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Comparing the closed almost disjointness and dominating numbers

by

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Abstract. We prove that if there is a dominating family of size $\aleph_1$, then there are $\aleph_1$ many compact subsets of ω^ω whose union is a maximal almost disjoint family of functions that is also maximal with respect to infinite partial functions.

1. Introduction. Recall that two infinite subsets a and b of ω are *almost disjoint* or *a.d.* if $a \cap b$ is finite. A family $\mathscr{A}$ of infinite subsets of ω is said to be *almost disjoint* or *a.d. in* $[\omega]^\omega$ if its members are pairwise almost disjoint. A *Maximal Almost Disjoint family*, or *MAD family in* $[\omega]^\omega$ is an infinite a.d. family in $[\omega]^\omega$ that is not properly contained in a larger a.d. family.

Two functions f and g in ω^ω are said to be *almost disjoint* or *a.d.* if they agree in only finitely many places. We say that a family $\mathscr{A} \subset \omega^\omega$ is *a.d. in* ω^ω if its members are pairwise a.d., and we say that an a.d. family $\mathscr{A} \subset \omega^\omega$ is *MAD in* ω^ω if $\forall f \in \omega^\omega\ \exists h \in \mathscr{A}\ [|f \cap h| = \aleph_0]$. Identifying functions with their graphs, every a.d. family in ω^ω is also an a.d. family in $[\omega \times \omega]^\omega$; however, it is never MAD in $[\omega \times \omega]^\omega$ because any function is a.d. from the vertical columns of $\omega \times \omega$. MAD families in ω^ω that become MAD in $[\omega \times \omega]^\omega$ when the vertical columns of $\omega \times \omega$ are thrown in were considered by Van Douwen.

We say that $p \subset \omega \times \omega$ is an *infinite partial function* if it is a function from some infinite set $A \subset \omega$ to ω. An a.d. family $\mathscr{A} \subset \omega^\omega$ is said to be *Van Douwen* if for any infinite partial function p there is $h \in \mathscr{A}$ such that $|h \cap p| = \aleph_0$. $\mathscr{A}$ is Van Douwen iff $\mathscr{A} \cup \{c_n : n \in \omega\}$ is a MAD family in $[\omega \times \omega]^\omega$, where c_n is the nth vertical column of $\omega \times \omega$. The first author showed in [3] that Van Douwen MAD families always exist.

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Recall that $\mathfrak{b}$ is the least size of an unbounded family in ω^ω, $\mathfrak{d}$ is the least size of a dominating family in ω^ω, and $\mathfrak{a}$ is the least size of a MAD family in $[\omega]^\omega$. It is well known that $\mathfrak{b} \leq \mathfrak{a}$. Whether $\mathfrak{a}$ could consistently be larger than $\mathfrak{d}$ was an open question for a long time, until Shelah achieved a breakthrough in [4] by producing a model where $\mathfrak{d} = \aleph_2$ and $\mathfrak{a} = \aleph_3$. However, it is not known whether $\mathfrak{a}$ can be larger than $\mathfrak{d}$ when $\mathfrak{d} = \aleph_1$; this is one of the few major remaining open problems in the theory of cardinal invariants posed during the earliest days of the subject (see [5] and [2]). In this note we take a small step towards resolving this question by showing that if $\mathfrak{d} = \aleph_1$, then there is a MAD family in $[\omega]^\omega$ which is the union of $\aleph_1$ compact subsets of $[\omega]^\omega$. More precisely, we will establish the following:

THEOREM 1. *Assume $\mathfrak{d} = \aleph_1$. Then there exist $\aleph_1$ compact subsets of ω^ω whose union is a Van Douwen MAD family.*

The cardinal invariant $\mathfrak{a}_{\text{closed}}$ was recently introduced and studied by Brendle and Khomskii [1] in connection with the possible descriptive complexities of MAD families in certain forcing extensions of $\mathbf{L}$.

DEFINITION 2. $\mathfrak{a}_{\text{closed}}$ is the least κ such that there are κ closed subsets of $[\omega]^\omega$ whose union is a MAD family in $[\omega]^\omega$.

Obviously, $\mathfrak{a}_{\text{closed}} \leq \mathfrak{a}$. Brendle and Khomskii showed in [1] that $\mathfrak{a}_{\text{closed}}$ behaves differently from $\mathfrak{a}$ by producing a model where $\mathfrak{a}_{\text{closed}} = \aleph_1 < \aleph_2 = \mathfrak{b}$. They asked whether $\mathfrak{s} = \aleph_1$ implies that $\mathfrak{a}_{\text{closed}} = \aleph_1$. As $\mathfrak{s} \leq \mathfrak{d}$, our result in this paper provides a partial positive answer to their question.

