is *Lipschitz* (resp., *asymptotically Lipschitz*) if there exists a function $\delta_f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $d_Y(f(x), f(x')) \leq \delta_f(d_X(x, x'))$ for all $x, x' \in X$ and $\delta_f(t) = ct$ for some $c > 0$ (resp., $\delta_f(t) = ct + b$ for some $b, c > 0$). It is *quasi-isometric* if:

(1) there exist functions $\delta_f, \gamma_f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\gamma_f(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \delta_f(d_X(x, x')) \quad \text{for all } x, x' \in X,$$

$\delta_f(t) = ct + b$, and $\gamma_f(t) = (1/c)t - b$ for some $b, c > 0$;
(2) $f(X)$ is *coarsely dense* in Y, i.e., there exists $R > 0$ such that $d_Y(y, f(X)) \leq R$ for every $y \in Y$.

Two maps $f, f' : (X, d_X) \to (Y, d_Y)$ are said to be *close* if there exists $S > 0$ such that $d_Y(f(x), g(x)) \leq S$ for all $x \in X$. A map $f : (X, d_X) \to (Y, d_Y)$ is called a *coarse equivalence* if there exists a coarse map $g : (Y, d_Y) \to (X, d_X)$ such that $f \circ g$ is close to id_Y and $g \circ f$ is close to id_X.

3.1. Dimension-raising type theorem for Assouad–Nagata dimension. For any map $f : X \to Y$, consider the following property [6]:

(B) There exists $d > 0$ such that for each $r > 0$ and for each $B \subset Y$ with $\operatorname{diam}(B) \le r$, there exists $A \subset X$ with $\operatorname{diam}(A) \le dr$ and $f(A) = B$.

LEMMA 3.1. *Let $f : X \to Y$ be a map, and let $\mathcal{U}$ be a cover of X. If $|f^{-1}(y)| \le n$ for each $y \in Y$, then*

$$\operatorname{mult}(f(\mathcal{U})) \le \operatorname{mult}(\mathcal{U}) \cdot n.$$

Proof. Let $k = \operatorname{mult}(\mathcal{U})$. Suppose to the contrary that $\operatorname{mult}(f(\mathcal{U})) > kn$. Then there exist $U_1, \dots, U_{kn+1} \in \mathcal{U}$ such that there exists $y \in f(U_1) \cap \dots \cap f(U_{kn+1})$. So, there exist $x_i \in U_i$ for $i = 1, \dots, kn+1$ such that $y = f(x_1) = \dots = f(x_{kn+1})$. Since $|f^{-1}(y)| \le n$, there exist at least $k+1$ indices $i_1, \dots, i_{k+1} \in \{1, \dots, kn+1\}$ such that $x_{i_1} = \dots = x_{i_{k+1}}$, implying that $U_{i_1} \cap \dots \cap U_{i_{k+1}} \ne \emptyset$. This contradicts $\operatorname{mult}(\mathcal{U}) \le k$. ∎

THEOREM 3.2. *Let X and Y be metric spaces, and let $f : X \to Y$ be a surjective Lipschitz map such that $|f^{-1}(y)| \le n$ for each $y \in Y$, and f has property* (B). *Then*

$$\dim_{\mathrm{AN}} Y \le (\dim_{\mathrm{AN}} X + 1)n - 1.$$

Proof. Since the assertion is trivial if $\dim_{\mathrm{AN}} X = \infty$, assume $m = \dim_{\mathrm{AN}} X < \infty$. Then there exists $c > 0$ such that for each $r > 0$ there exists a cover $\mathcal{U}_r$ of X with $\operatorname{mult}(\mathcal{U}_r) \le m+1$, $\operatorname{mesh}(\mathcal{U}_r) \le cr$, and $\operatorname{Leb}(\mathcal{U}_r) \ge r$. Let $d > 0$ be as in (B), and for each $r > 0$, let $\mathcal{V}_r = f(\mathcal{U}_{dr})$. Then Lemma 3.1 implies $\operatorname{mult}(\mathcal{V}_r) \le \operatorname{mult}(\mathcal{U}_{dr}) \cdot n$. Since f is Lipschitz, we have $\operatorname{mesh}(\mathcal{V}_r) \le \operatorname{Lip}(f) \operatorname{mesh}(\mathcal{U}_{dr}) \le \operatorname{Lip}(f) \cdot cdr$. To show that $\operatorname{Leb}(\mathcal{V}_r) \ge r$, let B be a subset of Y such that $\operatorname{diam}(B) \le r$. Then (B) implies that there exists a subset A of X such that $\operatorname{diam}(A) \le dr$ and $f(A) = B$. Since $\operatorname{Leb}(\mathcal{U}_{dr}) \ge dr$, $A \subset U$ for some $U \in \mathcal{U}_{dr}$. Hence $B = f(A) \subset f(U) \in \mathcal{V}_r$, showing that $\operatorname{Leb}(\mathcal{V}_r) \ge r$. Thus we have shown that $\dim_{\mathrm{AN}} X \le (m+1)n - 1$. ∎

3.2. Dimension-raising type theorem for asymptotic dimension. For any map $f : X \to Y$ and for each $n \in \mathbb{N}$, consider the following property:

$(\mathrm{B})_n$ For each $r < \infty$, there exists $d < \infty$ such that for each subset B of Y with $\operatorname{diam}(B) \le r$, $f^{-1}(B) = \bigcup_{i=1}^{n} A_i$ for some subsets A_i of X with $\operatorname{diam}(A_i) \le d$ for $i = 1, \dots, n$.

