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Measure-theoretic unfriendly colorings

by

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Abstract. We consider the problem of finding a measurable unfriendly partition of the vertex set of a locally finite Borel graph on standard probability space. After isolating a sufficient condition for the existence of such a partition, we show how it settles the dynamical analog of the problem (up to weak equivalence) for graphs induced by free, measure-preserving actions of groups with designated finite generating set. As a corollary, we obtain the existence of translation-invariant random unfriendly colorings of Cayley graphs of finitely generated groups.

1. Introduction. Given a graph G on a (possibly infinite) vertex set X, we say that a partition $X = X_1 \sqcup X_2$ is *unfriendly* if every vertex in X_i has at least as many neighbors in X_{3-i} as it has in X_i. A straightforward compactness argument grants the existence of unfriendly partitions for *locally finite* graphs, that is, graphs in which every vertex has finite degree; see [1] for a more general result allowing for finitely many vertices of infinite degree. On the other hand, by [6] there is a graph on an uncountable vertex set admitting no such partition. (The general case on a countable vertex set remains open.) In this paper we consider measure-theoretic analogs of the former result.

Using now a standard Borel space X as our vertex set, we say that a graph G on X is *Borel* if it is Borel as a (symmetric, irreflexive) subset of X^2. It will be more convenient to use the language of colorings rather than that of partitions. Towards that end, given $n \in \mathbb{N}^+$ and $\alpha \in [0,1]$, we say that $c\colon X \to n$ is an (n,α)*-coloring* of a locally finite graph G on X if for all $x \in X$ we have $|c^{-1}(c(x)) \cap G_x| \le \alpha |G_x|$, where G_x denotes the set of neighbors of x. So an $(n,0)$-coloring is what is normally called a proper n-coloring, i.e., no two adjacent vertices receive the same color. Furthermore, c is a $(2,1/2)$-coloring iff $c^{-1}(0) \sqcup c^{-1}(1)$ forms an unfriendly partition.

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If moreover μ is a Borel probability measure on X, we define (n,α,μ)-*colorings* to be those functions satisfying the condition $|c^{-1}(c(x))\cap G_x| \leq \alpha|G_x|$ on a μ-conull set of $x\in X$. Our focus in this paper is on the existence of Borel (n,α,μ)-colorings for various classes of Borel graphs on X. Note that (using the axiom of choice) the existence of a Borel (n,α,μ)-coloring is equivalent to the existence of a μ-measurable (n,α)-coloring.

In particular, suppose that Γ is a group with finite, symmetric generating set S (which we always assume does not contain the identity). Associated with any free, μ-preserving Borel action of Γ on (X,μ) is a graph relating distinct points of X if and only if an element of S sends one to the other. We then have (see Section 3 for a definition of weak equivalence), appearing as Theorem 3.4 in the text,

THEOREM. *Suppose that (X,μ) is a standard probability space, $n\in\mathbb{N}^+$, and Γ is a group with finite, symmetric generating set S. Then any free, μ-preserving Borel action of Γ on (X,μ) is weakly equivalent to one whose associated graph admits a Borel $(n,1/n,\mu)$-coloring.*

Recall that the (right) *Cayley graph* $\mathrm{Cay}(\Gamma,S)$ of a group Γ with designated generating set S has vertex set Γ and edges $(\gamma,\gamma s)$ for $\gamma\in\Gamma$ and $s\in S$. We may view the space of (n,α)-colorings of $\mathrm{Cay}(\Gamma,S)$ as a subset of n^Γ which is closed in the product topology, so a compact Polish space in its own right. Then Γ acts by (left) translations on the space of (n,α)-colorings by $(\gamma\cdot c)(\delta)=c(\gamma^{-1}\delta)$. Appearing as Corollary 4.1 in the text, we obtain

COROLLARY. *Suppose that Γ is a group with finite, symmetric generating set S. Then there is a translation-invariant Borel probability measure on the space of $(n,1/n)$-colorings of the Cayley graph* $\mathrm{Cay}(\Gamma,S)$.

Such a measure may be interpreted as a (translation-invariant) *random $(n,1/n)$-coloring* of $\mathrm{Cay}(\Gamma,S)$. In particular, in the case $n=2$ we obtain a random unfriendly partition of the Cayley graph.