2. The construction. Assume $\mathfrak{d} = \aleph_1$ in this section. We will build $\aleph_1$ compact subsets of ω^ω whose union is a Van Douwen MAD family. To this end, we will construct a sequence $\langle T_\alpha : \alpha < \omega_1 \rangle$ of finitely branching subtrees of $\omega^{<\omega}$ such that $\bigcup_{\alpha < \omega_1} [T_\alpha]$ has the required properties. Henceforth, $T \subset \omega^{<\omega}$ will mean T is a *subtree* of $\omega^{<\omega}$.

DEFINITION 3. Let $T \subset \omega^{<\omega}$. Let $A \in [\omega]^\omega$ and $p : A \to \omega$. For any ordinal ξ and $\sigma \in T$ define $\mathrm{rk}_{T,p}(\sigma) \geq \xi$ to mean

$$\forall \zeta < \xi \; \exists \tau \in T \; \exists l \in A \; [\tau \supset \sigma \wedge |\sigma| \leq l < |\tau| \wedge \tau(l) = p(l) \wedge \mathrm{rk}_{T,p}(\tau) \geq \zeta].$$

Note that if $\eta \leq \xi$ and $\mathrm{rk}_{T,p}(\sigma) \geq \xi$, then $\mathrm{rk}_{T,p}(\sigma) \geq \eta$, and that for a limit ordinal ξ, if $\forall \zeta < \xi \; [\mathrm{rk}_{T,p}(\sigma) \geq \zeta]$, then $\mathrm{rk}_{T,p}(\sigma) \geq \xi$. Also, for any $\sigma, \tau \in T$, if $\sigma \subset \tau$ and $\mathrm{rk}_{T,p}(\tau) \geq \xi$, then $\mathrm{rk}_{T,p}(\sigma) \geq \xi$. Moreover, if $\mathrm{rk}_{T,p}(\sigma) \ngeq \xi$ and if $\tau \in T$ and $l \in A$ are such that $\tau \supset \sigma$, $|\sigma| \leq l < |\tau|$, and $p(l) = \tau(l)$, then there is $\zeta < \xi$ such that $\mathrm{rk}_{T,p}(\tau) \ngeq \zeta$. Therefore, if there is $f \in [T]$ with $|f \cap p| = \aleph_0$, and if $\sigma \subset f$ and there is some ordinal ξ such that $\mathrm{rk}_{T,p}(\sigma) \ngeq \xi$, then there is some $\sigma \subset \tau \subset f$ and some ordinal $\zeta < \xi$ such that $\mathrm{rk}_{T,p}(\tau) \ngeq \zeta$, thus allowing us to construct an infinite, strictly descending sequence of ordinals. So if $f \in [T]$ with $|f \cap p| = \aleph_0$, then for any $\sigma \subset f$ and any ordinal ξ, $\mathrm{rk}_{T,p}(\sigma) \geq \xi$. On the other hand, suppose that $\sigma \in T$ with $\mathrm{rk}_{T,p}(\sigma) \geq \omega_1$. Then there is $\tau \in T$ with $\tau \supset \sigma$ and $l \in A$ such that $|\sigma| \leq l < |\tau|$, $p(l) = \tau(l)$, and $\mathrm{rk}_{T,p}(\tau) \geq \omega_1$, allowing us to construct $f \in [T]$ with $\sigma \subset f$ such that $|f \cap p| = \aleph_0$.

Definition 4. Suppose $T \subset \omega^{<\omega}$, $A \in [\omega]^{\omega}$, and $p : A \to \omega$. Assume that p is a.d. from each $f \in [T]$. Then define $H_{T,p} : T \to \omega_1$ by

$$H_{T,p}(\sigma) = \min\{\xi : \mathrm{rk}_{T,p}(\sigma) \not\geq \xi + 1\}.$$

Note the following features of this definition:

($*_1$) $\forall \sigma, \tau \in T\ [\sigma \subset \tau \Rightarrow H_{T,p}(\sigma) \geq H_{T,p}(\tau)]$.
($*_2$) For all $\sigma, \tau \in T$ with $\sigma \subset \tau$, if there exists $l \in A$ such that $|\sigma| \leq l < |\tau|$ and $p(l) = \tau(l)$, then $H_{T,p}(\tau) < H_{T,p}(\sigma)$.

On the other hand, notice that if there is a function $H : T \to \omega_1$ such that ($*_1$) and ($*_2$) hold with $H_{T,p}$ replaced with H, then p must be a.d. from $[T]$.