The following properties are useful in constructing maps with property $(\mathrm{B})_n$ in later sections.

PROPOSITION 3.3. *Suppose $f\colon X \to Y$ is a coarse map with property $(\mathrm{B})_n$ and $g\colon Y \to Z$ is a coarse map with property $(\mathrm{B})_m$. Then gf is a coarse map with property $(\mathrm{B})_{nm}$.*

Proof. Let $A \subset Z$ be an r-bounded set. Then $g^{-1}(A)$ is a union of d_g-bounded sets $A_1, \ldots, A_m$. Similarly, for each i the set $f^{-1}(A_i)$ is a union of d_f-bounded sets $A^i_1, \ldots, A^i_n$. Consequently, $(gf)^{-1}(A)$ is a union of nm d_f-bounded sets $\{A^i_j\}_{i=1,\ldots,m;\, j=1,\ldots,n}$. ■

PROPOSITION 3.4. *Suppose $f\colon X \to Y$ is a coarse map with property* (B)$_n$ *and $g\colon Z \to W$ is a coarse map with property* (B)$_m$. *Then $g \times f$ is a coarse map with property* (B)$_{nm}$.

Proof. Suppose p_Y and p_W are projections of $Y \times W$ to Y and W respectively. Given an r-bounded set $A \subset Y \times W$, $p_Y(A)$ and $p_W(A)$ are r-bounded as well. Furthermore, since $f^{-1}(p_Y(A))$ is a union of d_f bounded sets $A_1, \ldots, A_n$ and $g^{-1}(p_W(A))$ is a union of d_g bounded sets $B_1, \ldots, B_m$ we conclude that $(f \times g)^{-1}(A)$ is a union of $m \cdot n$ $(d_f + d_g)$-bounded sets $\{A_i \times B_j\}_{i=1,\ldots,n;\, j=1,\ldots,m}$. ■

PROPOSITION 3.5. *Suppose X is a metric space of asymptotic dimension* 0 *and Y is any metric space. Then* $\operatorname{asdim}(X \times Y) = \operatorname{asdim} Y$.

Proof. Let $n = \operatorname{asdim} Y$. Given $r < \infty$ there exist $d \in \mathbb{R}$, d-bounded r-disjoint families $\mathcal{U}_0, \ldots, \mathcal{U}_n$ of subsets of Y such that $\bigcup_{i=0}^n \mathcal{U}_i$ is a cover of Y, and a d-bounded r-disjoint cover $\mathcal{V}$ of X. Define $\mathcal{W}_i = \{V \times U : U \in \mathcal{U}_i, V \in \mathcal{V}\}$ for $i = 0, \ldots, n$ and note that $\mathcal{W}_0, \ldots, \mathcal{W}_n$ is a collection of $2d$-bounded r-disjoint families of subsets of $X \times Y$ such that $\bigcup_{i=0}^n \mathcal{W}_i$ is a cover of $X \times Y$. Hence $\operatorname{asdim}(X \times Y) \le n = \operatorname{asdim} Y$. Since $X \times Y$ contains an isometric copy of Y we also have $\operatorname{asdim}(X \times Y) \ge \operatorname{asdim} Y$. ■

LEMMA 3.6. *Let $f : X \to Y$ be a map, and let $\mathcal{U}$ be a cover of X. Suppose that f has property* (B)$_n$. *Let $r < \infty$, and let $d < \infty$ be as in* (B)$_n$. *Then*

$$r\text{-mult}(f(\mathcal{U})) \le d\text{-mult}(\mathcal{U}) \cdot n.$$

Proof. Let $m = d\text{-mult}(\mathcal{U})$. Suppose to the contrary that $r\text{-mult}(f(\mathcal{U})) > mn$. Then there exists a subset B of Y with $\operatorname{diam}(B) \le r$ such that $B \cap f(U_i) \ne \emptyset$ for some $U_1, \ldots, U_{mn+1} \in \mathcal{U}$. Then (B)$_n$ implies that $f^{-1}(B) = \bigcup_{j=1}^n A_j$ for some subsets A_j of X with $\operatorname{diam}(A_j) \le d$ for $i = 1, \ldots, n$. So, $\emptyset \ne f^{-1}(B) \cap U_i = (\bigcup_{j=1}^n A_j) \cap U_i$ for $i = 1, \ldots, mn+1$. This implies that there exists j_0 such that $A_{j_0} \cap U_i \ne \emptyset$ for some $i \in \{i_1, \ldots, i_{m+1}\} \subset \{1, \ldots, mn+1\}$. This contradicts the condition that $d\text{-mult}(\mathcal{U}) = m$. ■