2. Minimizing friendliness. We fix a standard probability space (X,μ); we denote by $\Delta(X)$ the set $\{(x,x): x\in X\}\subseteq X^2$. We say that a locally countable Borel graph G on X is *μ-preserving* if there are μ-preserving Borel automorphisms T_i, $i\in\omega$, of X such that $G\cup\Delta(X)=\bigcup_i \mathrm{graph}(T_i)$. This is the same as saying that the connectedness equivalence relation of G arises as the orbit equivalence relation of a μ-preserving group action. For convenience we restrict our attention to Borel graphs G with bounded degree, in the sense that there is some $d\in\mathbb{N}$ such that for all $x\in X$, $\deg(x)\leq d$. The results of this section actually hold under the somewhat weaker assumption that $\int \deg(x)\,d\mu(x)$ is finite, but the bounded-degree case suffices for later sections. For $A \subseteq X$ we define the *restriction* of G to A, written $G \restriction A$, by $G \cap A^2$. We say that A is G-*independent* if $G \restriction A = \emptyset$.

Fixing now a bounded-degree μ-preserving Borel graph G on the standard probability space (X, μ), for each Borel function $c\colon X \to n$ we will define a parameter $\mathrm{Friend}_n(c)$ recording the (n-)*friendliness* of the function. First let ν be the product $\mu \times$ (counting) measure on G, that is, for $A \subseteq G$ Borel, we have $\nu(A) = \int |A_x|\, d\mu$. The hypothesis that G has bounded degree ensures that ν is a finite measure. Next for each $c\colon X \to n$ define an auxiliary graph $G_c \subseteq G$ by $x\ G_c\ y$ if $x\ G\ y$ and $c(x) = c(y)$. Finally,

$$\mathrm{Friend}_n(c) = \nu(G_c) = \sum_{i<n} \nu(G \restriction c^{-1}(i)).$$

We also define the (n-)*friendliness* $\mathrm{Friend}_n(G)$ of the graph G as the infimum of $\mathrm{Friend}_n(c)$ over all Borel $c\colon X \to n$. Note that 2-friendliness may be viewed as a measure-theoretic analog of the size of a maximal cut.

PROPOSITION 2.1. *Suppose that G is a bounded-degree μ-preserving Borel graph on the standard probability space (X, μ) and $n \in \mathbb{N}^+$. Suppose moreover that $c\colon X \to n$ is a Borel function satisfying* $\mathrm{Friend}_n(c) = \mathrm{Friend}_n(G)$. *Then c is an $(n, 1/n, \mu)$-coloring of G.*

Proof. Suppose towards a contradiction that c is not an $(n, 1/n, \mu)$-coloring, i.e., that the set $Y = \{x \in X : |c^{-1}(c(x)) \cap G_x| > n^{-1}|G_x|\}$ has positive measure. Since G is locally finite, [5, Proposition 4.5] ensures that G has countable Borel chromatic number. In particular, there is a G-independent set $Y' \subseteq Y$ of positive measure. By the pigeonhole principle applied to G_x, for each $x \in Y'$ there is a least $d(x) \in n$ such that $|c^{-1}(d(x)) \cap G_x| < n^{-1}|G_x|$. Note of course that the assignment $x \mapsto d(x)$ is Borel. Then note that the coloring $c'\colon X \to 2$ defined by

$$c'(x) = \begin{cases} c(x) & \text{if } x \in X \setminus Y', \\ d(x) & \text{if } x \in Y', \end{cases}$$

satisfies $\mathrm{Friend}(c') \le \mathrm{Friend}(c) - 2\mu(Y') < \mathrm{Friend}(c) = \mathrm{Friend}(G)$, contradicting the definition of $\mathrm{Friend}(G)$. ∎

REMARK 2.2. While the minimization of 2-friendliness is sufficient for a coloring to induce an unfriendly partition, it is far from necessary. For instance, for fixed irrational $\alpha \in (0, 1)$ consider the graph G_α on $[0, 1)$, where $x\ G_\alpha\ y$ iff $x - y = \pm\alpha \mod 1$. Then $\mathrm{Friend}_2(G_\alpha) = 0$, but it is not hard to show that $\mathrm{Friend}_2(c) > 0$ for each Borel $c\colon [0, 1) \to 2$. Nevertheless, the 2-regularity of G_α makes it straightforward to find a Borel unfriendly partition for G_α: indeed, by [5, Proposition 4.2] there is a maximal G_α independent set which is Borel, so it and its complement form an unfriendly partition. In the interest of full disclosure, we actually do not know whether every locally finite Borel graph admits a Borel unfriendly partition.