Definition 5. I is said to be an *interval partition* if $I = \langle i_n : n \in \omega\rangle$, where $i_0 = 0$, and $\forall n \in \omega\ [i_n < i_{n+1}]$. For $n \in \omega$, I_n denotes the interval $[i_n, i_{n+1})$.

Given two interval partitions I and J, we say that I *dominates* J and write $J \leq^* I$ if $\forall^\infty n \in \omega\ \exists k \in \omega\ [J_k \subset I_n]$.

It is well known that $\mathfrak{d}$ is also the size of the smallest family of interval partitions dominating any interval partition. So fix a sequence $\langle I^\alpha : \alpha < \omega_1\rangle$ of interval partitions such that:

(1) $\forall \alpha \leq \beta < \omega_1\ [I^\alpha \leq^* I^\beta]$.
(2) For any interval partition J, there exists $\alpha < \omega_1$ such that $J \leq^* I^\alpha$.

Fix an ω_1-scale $\langle f_\alpha : \alpha < \omega_1\rangle$ such that $\forall \alpha < \omega_1\ \forall n \in \omega\ [f_\alpha(n) < f_\alpha(n+1)]$. For each $\alpha \geq 1$, define e_α and g_α by induction on α as follows. If α is a successor, then $e_\alpha : \omega \to \alpha$ is any onto function, and $g_\alpha = f_\alpha$. If α is a limit, then let $\{e_n : n \in \omega\}$ enumerate $\{e_\xi : \xi < \alpha\}$. Now, define $e_\alpha : \omega \to \alpha$ and $g_\alpha \in \omega^\omega$ such that

(3) $\forall n \in \omega\ [g_\alpha(n) \leq g_\alpha(n+1)]$.
(4) $\forall n \in \omega\ \forall i \leq n\ \forall j \leq f_\alpha(n)\ \exists k < g_\alpha(n)\ [e_\alpha(k) = e_i(j)]$.

Observe that such an e_α must be a surjection. For each $n \in \omega$, put

$$w_\alpha(n) = \{e_\alpha(i) : i \leq g_\alpha(n)\}.$$

Now fix $\alpha < \omega_1$ and assume that $T_\epsilon \subset \omega^{<\omega}$ has been defined for each $\epsilon < \alpha$ such that each T_ϵ is finitely branching and $\bigcup_{\epsilon<\alpha}[T_\epsilon]$ is an a.d. family in ω^ω. Let $\langle \epsilon_n : n \in \omega\rangle$ enumerate α, possibly with repetitions. For a tree

$T \subset \omega^{<\omega}$ and $l \in \omega$, write

$$T \restriction l = \{\sigma \in T : |\sigma| \leq l\} \quad \text{and} \quad T(l) = \{\sigma \in T : |\sigma| = l\}.$$

We will define a sequence of natural numbers $0 = l_0 < l_1 < \cdots$ and determine $T_\alpha \restriction l_n$ by induction on n. First, $T_\alpha \restriction l_0 = \{0\}$. Assume that l_n and $T_\alpha \restriction l_n$ are given. Suppose also that we are given a sequence of natural numbers $\langle k_i : i < n \rangle$ such that

(5) $\forall i < i+1 < n\ [k_i < k_{i+1}]$.
(6) $I^\alpha_{k_i} \subset [0, l_n)$.

Let σ^* denote the member of $T_\alpha(l_n)$ that is rightmost with respect to the lexicographical ordering on ω^{l_n}. Suppose we are also given $L_n : T_\alpha(l_n) \setminus \{\sigma^*\} \to W_n$, an injection. Here W_n is the set of all pairs $\langle p_0, \bar{h} \rangle$ such that:

(7) There are $s \in [\omega]^{<\omega}$ and numbers $i_0 < j_0 \leq n$ such that
   (a) $s \subset \bigcup_{i \in [i_0, j_0)} I^\alpha_{k_i}$,
   (b) for each $i \in [i_0, j_0)$, $|s \cap I^\alpha_{k_i}| = 1$,
   (c) $p_0 : s \to \omega$ such that $\forall m \in s\ [p_0(m) \leq f_\alpha(m)]$.
(8) There is $j_1 < n$ such that $\bar{h} = \langle h_{\epsilon_i} : i \leq j_1 \rangle$ (if $\alpha = 0$, this means that $\bar{h} = 0$). For each $i \leq j_1$, $h_{\epsilon_i} : T_{\epsilon_i} \restriction \max(s)+1 \to w_\alpha(\max(s)+1)$ such that $(*_1)$ and $(*_2)$ hold with T replaced with $T_{\epsilon_i} \restriction \max(s)+1$, $H_{T,p}$ replaced with h_{ϵ_i}, A with s, and p with p_0.