THEOREM 3.7. *Let X and Y be metric spaces, and let $f : X \to Y$ be a coarse map with property* (B)$_n$. *Then*

$$\operatorname{asdim} Y \le (\operatorname{asdim} X + 1)n - 1.$$

Proof. Since the assertion is trivial if $\operatorname{asdim} X = \infty$, assume $m = \operatorname{asdim} X < \infty$. Let $r > 0$, and let $d > 0$ be as in (B)$_n$. Then, by Proposition 2.1(3), there exists a uniformly bounded cover $\mathcal{U}_d$ of X such that $d\text{-mult}(\mathcal{U}_d) \le m+1$.

Consider $\mathcal{V} = f(\mathcal{U}_d)$. By Lemma 3.6, r-mult$(\mathcal{V}) \leq d$-mult$(\mathcal{U}_d) \cdot n \leq (m+1)n$. Since f is bornologous, $\mathcal{V}$ is uniformly bounded. Consequently, $\operatorname{asdim} Y \leq (m+1)n - 1 = (\operatorname{asdim} X + 1)n - 1$, as required. ∎

3.3. Dimension-raising type theorem for asymptotic Assouad–Nagata dimension. We can modify the argument for asymptotic dimension to obtain the dimension-raising theorem for asymptotic Assouad–Nagata dimension.

For any map $f : X \to Y$ and for each $n \in \mathbb{N}$, consider the following condition:

(C)$_n$ There exist $c, r_0 > 0$ such that for each $r \geq r_0$ and for each subset B of Y with $\operatorname{diam}(B) \leq r$, $f^{-1}(B) = \bigcup_{i=1}^{n} A_i$ for some subsets A_i of X with $\operatorname{diam}(A_i) \leq cr$ for $i = 1, \ldots, n$.

Remark 3.8. It can be verified that Propositions 3.3–3.5 hold for asymptotic Assouad–Nagata dimension if "coarse map" is replaced by "asymptotic Lipschitz map".

Lemma 3.9. *Let $f : X \to Y$ be a map, and let $\mathcal{U}$ be a cover of X. Suppose that f satisfies condition* (C)$_n$. *Let $c, r_0 > 0$ be as in* (C)$_n$. *Then for each $r \geq r_0$,*

$$r\text{-mult}(f(\mathcal{U})) \leq cr\text{-mult}(\mathcal{U}) \cdot n.$$

Proof. Use the same technique as in the proof of Lemma 3.6. ∎

Theorem 3.10. *Let X and Y be metric spaces, and let $f : X \to Y$ be an asymptotically Lipschitz map with property* (C)$_n$. *Then*

$$\operatorname{asdim}_{\mathrm{AN}} Y \leq (\operatorname{asdim}_{\mathrm{AN}} X + 1)n - 1.$$

Proof. We can assume $m = \operatorname{asdim}_{\mathrm{AN}} X < \infty$. Let $c, r_0 > 0$ be as in (B)$_n$, and let $r \geq r_0$. Then, by Proposition 2.2(2), there exists a cover $\mathcal{U}_r$ of X such that $\operatorname{mesh}(\mathcal{U}_r) \leq cr$ and d-mult$(\mathcal{U}_r) \leq m + 1$. Consider $\mathcal{V} = f(\mathcal{U}_r)$. By Lemma 3.9, r-mult$(\mathcal{V}) \leq cr$-mult$(\mathcal{U}_r) \cdot n \leq (m+1)n$. Since f is asymptotically Lipschitz, $\operatorname{mesh}(\mathcal{V}) \leq c'cr + b$ for some $b, c' > 0$. If $r \geq \max\{r_0, b/c\}$, then $\operatorname{mesh}(\mathcal{V}) \leq c''cr$, where $c'' = c' + 1$. Hence $\operatorname{asdim}_{\mathrm{AN}} Y \leq (m+1)n - 1 = (\operatorname{asdim}_{\mathrm{AN}} X + 1)n - 1$, as required. ∎

4. n-Precode structure for asymptotic dimension. A metric space (X, d) is said to be *ultrametric* if $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for all $x, y, z \in X$. Every ultrametric space has asymptotic dimension 0. Indeed, for each $r < \infty$, there exists an r-disjoint cover $\mathcal{U}$ which consists of r-components. Since each r-component of an ultrametric space is an r-ball, $\mathcal{U}$ is uniformly bounded.

In this section, we present a procedure to construct coarse maps from ultrametric spaces with property (B)$_n$.