QUESTION 2.3. *Suppose that G is a locally finite Borel graph on a standard Borel space X and $n \in \mathbb{N}^+$. Does G admit a Borel $(n, 1/n)$-coloring?*

There is a natural measure-theoretic weakening of Question 2.3 which also remains open.

QUESTION 2.4. *Suppose that G is a locally finite, μ-preserving graph on a standard probability space (X, μ) and $n \in \mathbb{N}^+$. Does G admit a Borel $(n, 1/n, \mu)$-coloring?*

The results of the next section rule out certain possible counterexamples arising from combinatorial information invariant under weak equivalence of group actions.

3. Group actions. We next narrow our focus to graphs arising from countable groups acting by free μ-preserving automorphisms on (X, μ). Given a countable group Γ with symmetric generating set S and such an action a of Γ on (X, μ), we define the graph $G(S, a)$ on the vertex set X with edge (x, y) iff $x \neq y$ and $\exists s \in S\ (y = s \cdot x)$. By freeness of the action (and assuming that S does not contain the identity element of the group), each vertex has degree $|S|$, so if S is a finite generating set, the graph is locally finite (and in fact has bounded degree). Combinatorial parameters associated with $G(S, a)$ reflect various dynamical properties of the action a; for more see [2].

Recall that we may equip the space $\mathrm{Aut}(X, \mu)$ of μ-preserving Borel automorphisms of the standard probability space (X, μ) with the *weak topology*, the weakest topology rendering for each Borel $A \subseteq X$ the map $T \mapsto T(A)$ continuous. This topology makes $\mathrm{Aut}(X, \mu)$ a Polish group. Then, following [4, II.10] we equip the space $\mathrm{A}(\Gamma, X, \mu)$ of μ-preserving actions of Γ on X with its own *weak topology* inherited as a (closed, thus Polish) subset of $\mathrm{Aut}(X, \mu)^\Gamma$. We let $\mathrm{FR}(\Gamma, X, \mu)$ denote the subset of μ-a.e. free actions.

We also recall the notion of weak containment among the elements of $\mathrm{A}(\Gamma, X, \mu)$. We say that a is *weakly contained* in b, denoted by $a \prec b$, if for any $\varepsilon > 0$, any finite sequence $(A_i)_{i \leq n}$ of Borel subsets of X and any finite $F \subseteq \Gamma$, there is a finite sequence $(B_i)_{i \leq n}$ of Borel subsets of X such that for all $i, j \leq n$ and $\gamma \in F$,

$$|\mu(A_i \cap \gamma^a \cdot A_j) - \mu(B_i \cap \gamma^b \cdot B_j)| < \varepsilon.$$

Equivalently, $a \prec b$ exactly when a is in the weak closure of the set of actions in $\mathrm{A}(\Gamma, X, \mu)$ conjugate to b. Finally, a and b are *weakly equivalent*, written $a \sim b$, if $a \prec b$ and $b \prec a$.

In [2, Corollary 4.2 and Theorem 4.3] and [3, Proposition 5.1] it is shown that many measure-theoretic combinatorial parameters of $G(S,a)$ respect weak containment. We next see that friendliness is another such parameter.

PROPOSITION 3.1. *Suppose that Γ is a group with finite, symmetric generating set S, and $a, b \in \mathrm{FR}(\Gamma, X, \mu)$ with $a \prec b$. Then*