Assume that for each $i < n$, we are also given $\sigma_i \in T_\alpha(l_i)$, which we will call *the active node at stage* i. Note that $T_\alpha(l_0) = \{0\}$, and so $\sigma_0 = 0$. For each $\sigma \in T_\alpha(l_n)$, let $\Delta(\sigma) = \max(\{0\} \cup \{i < n : \sigma_i = \sigma \restriction l_i\})$. For, $\sigma, \tau \in T_\alpha(l_n)$, say $\sigma \lhd \tau$ if either $\Delta(\sigma) < \Delta(\tau)$, or $\Delta(\sigma) = \Delta(\tau)$ and σ is to the left of τ in the lexicographic ordering on ω^{l_n}. Let σ_n be the $\lhd$-minimal member of $T_\alpha(l_n)$. Then σ_n will be active at stage n. The meaning of this is that none of the other nodes in $T_\alpha(l_n)$ will be allowed to branch at stage n. Choose k_n greater than all k_i for $i < n$ such that $I^\alpha_{k_n} \subset [l_n, \infty)$. Let V_n be the set of all pairs $\langle p_1, \bar{\mathbf{h}} \rangle$ such that:

(9) There exist s and a natural number $i_1 \leq n$ such that
   (a) $s \subset \bigcup_{i \in [i_1, n+1)} I^\alpha_{k_i}$,
   (b) for each $i \in [i_1, n+1)$, $|s \cap I^\alpha_{k_i}| = 1$,
   (c) $p_1 : s \to \omega$ such that $\forall m \in s\ [p_1(m) \leq f_\alpha(m)]$.
(10) There is $j_2 \leq n$ such that $\bar{\mathbf{h}} = \langle \mathbf{h}_{\epsilon_i} : i \leq j_2 \rangle$. For each $i \leq j_2$, $\mathbf{h}_{\epsilon_i} : T_{\epsilon_i} \restriction \max(s)+1 \to w_\alpha(\max(s)+1)$ such that $(*_1)$ and $(*_2)$ are satisfied with T replaced with $T_{\epsilon_i} \restriction \max(s)+1$, $H_{T,p}$ replaced with $\mathbf{h}_{\epsilon_i}$, A with s, and p with p_1.

Note that V_n is always finite. Now, the construction splits into two cases.

CASE I: $\sigma_n \neq \sigma^*$. Put $\langle p_0, \bar{h}\rangle = L_n(\sigma_n)$. Let $i_0 < n$ be as in (7) above, and let $j_1 < n$ be as in (8). Let

$$U_n = \{\langle p_1, \bar{\mathbf{h}}\rangle \in V_n : p_0 \subset p_1 \wedge i_0 = i_1 \wedge j_1 < j_2 \\ \wedge \forall i \leq j_1 \, [\mathbf{h}_{\epsilon_i} \restriction \mathrm{dom}(h_{\epsilon_i}) = h_{\epsilon_i}]\}.$$

Here i_1 is as in (9), and j_2 is as in (10) with respect to $\langle p_1, \bar{\mathbf{h}}\rangle$. Now choose $l_{n+1} > l_n$ large enough so that $I^{\alpha}_{k_n} \subset [l_n, l_{n+1})$ and so that it is possible to pick $\{\tau_x : x \in U_n\} \subset \omega^{l_{n+1}}$ and $\{\tau_\sigma : \sigma \in T_\alpha(l_n)\} \subset \omega^{l_{n+1}}$ such that the following conditions are satisfied:

(11) For each $x \in U_n$, $\tau_x \supset \sigma_n$, and for each $\sigma \in T_\alpha(l_n)$, $\tau_\sigma \supset \sigma$.
(12) For each $x, y \in U_n$, if $x \neq y$, then there exists $m \in [l_n, l_{n+1})$ such that $\tau_x(m) \neq \tau_y(m)$. For each $x \in U_n$, there exists $m \in [l_n, l_{n+1})$ such that $\tau_x(m) \neq \tau_{\sigma_n}(m)$. For $x = \langle p_1, \bar{\mathbf{h}}\rangle \in U_n$, if $\{i^*\} = \mathrm{dom}(p_1) \cap I^{\alpha}_{k_n}$, then $p_1(i^*) = \tau_x(i^*)$.
(13) For each $x \in U_n$ and $\sigma \in T_\alpha(l_n)$, $\forall m \in [l_n, l_{n+1})$ $[\tau_x(m) \neq \tau_\sigma(m)]$. For $\sigma, \eta \in T_\alpha(l_n)$, if $\sigma \neq \eta$, then $\forall m \in [l_n, l_{n+1})$ $[\tau_\sigma(m) \neq \tau_\eta(m)]$.
(14) For each $i \leq n$, $\tau \in T_{\epsilon_i}(l_{n+1})$, $\sigma \in T_\alpha(l_n)$ and $m \in [l_n, l_{n+1})$, $\tau(m) \neq \tau_\sigma(m)$. For each $x \in U_n$, $i \leq j_2$, $\tau \in T_{\epsilon_i}(l_{n+1})$ and $m \in [l_n, l_{n+1})$, if $\tau_x(m) = \tau(m)$, then $m \in \mathrm{dom}(p_1)$ and $p_1(m) = \tau_x(m)$.

Define L_{n+1} as follows. For any $x \in U_n$, $L_{n+1}(\tau_x) = x$. For any $\sigma \in T_\alpha(l_n) \setminus \{\sigma^*\}$, $L_{n+1}(\tau_\sigma) = L_n(\sigma)$. This finishes Case I.

CASE II: $\sigma_n = \sigma^*$. For each $\sigma \in T_\alpha(l_n) \setminus \{\sigma_n\}$, let $\langle p_0(\sigma), \bar{h}(\sigma)\rangle = L_n(\sigma)$. Let $i_0(\sigma) < n$ witness (7) for $L_n(\sigma)$ and let $j_1(\sigma) < n$ witness (8) for $L_n(\sigma)$. Let U_n be the set of all $\langle p_1, \bar{\mathbf{h}}\rangle \in V_n$ such that there is no $\sigma \in T_\alpha(l_n) \setminus \{\sigma_n\}$ so that

$$p_0(\sigma) \subset p_1 \wedge i_0(\sigma) = i_1 \wedge j_1(\sigma) < j_2 \wedge \forall i \leq j_1(\sigma) \, [\mathbf{h}_{\epsilon_i} \restriction \mathrm{dom}(h_{\epsilon_i}) = h_{\epsilon_i}].$$

Here $i_1 \leq n$ and $j_2 \leq n$ witness (9) and (10) respectively with respect to $\langle p_1, \bar{\mathbf{h}}\rangle$. Choose $l_{n+1} > l_n$ large enough so that $I^{\alpha}_{k_n} \subset [l_n, l_{n+1})$ and so that it is possible to choose $\{\tau^*\}$, $\{\tau_x : x \in U_n\}$, and $\{\tau_\sigma : \sigma \in T_\alpha(l_n) \setminus \{\sigma_n\}\}$, subsets of $\omega^{l_{n+1}}$, satisfying the following conditions:

(15) $\tau^* \supset \sigma_n$. For each $x \in U_n$, $\tau_x \supset \sigma_n$. For each $\sigma \in T_\alpha(l_n) \setminus \{\sigma_n\}$, $\tau_\sigma \supset \sigma$.
(16) τ^* is the rightmost branch of $T_\alpha(l_{n+1})$. For each $x \in U_n$, there exists $m \in [l_n, l_{n+1})$ such that $\tau^*(m) \neq \tau_x(m)$. For each $x, y \in U_n$, if $x \neq y$, then there is $m \in [l_n, l_{n+1})$ so that $\tau_x(m) \neq \tau_y(m)$. For each $x = \langle p_1, \bar{\mathbf{h}}\rangle \in U_n$, if $\{i^*\} = I^{\alpha}_{k_n} \cap \mathrm{dom}(p_1)$, then $p_1(i^*) = \tau_x(i^*)$.
(17) For each $x \in U_n$ and $m \in [l_n, l_{n+1})$, $\tau_x(m) \neq \tau^*(m)$. For each $\sigma \in T_\alpha(l_n) \setminus \{\sigma_n\}$ and for each $m \in [l_n, l_{n+1})$, $\tau^*(m) \neq \tau_\sigma(m)$, and for each $x \in U_n$, $\tau_\sigma(m) \neq \tau_x(m)$. For each $\sigma, \eta \in T_\alpha(l_n) \setminus \{\sigma_n\}$, if $\sigma \neq \eta$, then for all $m \in [l_n, l_{n+1})$, $\tau_\sigma(m) \neq \tau_\eta(m)$.