THEOREM 4.1. *Suppose $\mathcal{U}_0, \mathcal{U}_1, \ldots$ is a sequence of uniformly bounded covers of a metric space X and fix $n \in \mathbb{N}$.*

(1) *If for every i and every $U \in \mathcal{U}_i$ there exists exactly one $V \in \mathcal{U}_{i+1}$ satisfying $U \subset V$ then every $W^0 \in \mathcal{U}_0$ defines a unique sequence $(W^0, W^1, \ldots)$ with $W^i \in \mathcal{U}_i$ and $W^i \subset W^{i+1}$.*
(2) *Assume the conditions of the previous case along with the following additional condition: for every bounded subset $D \subset X$ there exist i and $U \in \mathcal{U}_i$ such that $D \subset U$. Then the following rule defines an ultrametric on $\mathcal{U}_0$: $d_B(V, V) = 0$, and for $V \neq W$,*

$$d_B(V, W) = 3^{p(V,W)}, \quad p(V, W) = \min\{k \in \mathbb{Z} : \exists \widetilde{U} \in \mathcal{U}_k : V \cup W \subset \widetilde{U}\}.$$

Furthermore, $\operatorname{asdim}(\mathcal{U}_0, d_B) = 0$ *and a map $q \colon \mathcal{U}_0 \to X$ sending $U \in \mathcal{U}_0$ to any chosen point $x \in U$ is coarse.*
(3) *Assume the conditions of the previous case along with the following additional condition: for every $r < \infty$ there exists i such that $r\text{-mult}(\mathcal{U}_i) \le n$. Then q has property* $(\mathrm{B})_n$.

Proof. (1) is obvious.

(2) The distance d_B is finite (as the union of every pair of elements of $\mathcal{U}_0$ is contained in some $U \in \mathcal{U}_i$), symmetric and equals 0 exactly for two identical elements of $\mathcal{U}_0$. It is easy to see that the uniqueness of sequences in (1) implies that d_B is an ultrametric.

It has been remarked that the asymptotic dimension of an ultrametric space is 0. To see that q is coarse observe that if $d_B(U, V) \le 3^n$ then $d(q(U), q(V)) \le \operatorname{mesh}(\mathcal{U}_n)$.

(3) Fix $r < \infty$ and choose i such that the r-multiplicity of $\mathcal{U}_i$ is at most n. Suppose $B \subset Y$ is of diameter at most r and let $U_1, \ldots, U_n$ denote the collection of all elements of $\mathcal{U}_i$ that intersect B (some elements may be identical since B might intersect less than n elements of $\mathcal{U}_i$). Then $q^{-1}(B)$ is the union of the sets $A_j = \{U \in \mathcal{U}_0 : U \subset U_j\}$, which are of diameter at most 3^i.

We mention the following important technical detail: if $U^0 \in \mathcal{U}_0$ intersects B then (using the convention of (1)) U^i contains U^0, hence is listed as U_j for some j. In particular, $U^0 \in A_j$. ∎

DEFINITION 4.2. Any sequence of uniformly bounded covers satisfying (1)–(3) of Theorem 4.1 is called the *n-precode structure for asymptotic dimension.*

COROLLARY 4.3. *If a metric space X admits an n-precode structure for asymptotic dimension then there exists an ultrametric space Z and a coarse map $f \colon Z \to X$ with property* $(\mathrm{B})_n$.

Corollary 4.4. *If a metric space X admits a 1-precode structure for asymptotic dimension then there exists an ultrametric space Z and a coarse equivalence $f\colon Z \to X$.*

Proof. Suppose X admits a 1-precode structure $\mathcal{U}_0, \mathcal{U}_1, \ldots$. Let $Z = \mathcal{U}_0$ and let $f : Z \to X$ be the coarse map with property $(\mathrm{B})_1$ defined as in (2) of Theorem 4.1. To verify that f is a coarse equivalence, we define a map $g : X \to Z$ by $g(x) = U_x$ for each $x \in X$, where U_x is an element of $\mathcal{U}_0$ with $x \in U$.

To show that g is bornologous, let $R < \infty$, and let $d(x, y) < R$. Take $k \in \mathbb{Z}$ such that $R \le 3^k$. Then $g(x) = U_x$ and $g(y) = U_y$, where U_x and U_y are elements of $\mathcal{U}_0$ with $x \in U_x$ and $y \in U_y$, respectively. Condition (3) of Theorem 4.1 implies that 3^k-$\mathrm{mult}(\mathcal{U}_i) \le 1$ for some $i \in \mathbb{N}$. Condition (1) of Theorem 4.1 implies that there exist unique elements U'_x and U'_y of $\mathcal{U}_i$ such that $U_x \subset U'_x$ and $U_y \subset U'_y$. Since $d(x, y) < 3^k$ and 3^k-$\mathrm{mult}(\mathcal{U}_i) \le 1$, $U'_x = U'_y$. This means that $d_B(U_x, U_y) \le 3^i$.