$$\mathrm{Friend}_n(G(S,a)) \geq \mathrm{Friend}_n(G(S,b)).$$

Proof. Fix $\varepsilon > 0$, and choose a Borel function $c\colon X \to n$ with $\mathrm{Friend}_n(c) < \mathrm{Friend}_n(G(S,a))+\varepsilon$. For each $i < n$ set $A_i = c^{-1}(i)$. By weak containment, we may find Borel sets $B_i \subseteq X$ such that for all $\gamma \in S \cup \{1_\Gamma\}$ and $i, j < n$, $|\mu(A_i \cap \gamma^a \cdot A_j) - \mu(B_i \cap \gamma^b \cdot B_j)| < \varepsilon$. Note in particular that for $i \neq j$, the disjointness of A_i and A_j implies $\mu(B_i \cap B_j) < \varepsilon$. Let $d\colon X \to n$ be any Borel function satisfying $d(x) = \min\{i < n : x \in B_i\}$ for $x \in \bigcup_i B_i$. Note that the above considerations show for each $i < n$ that $\mu(B_i \mathbin{\triangle} d^{-1}(i)) < 2n\varepsilon$. By aiming for a smaller ε, we may assume that in fact $\mu(B_i \mathbin{\triangle} d^{-1}(i)) < \varepsilon$ to clean up some inequalities.

We now estimate the n-friendliness of d with respect to $G(S,b)$. For each $i < n$, we have

$$\begin{aligned}
\nu(G(S,b)\restriction d^{-1}(i)) &= \sum_{s\in S} \mu(d^{-1}(i) \cap s^b \cdot d^{-1}(i)) \\
&< \sum_{s\in S} (\mu(B_i \cap s^b \cdot B_i) + 2\varepsilon) \\
&< \sum_{s\in S} (\mu(A_i \cap s^a \cdot A_i) + 3\varepsilon) \\
&= \nu(G(S,a)\restriction A_i) + 3|S|\varepsilon.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathrm{Friend}_n(d) &= \sum_{i<n} \nu(G(S,b)\restriction d^{-1}(i)) < \sum_{i<n} \nu(G(S,a)\restriction A_i) + 3|S|\varepsilon \\
&= \mathrm{Friend}_n(c) + 3n|S|\varepsilon < \mathrm{Friend}_n(G(S,a)) + (3n|S|+1)\varepsilon.
\end{aligned}$$

As ε may be chosen to be arbitrarily small, we see that $\mathrm{Friend}_n(G(S,a)) \geq \mathrm{Friend}_n(G(S,b))$, as desired. ∎

COROLLARY 3.2. *Suppose that Γ is a group with finite, symmetric generating set S, and $a, b \in \mathrm{FR}(\Gamma, X, \mu)$ with $a \sim b$. Then*

$$\mathrm{Friend}_n(G(S,a)) = \mathrm{Friend}_n(G(S,b)).$$

Next, we record a version of [3, Theorem 5.2] allowing us to realize the infimum in the definition of friendliness within any weak equivalence class.

PROPOSITION 3.3. *Suppose that Γ is a group with finite, symmetric generating set S. For any $a \in \mathrm{FR}(\Gamma, X, \mu)$, there is an action $b \in \mathrm{FR}(\Gamma, X, \mu)$ with $b \sim a$ and a μ-measurable function $c\colon X \to n$ such that*

$$\mathrm{Friend}_n(G(S, b)) = \mathrm{Friend}_n(c).$$

Proof. We use the notation of [3]. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. We obtain the action b as an appropriately chosen factor of the ultrapower action $a_{\mathcal{U}}$ on $(X_{\mathcal{U}}, \mu_{\mathcal{U}})$. First, for each $k \in \mathbb{N}$ fix Borel functions $c_k\colon X \to n$ satisfying $\mathrm{Friend}_n(c_k) \leq \mathrm{Friend}_n(G(S, a)) + 1/k$. For each $i < n$ set $C_i = [(c_k^{-1}(i))_k]_{\mathcal{U}}$. Then $\{C_i : i < n\}$ form a $\mu_{\mathcal{U}}$-a.e. partition of $X_{\mathcal{U}}$. Defining $c\colon X_{\mathcal{U}} \to n$ so that $x \in C_{c(x)}$ $\mu_{\mathcal{U}}$-a.e., we see that $\mathrm{Friend}_n(c) = \mathrm{Friend}_n(G(S, a))$ (as computed by $\mu_{\mathcal{U}}$). Restricting to a sufficiently large countably generated nonatomic, Γ-invariant subalgebra of the ultrapower measure algebra containing C_i and a countable collection of subsets of X generating its Borel σ-algebra, we obtain (up to isomorphism) an action $b \in \mathrm{FR}(\Gamma, X, \mu)$. Then [3, §4(B)] implies that $b \sim a$, and hence, by Corollary 3.2, $\mathrm{Friend}_n(G(S, b)) = \mathrm{Friend}_n(G(S, a))$. ∎