(18) For each $i \leq n$, $\tau \in T_{\epsilon_i}(l_{n+1})$, $m \in [l_n, l_{n+1})$, and $\sigma \in T_\alpha(l_n) \setminus \{\sigma_n\}$, $\tau^*(m) \neq \tau(m)$ and $\tau_\sigma(m) \neq \tau(m)$. For each $x = \langle p_1, \bar{\mathbf{h}} \rangle \in U_n$, $i \leq j_2$, $\tau \in \tau_{\epsilon_i}(l_{n+1})$ and $m \in [l_n, l_{n+1})$, if $\tau_x(m) = \tau(m)$, then $m \in \operatorname{dom}(p_1)$ and $p_1(m) = \tau_x(m)$.

For each $\sigma \in T_\alpha(l_n) \setminus \{\sigma_n\}$, define $L_{n+1}(\tau_\sigma) = L_n(\sigma)$. For each $x \in U_n$, set $L_{n+1}(\tau_x) = x$. This completes the construction. We now check that it is as required.

Lemma 6. *For each $f \in [T_\alpha]$, there are infinitely many $n \in \omega$ such that $\sigma_n = f \restriction l_n$.*

Proof. For each $n \in \omega$ put $\Theta(n) = \min\{\Delta(\sigma) : \sigma \in T_\alpha(l_n)\}$. It is clear from the construction that $\Theta(n+1) \geq \Theta(n)$. If the lemma fails, then there are m and $\tau \in T_\alpha(l_{m+1})$ with the property that for infinitely many $n > m+1$, there is a $\sigma \in T_\alpha(l_n)$ such that $\Theta(n) = \Delta(\sigma) = m$ and $\sigma \restriction l_{m+1} = \tau$. Let τ be the leftmost node in $T_\alpha(l_{m+1})$ with this property. Choose $n_1 > n_0 > m+1$ and $\sigma \in T_\alpha(l_{n_1})$ such that $\Theta(n_1) = \Theta(n_0) = \Delta(\sigma) = m$, $\sigma \restriction l_{m+1} = \tau$, and there is no $\eta \in T_\alpha(l_{n_0})$ such that $\Delta(\eta) = m$ and $\eta \restriction l_{m+1}$ is to the left of τ. Note that $\Delta(\sigma \restriction l_{n_0}) = m$. So σ_{n_0} is to the left of $\sigma \restriction l_{n_0}$, and $\sigma_{n_0} \restriction l_{m+1}$ is not to the left of τ, whence $\sigma_{n_0} \restriction l_{m+1} = \tau$. But then there is some $n \in [m+1, n_0)$ where $\sigma \restriction l_n$ was active, a contradiction. ∎

Note that Lemma 6 implies that for any $\sigma \in T_\alpha$, there is a unique minimal extension of σ which is active. Lemma 6 also implies that there are infinitely many n where Case II occurs.

Lemma 7. *T_α is finitely branching and $\bigcup_{\epsilon \leq \alpha} [T_\epsilon]$ is a.d. in ω^ω.*

Proof. It is clear from the construction that T_α is finitely branching. Fix $f, g \in [T_\alpha]$ with $f \neq g$. Let $n = \max\{i \in \omega : f \restriction l_i = g \restriction l_i\}$. It is clear from the construction that $\forall m \geq l_{n+1}\ [f(m) \neq g(m)]$.

Next, fix $\epsilon < \alpha$. Suppose $\epsilon = \epsilon_i$. Let $h \in [T_{\epsilon_i}]$ and $f \in [T_\alpha]$, and suppose for a contradiction that $|h \cap f| = \aleph_0$. So there are infinitely many $n \in \omega$ such that $f \restriction [l_n, l_{n+1}) \cap h \restriction [l_n, l_{n+1}) \neq 0$. For any $n \geq i$, this can only happen if $f \restriction l_n = \sigma_n$ and $f \restriction l_{n+1} = \tau_{x_n}$ for some $x_n \in U_n$. This is because if $n \geq i$ and $f \restriction [l_n, l_{n+1}) \cap h \restriction [l_n, l_{n+1}) \neq 0$, then when Case I occurs, (14) says that $f \restriction l_{n+1} \neq \tau_\sigma$ for any $\sigma \in T_\alpha(l_n)$, while when Case II occurs, by (18), $f \restriction l_{n+1} \neq \tau^*$ and also $f \restriction l_{n+1} \neq \tau_\sigma$ for any $\sigma \in T_\alpha(l_n) \setminus \{\sigma_n\}$. So $f \restriction l_{n+1} = \tau_{x_n}$ for some $x_n \in U_n$, and $f \restriction l_n = \sigma_n$. Now, put $x_n = \langle p_{1,n}, \bar{\mathbf{h}}_n \rangle$. Note that in this case $L_{n+1}(f \restriction l_{n+1}) = x_n$. For such n, let $j_2(n)$ be as in (10) with respect to x_n. So for infinitely many such n, $j_2(n) \geq i$. But then for infinitely many such n, $\mathbf{h}_{\epsilon_i,n}(h \restriction \max(\operatorname{dom}(p_{1,n})) + 1) < \mathbf{h}_{\epsilon_i,n}(h \restriction l_n)$, producing an infinite strictly descending sequence of ordinals. ∎