To verify that g is proper, let $R < \infty$. Suppose A is a subset of Z such that $\mathrm{diam}(A) \le R$, and take $k \in \mathbb{Z}$ such that $R \le 3^k$. Let $x, y \in g^{-1}(A)$. Then $g(x) = U_x$ and $g(y) = U_y$, where U_x and U_y are elements of $\mathcal{U}_0$ with $x \in U_x$ and $y \in U_y$, respectively. Since $d_B(U_x, U_y) \le 3^k$, we have $d(x, y) \le \mathrm{mesh}(\mathcal{U}_k)$, showing that $\mathrm{diam}\, g^{-1}(A) \le \mathrm{mesh}(\mathcal{U}_k)$.

To show that $f \circ g$ is close to id_X, let $x \in X$. Then $g(x) = U_x$, where U_x is an element of $\mathcal{U}_0$ such that $x \in U_x$, and so $f(g(x)) \in U_x$. This means that $d(f(g(x)), x) \le \mathrm{mesh}(\mathcal{U}_0)$. Also $g \circ f = \mathrm{id}_Z$. This shows that Z and X are coarse equivalent. ■

Example 4.5. The metric space $(\mathbb{N}, d_\varepsilon)$, where d_ε is the Euclidean metric, admits a 2-precode structure for asymptotic dimension. Indeed, we define $\mathcal{U}_0 = \{U^0_n : n \in \mathbb{N}\}$, where $U^0_n = \{n\}$ for each $n \in \mathbb{Z}$. Assuming that $\mathcal{U}_i = \{U^i_n : n \in \mathbb{N}\}$ has been defined, we define $\mathcal{U}_{i+1} = \{U^{i+1}_n : n \in \mathbb{N}\}$, where $U^{i+1}_n = U^i_{2n} \cup U^i_{2n+1}$ for each $n \in \mathbb{N}$. The sequence of covers $\mathcal{U}_i$ thus defined satisfies conditions (1)–(3) of Theorem 4.1.

Hence there exist an ultrametric space (X, d) and a coarse map $f : (X, d) \to (\mathbb{N}, d_\varepsilon)$ with property $(\mathrm{B})_2$. Note $\mathrm{asdim}\, X = 0$ and $\mathrm{asdim}(\mathbb{N}, d_\varepsilon) = 1$.

Proposition 3.4 implies that $f \times \mathrm{id}_{\mathbb{N}^n} : (X, d) \times (\mathbb{N}^n, d_\varepsilon) \to (\mathbb{N}^{n+1}, d_\varepsilon)$ is a coarse map with property $(\mathrm{B})_2$. Note that $\mathrm{asdim}\, X \times \mathbb{N}^n = \mathrm{asdim}\, \mathbb{N}^n = n$ (Proposition 3.5) and $\mathrm{asdim}(\mathbb{N}^{n+1}, d_\varepsilon) = n + 1$.

Example 4.6. In this example we present a 2-precode structure for asymptotic dimension on the metric space $(\mathbb{N}, d_\varepsilon)$, where d_ε is the Euclidean metric. The example is closely related to Example 4.5 (and analogous conclusions can easily be drawn) although the formal description is somewhat different. Define $a^k(n) = \{n, n+1, \ldots, n+3^k-1\} \subset \mathbb{Z}$. The 2-precode structure for asymptotic dimension is given by the covers $\mathcal{U}_k = \{a^k(n) : \exists j \in \mathbb{Z}\colon n = (3^{k+1} - 1) \cdot \frac{1}{2} + j \cdot 3^k\}$.

Note that $\mathcal{U}_k$ is a cover of $\mathbb{Z}$ by disjoint intervals of length 3^k, the element 0 being approximately in the middle of one such interval. The cover $\mathcal{U}_{k+1}$ is obtained by taking unions of three consecutive intervals so that the resulting cover is disjoint and 0 is approximately in the middle of one such union (i.e., three times larger interval).

5. Finite-to-one mapping theorem for asymptotic dimension. In this section, using the n-precode structure, we prove a finite-to-one mapping type theorem for asymptotic dimension.

THEOREM 5.1. *Let X be a metric space. If* $\operatorname{asdim} X \leq n$ *then X admits an $(n+1)$-precode structure for asymptotic dimension.*

Proof. We provide an inductive construction of covers $\mathcal{U}_i$. Fix $x_0 \in X$ and let $\mathcal{U}_0 = \{\{x\}\}_{x \in X}$ be a cover by singletones.