THEOREM 3.4. *Suppose that Γ is a group with finite, symmetric generating set S. For any $a \in \mathrm{FR}(\Gamma, X, \mu)$ there is an action $b \in \mathrm{FR}(\Gamma, X, \mu)$ with $b \sim a$ such that $G(S, b)$ admits a Borel $(n, 1/n, \mu)$-coloring.*

Proof. By Proposition 3.3, there is a $b \in \mathrm{FR}(\Gamma, X, \mu)$ with $b \sim a$ such that the infimum in the definition of $\mathrm{Friend}(G(S, b))$ is attained. Then Proposition 2.1 implies that any function attaining that infimum is an $(n, 1/n, \mu)$-coloring of $G(S, b)$. ∎

QUESTION 3.5. *For a group Γ with finite generating set S, consider the Bernoulli shift action s of Γ on $[0, 1]^\Gamma$ with product Lebesgue measure μ defined by $(\gamma \cdot x)(\delta) = x(\gamma^{-1}\delta)$. Does $G(S, s)$ admit a Borel $(n, 1/n, \mu)$-coloring? An affirmative answer, in conjunction with* [7, *Corollary* 1.6], *would provide an alternate proof of Theorem* 3.4.

REMARK 3.6. Theorem 3.4 rules out various approaches to producing a counterexample for Question 2.4. In particular, analysis of any combinatorial quantity of group actions invariant under weak equivalence cannot establish the inexistence of a Borel $(n, 1/n, \mu)$-coloring of $G(S, a)$. This is in contrast to [2, Theorem 4.17], in which an analysis of the norm of the averaging operator ruled out various proper colorings in the measure-theoretic context (or $(n, 0, \mu)$-colorings in the current vernacular).

4. Random colorings of Cayley graphs. Given a group Γ with generating set S, a positive natural number n, and $\alpha \in [0, 1]$, we may view the space $\mathrm{Col}(\Gamma, S, n, \alpha)$ of (n, α)-colorings of the (right) Cayley graph $\mathrm{Cay}(\Gamma, S)$ as a closed (thus Polish) subset of n^Γ. The action of Γ by left translations on $\mathrm{Cay}(\Gamma, S)$ induces an action on $\mathrm{Col}(\Gamma, S, n, \alpha)$. A *translation-invariant random* (n, α)*-coloring* of $\mathrm{Cay}(\Gamma, S)$ is a Borel probability measure on $\mathrm{Col}(\Gamma, S, n, \alpha)$ which is invariant under this Γ action.

In particular, a translation-invariant random $(2, 1/2)$-coloring may be viewed as a random unfriendly partition of the Cayley graph, where the translation invariance means that the likelihood of choosing a partition is independent of the selection of a vertex of the Cayley graph as the identity.

COROLLARY 4.1. *Suppose that Γ is a group with finite, symmetric generating set S, and $n \in \mathbb{N}^+$. Then there is a translation-invariant random $(n, 1/n)$-coloring of* $\mathrm{Cay}(\Gamma, S)$.

Proof. Fix a nonatomic standard probability space (X, μ). By Theorem 3.4, there is some $b \in \mathrm{FR}(\Gamma, X, \mu)$ such that $G(S, b)$ admits a Borel $(n, 1/n, \mu)$-coloring $c\colon X \to n$ (in fact b may be chosen from any weak equivalence class). Define $\pi\colon X \to \mathrm{Col}(\Gamma, S, n, 1/n)$ by $(\pi(x))(\gamma) = c(\gamma^{-1}\cdot x)$. Then $\pi_*\mu$ is a translation-invariant random $(n, 1/n)$-coloring, where as usual $\pi_*\mu(A) = \mu(\pi^{-1}(A))$. ■

We close with a question which is essentially a probabilistic version of Question 3.5.

QUESTION 4.2. *Can such a translation-invariant random $(n, 1/n)$-coloring be found as a factor of IID?*

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