LEMMA 8. *For each $A \in [\omega]^\omega$ and $p : A \to \omega$, there are $\alpha < \omega_1$ and $f \in [T_\alpha]$ such that $|p \cap f| = \aleph_0$.*

Proof. Suppose for a contradiction that there are $A \in [\omega]^\omega$ and $p : A \to \omega$ such that p is a.d. from $[T_\alpha]$, for each $\alpha < \omega_1$. Let $M \prec H(\theta)$ be a countable elementary submodel containing everything relevant. Put $\alpha = M \cap \omega_1$. For each $\epsilon < \alpha$, let H_ϵ denote $H_{T_\epsilon,p}$, and note that H_ϵ and $\mathrm{ran}(H_\epsilon)$ are members of M. Let $\xi_\epsilon = \sup(\mathrm{ran}(H_\epsilon)) + 1 < \alpha$. Find $g \in M \cap \omega^\omega$ such that for $n \in \omega$, $H''_\epsilon T_\epsilon{\restriction}n \subset \{e_{\xi_\epsilon}(j) : j \le g(n)\}$. Since $\forall^\infty n \in \omega\ [g(n) \le f_\alpha(n)]$, it follows from (4) that for all but finitely many $n \in \omega$, and all $\sigma \in T_\epsilon{\restriction}n$, $H_\epsilon(\sigma) \in w_\alpha(n)$. Now, find an infinite $q \subset p$ such that $\forall m \in \mathrm{dom}(q)\ [q(m) \le f_\alpha(m)]$ and $\forall^\infty n \in \omega\ [|\mathrm{dom}(q) \cap I^\alpha_n| = 1]$. Note that for any $\epsilon < \alpha$, $(*_1)$ and $(*_2)$ are satisfied with T replaced with T_ϵ, $H_{T,p}$ replaced with H_ϵ, A with $\mathrm{dom}(q)$, and p with q. But now, it follows from the construction that there is $f \in [T_\alpha]$ such that for infinitely many $n \in \omega$, there is $m \in [l_n, l_{n+1}) \cap \mathrm{dom}(q)$ such that $q(m) = f(m)$.

We describe how to find such an $f \in [T_\alpha]$. We have $\forall^\infty n \in \omega\ [|\mathrm{dom}(q) \cap I^\alpha_{k_n}| = 1]$. For each $n \in \omega$ such that $|\mathrm{dom}(q) \cap I^\alpha_{k_n}| = 1$, let m_n be the unique member of $\mathrm{dom}(q) \cap I^\alpha_{k_n}$. We observed above that for any $\epsilon < \alpha$, for all but finitely many $n \in \omega$, and each $\sigma \in T_\epsilon{\restriction}n$, $H_\epsilon(\sigma) \in w_\alpha(n)$. It follows that for any $i \in \omega$, there is $u_i \ge i$ such that for each $j \le i$ and each $n \ge u_i$, m_n is defined and $\forall \sigma \in T_{\epsilon_j}{\restriction}m_n + 1\ [H_{\epsilon_j}(\sigma) \in w_\alpha(m_n + 1)]$. Choose $n^* \ge u_0$ so that Case II occurs at stage n^*. Put $\eta_0 = \sigma_{n^*}$. Define $s_0 = \{m_{n^*}\}$ and $q_0 = q{\restriction}s_0$. Put $\bar{h}_0 = \langle h_0 \rangle$, where $h_0 = H_{\epsilon_0}{\restriction}(T_{\epsilon_0}{\restriction}\max(s_0) + 1)$. Note that h_0 is a map from $T_{\epsilon_0}{\restriction}\max(s_0) + 1$ to $w_\alpha(\max(s_0) + 1)$, and so $x_0 = \langle q_0, \bar{h}_0 \rangle \in V_{n^*}$. Since $m_{n^*} \notin I^\alpha_{k_i}$ for any $i < n^*$, it follows that $x_0 \in U_{n^*}$. Put $\eta_1 = \tau_{x_0} \supsetneq \eta_0$. Notice that $\eta_1(m_{n^*}) = q(m_{n^*})$. Notice also that η_1 is not the rightmost branch of $T_\alpha(l_{(n^*+1)})$, and so if σ is any extension of η_1 that happens to be active at a certain stage, then Case I necessarily occurs at that stage. Finally, note that $L_{n^*+1}(\eta_1) = x_0$.