Let $k \in \mathbb{N}$ and suppose we have constructed covers $\mathcal{U}_0, \ldots, \mathcal{U}_k$ with the following properties:

(1) $\mathcal{U}_i$ is an M_i-bounded cover for $i = 0, \ldots, k$;
(2) the i-multiplicity of $\mathcal{U}_i$ is at most $n+1$ for $i = 0, \ldots, k$;
(3) elements of $\mathcal{U}_i$ are disjoint for $i = 0, \ldots, k$;
(4) given $i < k$ and $U \in \mathcal{U}_i$ there exists $V \in \mathcal{U}_{i+1}$ containing U (such a V is unique by the previous property);
(5) given $i < k$ there exists $U_{\alpha_i} \in \mathcal{U}_i$ containing the closed ball $B(x_0, i)$ (again, such an element is unique by (3)).

The cover $\mathcal{U}_{k+1}$ is constructed as follows. By Proposition 2.3 there exists an N_{k+1}-bounded cover $\mathcal{V}_{k+1} = \{V_\beta\}_{\beta \in \Sigma}$ of $(k+1+2M_k)$-multiplicity at most $n+1$ and of Lebesgue number at least $2(k+1)$. Let $V_{\alpha_{k+1}} \in \mathcal{V}_{k+1}$ be a set containing the closed ball $\mathrm{B}(x_0, k+1)$. For every $U \in \mathcal{U}_k$ define $\tau(U) \in \Sigma$ in the following way:

- if $U \cap V_{\alpha_{k+1}} \neq \emptyset$ then $\tau(U) = \alpha_{k+1}$;
- else $\tau(U)$ is any index in Σ such that $U \cap V_{\tau(U)} \neq \emptyset$.

Define $\mathcal{U}_{k+1} = \{U_\beta\}_{\beta \in \Sigma}$, where

$$U_\beta = \bigcup_{W \in \mathcal{U}_k,\, \tau(W) = \beta} W.$$

We now verify that the cover $\mathcal{U}_{k+1}$ satisfies the required conditions:

(1) $\mathcal{U}_{k+1}$ is $(2M_k + N_{k+1})$-bounded by construction;
(2) the $(k+1)$-multiplicity of $\mathcal{U}_{k+1}$ is at most $n+1$ (this is a consequence of two facts: for every $\beta \in \Sigma$ the M_k-neighborhood of V_β contains U_β; and the $(k+1+2M_k)$-multiplicity of $\mathcal{V}_{k+1}$ is at most $n+1$);

(3) the elements of $\mathcal{U}_{k+1}$ are disjoint by construction as the elements of $\mathcal{U}_k$ are disjoint and each $U \in \mathcal{U}_k$ is assigned exactly one $\tau(U)$;
(4) obviously, $U \subset U_{\tau(U)}$ for every $U \in \mathcal{U}_k$;
(5) $U_{\alpha_{k+1}} \in \mathcal{U}_{k+1}$ contains the closed ball $B(x_0, k+1)$ by construction.

It is apparent from the properties listed above that the covers $\mathcal{U}_i$ form an $(n+1)$-precode structure for asymptotic dimension on X. ∎

Corollary 5.2 is a large scale version of the finite-to-one mapping theorem.

COROLLARY 5.2. *For every metric space X, $\operatorname{asdim} X \le n$ if and only if there exist a metric space Y of $\operatorname{asdim} Y = 0$ and a coarse map $q\colon Y \to X$ with property* $(\mathrm{B})_{n+1}$.

COROLLARY 5.3. *For every $n \in \mathbb{N}$ and $m \ge n$ there exist metric spaces X and Y with $\operatorname{asdim} Y = m$ and $\operatorname{asdim} X = n+m$, respectively, and a coarse map $q\colon Y \to X$ with property* $(\mathrm{B})_{n+1}$.

COROLLARY 5.4. *For every metric space (X,d), $\operatorname{asdim}(X,d) = 0$ if and only if there exists an ultrametric ρ on X such that $\mathrm{id} : (X,d) \to (X,\rho)$ is a coarse equivalence.*

Proof. The corollary easily follows from Theorems 5.1 and 4.4. ∎

Corollary 5.4 generalizes the result by Brodskiy, Dydak, Levin, and Mitra [3], which states that $\dim_{\mathrm{AN}}(X,d) = 0$ if and only if there is an ultrametric ρ such that the identity map $\mathrm{id} : (X,d) \to (X,\rho)$ is bi-Lipschitz.

6. Finite-to-one mapping theorem for asymptotic Assouad–Nagata dimension. In this section, we generalize the results of Sections 5 and 6 to the case of asymptotic Assouad–Nagata dimension. The following is an analogue of Theorem 4.1 which provides a general way to construct asymptotically Lipschitz maps from ultrametric spaces with property $(\mathrm{C})_n$.

THEOREM 6.1. *Suppose $\mathcal{U}_0, \mathcal{U}_1, \ldots$ is a sequence of uniformly bounded covers of a metric space X which satisfies conditions* (1) *and* (2) *in Theorem* 4.1, *and fix $n \in \mathbb{N}$.*

(1) *Assume the following condition: there exist $a > 1$ and $i_0 \in \mathbb{N}$ such that $\operatorname{mesh}(\mathcal{U}_i) \le a^i$ for $i \ge i_0$. Then there exists an ultrametric d_C on $\mathcal{U}_0$ such that a map $q : \mathcal{U}_0 \to X$ sending $U \in \mathcal{U}_0$ to any chosen point $x \in U$ is asymptotically Lipschitz.*
(2) *Assume the condition of the previous case along with the following additional condition: there exist $c, r_0 > 0$ such that for every $r \ge r_0$ there exists $i \in \mathbb{N}$ such that $a^i \le cr$ and $r\text{-}\operatorname{mult}(\mathcal{U}_i) \le n$. Then q has property* $(\mathrm{C})_n$.

Proof. (1) Let d_C be the ultrametric d_B obtained in Theorem 4.1(2) with the base number 3 being replaced by a, i.e., $d_C(V,V)=0$, and for $V\neq W$,

$$d_C(V,W)=a^{p(V,W)},\quad p(V,W)=\min\{k\in\mathbb{Z}:\exists\widetilde{U}\in\mathcal{U}_k\colon V\cup W\subset\widetilde{U}\}.$$

To see that q is asymptotically Lipschitz, observe that if $d_C(U,V)=a^n$ then $d(q(U),q(V))\le \mathrm{mesh}(\mathcal{U}_n)\le d_C(U,V)+a^{i_0}$.

(2) Let $c,r_0>0$ be as in the hypothesis. Fix $r\ge r_0$, and choose i so that $a^i\le cr$ and r-mult$(\mathcal{U}_i)\le n$. Suppose $B\subset Y$ is of diameter at most r, and let $U_1,\dots,U_n$ denote the collection of all elements of $\mathcal{U}_i$ that have a nonempty intersection with B. Then $q^{-1}(B)$ is the union of the sets $A_j=\{U\in\mathcal{U}_0:U\subset U_j\}$, which have $\mathrm{diam}(A_j)\le a^i\le cr$. ∎

Definition 6.2. Any sequence of uniformly bounded covers satisfying (1)–(2) of Theorem 6.1 is called an *n-precode structure for asymptotic Assouad–Nagata dimension.*

The following is an analogue of Theorem 5.1 for asymptotic Assouad–Nagata dimension.

Theorem 6.3. *Let X be a metric space. If* $\mathrm{asdim_{AN}}\,X\le n$ *then X admits an $(n+1)$-precode structure for asymptotic Assouad–Nagata dimension.*

Proof. We inductively construct covers $\mathcal{U}_i$ which satisfy all the required conditions in Theorem 6.1. Their construction follows the steps used for Theorem 5.1.

Fix $x_0\in X$ and let $\mathcal{U}_0=\{\{x\}\}_{x\in X}$ be the cover by singletones.

Proposition 2.4 implies that there exist $c,d>0$ such that for all $s,t<\infty$ there exists a cover $\mathcal{U}_{s,t}$ of X with $\mathrm{mesh}(\mathcal{U}_{s,t})\le c(s+4t)+d$, s-mult$(\mathcal{U}_{s,t})\le n+1$, and $\mathrm{Leb}(\mathcal{U}_{s,t})\ge t$. Without loss of generality, we can assume $c\ge d\ge 2$.

Let $k\in\mathbb{N}$ and suppose we have constructed covers $\mathcal{U}_0,\dots,\mathcal{U}_k$ with the following properties:

(1) $\mathrm{mesh}(\mathcal{U}_i)\le(14c)^i$ for $i=0,\dots,k$;
(2) $((3^i-1)/3)$-mult$(\mathcal{U}_i)\le n+1$ for $i=0,\dots,k$;
(3) elements of $\mathcal{U}_i$ are disjoint for $i=0,\dots,k$;
(4) given $i<k$ and $U\in\mathcal{U}_i$ there exists a unique $V\in\mathcal{U}_{i+1}$ containing U;
(5) given $i<k$ there exists a unique $U_{\alpha_i}\in\mathcal{U}_i$ containing $\mathrm{B}(x_0,(3^i-1)/3)$.

To define $\mathcal{U}_{k+1}$, let $\mathcal{V}_{k+1}=\{V_\beta\}_{\beta\in\Sigma}$ be the cover $\mathcal{U}_{s,t}$, where $s=3^k+2\cdot(14c)^k$ and $t=2\cdot 3^k$. Then $\mathcal{V}_k$ satisfies the following conditions:

$$\mathrm{mesh}(\mathcal{V}_{k+1})\le c(3^{k+2}+2\cdot(14c)^k)+d, \tag{6.1}$$

$$(3^k+2\cdot(14c)^k)\text{-mult}(\mathcal{V}_{k+1})\le n+1, \tag{6.2}$$

$$\mathrm{Leb}(\mathcal{V}_{k+1})\ge 2\cdot 3^k. \tag{6.3}$$

Note that (6.1) holds since $s+4t=3^{k+2}+2\cdot(14c)^k$.