Now, for each $n > n^*$, let $s_n = \{m_j : n^* \le j \le n\}$, and put $q_n = q{\restriction}s_n$. For any $i > 0$ and $n > n^*$, if $n \ge u_i$, then for each $j \le i$, define $h^n_j = H_{\epsilon_j}{\restriction}(T_{\epsilon_j}{\restriction}\max(s_n) + 1)$. Put $\bar{h}^n_i = \langle h^n_j : j \le i \rangle$ and $x^n_i = \langle q_n, \bar{h}^n_i \rangle$. Note that for any $i > 0$ and $n > n^*$, if $n \ge u_i$, then $x^n_i \in V_n$. Moreover, if at stage n, Case I occurs and $L_n(\sigma_n) = x^v_{i-1}$ for some $v \in \omega$, then $x^n_i \in U_n$; here $x^v_0 = x_0$ for all $v \in \omega$. Now, it is easy to see that there is a branch $g \in [T_\alpha]$ such that $\eta_1 \subset g$ and $\forall n \ge n^* + 1\ [L_n(g{\restriction}l_n) = x_0]$. This is because for any $n \ge n^* + 1$, given $g{\restriction}l_n$ such that $\eta_1 \subset g{\restriction}l_n$ and $L_n(g{\restriction}l_n) = x_0$, if σ is the unique minimal extension of $g{\restriction}l_n$ that is active, then $\tau_\sigma \supsetneq g{\restriction}l_n$ and $L_{u+1}(\tau_\sigma) = x_0$, where u is the stage at which σ is active. Applying Lemma 6 to g, find n^{**} such that $n^{**} > n^*$, $n^{**} \ge u_1$, and $\sigma_{n^{**}} = g{\restriction}l_{n^{**}}$. It follows that $x^{n^{**}}_1 \in U_{n^{**}}$. Let $\eta_2 = \tau_{x^{n^{**}}_1} \supsetneq \eta_1$. Note that $\eta_2(m_{n^{**}}) = q(m_{n^{**}})$ and that $L_{n^{**}+1}(\eta_2) = x^{n^{**}}_1$.

Continuing in this fashion, we get

$$f = \bigcup_{n\in\omega} \eta_n \in [T_\alpha] \quad \text{with} \quad |f \cap q| = \omega. \ \blacksquare$$

3. Remarks and questions. The construction in this paper is very specific to ω_1; indeed, it is possible to show that $\mathfrak{d}$ is not always an upper bound for $\mathfrak{a}_{\mathrm{closed}}$. A modification of the methods of Section 4 of [4] shows that if κ is a measurable cardinal and if

$$\lambda = \mathrm{cf}(\lambda) = \lambda^\kappa > \mu = \mathrm{cf}(\mu) > \kappa,$$

then there is a c.c.c. poset $\mathbb{P}$ such that $|\mathbb{P}| = \lambda$, and $\mathbb{P}$ forces that $\mathfrak{b} = \mathfrak{d} = \mu$ and $\mathfrak{a} = \mathfrak{a}_{\mathrm{closed}} = \mathfrak{c} = \lambda$.

As mentioned in Section 1, we see the result in this paper as providing a weak positive answer to the following basic question, which has remained open for long.

Question 9. If $\mathfrak{d} = \aleph_1$, then is $\mathfrak{a} = \aleph_1$?

There are also several open questions about upper and lower bounds for $\mathfrak{a}_{\mathrm{closed}}$.

Question 10 (Brendle and Khomskii [1]). If $\mathfrak{s} = \aleph_1$, then is $\mathfrak{a}_{\mathrm{closed}} = \aleph_1$?

Question 11. Is $\mathfrak{h} \leq \mathfrak{a}_{\mathrm{closed}}$?

Regarding Question 10, it is proved in Brendle and Khomskii [1] that if **V** is any ground model satisfying CH, then any finite support iteration of Suslin c.c.c. posets in **V** forces that $\mathfrak{a}_{\mathrm{closed}} = \aleph_1$. It is well known that **V** remains a splitting family after such a finite support iteration of Suslin c.c.c. posets. Showing a positive answer to Question 10 would be an improvement of the result in this paper.

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