Let $V_{\alpha_{k+1}} \in \mathcal{V}_{k+1}$ be a set containing $\mathrm{B}(x_0, (3^{k+1}-1)/3)$. For every $U \in \mathcal{U}_k$ define $\tau(U) \in \Sigma$ in the following way:

- if $U \cap V_{\alpha_{k+1}} \neq \emptyset$ then $\tau(U) = \alpha_{k+1}$;
- else $\tau(U)$ is any index in Σ such that $U \cap V_{\tau(U)} \neq \emptyset$.

Define $\mathcal{U}_{k+1} = \{U_\beta\}_{\beta \in \Sigma}$, where

$$U_\beta = \bigcup_{W \in \mathcal{U}_k,\, \tau(W) = \beta} W.$$

We claim that $\mathcal{U}_{k+1}$ satisfies the following conditions:

(1) $\mathrm{mesh}(\mathcal{U}_{k+1}) \leq (14c)^{k+1}$;
(2) $((3^{k+1}-1)/3)$-$\mathrm{mult}(\mathcal{U}_{k+1}) \leq n+1$;
(3) elements of $\mathcal{U}_{k+1}$ are disjoint;
(4) $U \subset U_{\tau(U)}$ for every $U \in \mathcal{U}_k$;
(5) $U_{\alpha_{k+1}} \in \mathcal{U}_{k+1}$ contains $\mathrm{B}(x_0, (3^{k+1}-1)/3)$.

To see (1), observe that

$$\begin{aligned} \mathrm{mesh}(\mathcal{U}_{k+1}) &\leq 2\,\mathrm{mesh}(\mathcal{U}_k) + \mathrm{mesh}(\mathcal{V}_{k+1}) \\ &\leq 2 \cdot (14c)^k + c \cdot (3^{k+2} + 2 \cdot (14c)^k) + d \\ &= (2 \cdot (14c)^k + 3^{k+2} \cdot c + d) + 2 \cdot 14^k \cdot c^{k+1} \\ &\leq 14^k \cdot (2c^k + 3^2 \cdot c + c) + 2 \cdot 14^k \cdot c^{k+1} \\ &\leq 14^k \cdot 12 c^{k+1} + 2 \cdot 14^k \cdot c^{k+1} = (14c)^{k+1}. \end{aligned}$$

Condition (2) follows from (6.2) and $(3^{k+1}-1)/3 < 3^k$. All the other conditions follow by construction. ■

Corollary 6.4. *For every metric space X, $\mathrm{asdim}_{\mathrm{AN}} X \leq n$ if and only if there exist a metric space Y of $\mathrm{asdim}_{\mathrm{AN}} Y = 0$ and an asymptotically Lipschitz map $q \colon Y \to X$ with property $(\mathrm{C})_{n+1}$.*

Corollary 6.5. *For every $n \in \mathbb{N}$ and $m \geq n$ there exist metric spaces X and Y with $\mathrm{asdim}_{\mathrm{AN}} Y = m$ and $\mathrm{asdim}_{\mathrm{AN}} X = n + m$, respectively, and an asymptotically Lipschitz map $q \colon Y \to X$ with property $(\mathrm{C})_{n+1}$.*

Corollary 6.6. *If a metric space X admits a 1-precode structure for asymptotic Assouad–Nagata dimension then there exists an ultrametric space Z and a quasi-isometric map $f \colon Z \to X$.*

Proof. Let $\mathcal{U}_0, \mathcal{U}_1, \ldots$ be a 1-precode structure, and let $f : Z \to X$ be the asymptotic Lipschitz map defined as in Theorem 6.1. It suffices to show that f is a quasi-isometry. Let $U, V \in \mathcal{U}_0, U \neq V$. Let $n \in \mathbb{N}$ be such that $a^{n-1} \leq d(f(U), f(V)) \leq a^n$. Let $c, r_0 > 0$ be as in condition (2) of Theorem 6.1. Then there exists $i \in \mathbb{N}$ such that $a^i \leq c(a^n + r_0)$ and a^n-$\mathrm{mult}(\mathcal{U}_i) \leq 1$. Let U' and V' be the unique elements of $\mathcal{U}_i$ such that $U \subset U'$ and $V \subset V'$, respectively. Then $U = U'$ and $V = V'$. This implies that $d_C(U,V) \leq a^i \leq ca^n + cr_0 \leq (ca)d(f(U), f(V)) + cr_0$. This shows that f is quasi-isometric since the image of f is apparently coarsely dense. ∎

COROLLARY 6.7. *For every metric space (X,d),* $\operatorname{asdim}_{\mathrm{AN}}(X,d) = 0$ *if and only if there exists an ultrametric ρ on X such that* $\mathrm{id} : (X,d) \to (X,\rho)$ *is a quasi-isometric map.*

Proof. This easily follows from Theorem 6.3 and Corollary 6.6. ∎

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Received 29 May 2013