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Dynamics of a Lotka–Volterra map

by

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Abstract. Given the plane triangle with vertices $(0,0)$, $(0,4)$ and $(4,0)$ and the transformation $F:(x,y)\mapsto(x(4-x-y),xy)$ introduced by A. N. Sharkovskiĭ, we prove the existence of the following objects: a unique invariant curve of spiral type, a periodic trajectory of period 4 (given explicitly) and a periodic trajectory of period 5 (described approximately). Also, we give a decomposition of the triangle which helps to understand the global dynamics of this discrete system which is linked with the behavior of the Schrödinger equation.

1. Introduction and statement of the main results. Two-dimensional continuous transformations of the plane, $G:(x,y)\mapsto(f(x,y),g(x,y))$, have been considered for a long time to describe many phenomena coming from population dynamics, economy theory, social sciences and engineering.

In most cases there exist compact subsets $X\subset\mathbb{R}^2$, invariant under the action of the transformation (i.e., $G(X)\subseteq X$), where the most interesting part of the dynamics of the system is developed. If we see them as two-dimensional discrete dynamical systems, i.e. couples of the form $(X,G|_X)$, the interest is focused on the behavior of points of X, i.e., how the trajectories of all points evolve under the action of G.

In applications, the maps f and g are usually piecewise polynomial on X, i.e., there exists a finite partition of X, $\{X_i\}_{i=1}^n$, such that f,g restricted to

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X_i are polynomials. See, for instance, the models of stability of synchronized states of Glendinning [4] or the Duffing transformation [12].

More frequently, maps f and g are quadratic polynomials (or piecewise quadratic) and within the quadratic case, the Lotka–Volterra transformations (or piecewise Lotka–Volterra) of the form

$$G:(x,y)\mapsto(x(a_1+b_1x+c_1y),y(a_2+b_2x+c_2y)) \tag{1}$$

where $a_i,b_i,c_i\in\mathbb{R}$ for $i\in\{1,2\}$. In particular, the case $b_1=c_1=b_2=c_2=-1$ appears in several applications (see [3]).

When we try to understand the dynamics of such systems we concentrate on two facts. Firstly, we look for invariant sets and consider the dynamics only on them if we find that outside them the behavior is easy to describe. Secondly, we study the dynamics on the boundaries of such invariant sets. When the boundaries are composed of segments, the dynamics on them can be as complicated as that of some interval maps. Additionally, it could be interesting to explore connections between the dynamics on the boundaries and interiors of the invariant sets.

This is what happens in the example suggested by A. N. Sharkovskiĭ in 1993 (see [9]) for the case $a_1=4$, $b_1=c_1=-1$, $a_2=b_2=0$ and $c_2=1$, that is,

$$F:(x,y)\mapsto(x(4-x-y),xy). \tag{2}$$

It is easy to see that the triangle $\varDelta\subset\mathbb{R}^2$ with vertices $(0,0)$, $(4,0)$ and $(0,4)$ is strongly invariant under F ($F(\varDelta)=\varDelta$) while if we set $y=0$ the dynamics on $[0,4]$ is that of the full parabola $x(4-x)$ and on the other sides of $\varDelta$ the dynamics is trivial.

Outside $\varDelta$ the dynamics is easy to follow. All points except some periodic ones (if they exist) go to infinity and there is no connection between the dynamics outside and inside $\varDelta$. In fact all preimages of all points in $\mathrm{Int}(\varDelta)$ are also in $\mathrm{Int}(\varDelta)$.

The system (2) is the result of some reductions made by Sharkovskiĭ of a system given by Y. Avishai and D. Berend [1] linked with the dynamics of the Schrödinger equation.

G. Świrszcz [10] answers some of the questions posed by Sharkovskiĭ for (2). In particular, he constructs an absolutely continuous σ-finite invariant measure for F and proves that the preimages of the side $I=\varDelta\cap\{y=0\}$ form a dense subset of $\varDelta$ and there is another dense set $\varLambda$ consisting of points whose trajectories approach the interval I but are not attracted by I.

The aim of this paper is to continue Sharkovskiĭ's syllabus for (2) by proving the existence of a unique invariant curve joining the points $(1,2)$ and $(0,0)$ which is simultaneously the unstable manifold of $(1,2)$ and the stable one of $(0,0)$. This curve is of spiral type and strongly invariant. The key point in the proof is to decompose Δ into what we call ω-regions. The decomposition additionally allows us to prove that if a periodic trajectory exists then it must have a part on Δ_l and another part on Δ_i where $\Delta_i = \omega_0$ and Δ_l is the rest of $\mathrm{Int}(\Delta)$ (for definitions see the next section).

Using algebraic systems of non-linear equations, it is immediate that there are no periodic points of period 2 or 3. Using algebraic properties of the resultant associated to such systems it can be proved that

$$\left\{\left(2-\sqrt{2},\frac{1}{2}\right),\left(1+\frac{1}{\sqrt{2}},1-\frac{1}{\sqrt{2}}\right),\left(2+\sqrt{2},\frac{1}{2}\right),\left(1-\frac{1}{\sqrt{2}},1+\frac{1}{\sqrt{2}}\right)\right\}$$

is the unique cycle of period 4 in $\mathrm{Int}(\Delta)$. This is the first time in the literature on quadratic systems where a periodic trajectory of period 4 is explicitly obtained. For example in [11] some numerical work is needed to understand the genealogies of periodic points of periods less than or equal to 5 in the two-parameter family

$$F_{(a,b)}(x,y) = (y, ay + b - x^2).$$

In other cases periodic points have only been claimed to exist (see for instance [8] for the two-dimensional logistic family of maps $F_a(x,y) = (y, ay(1-x))$).

Additionally it is proved (using the same procedure for the conjugate system, i.e., $(x,y) \mapsto (y|x|, x^2-2)$) that there also exists a unique periodic point of period 5 in $\mathrm{Int}(D)$ where D is the image of Δ under the conjugacy map (i.e. $C : \Delta \to D$ given by $(x,y) \mapsto ((x-2)\sqrt{x(4-x-y)}, x(4-x-y)-2)$):

$$(x,y) = (-0.7873282213706032, -1.5245690977552053).$$

In this case, to give explicitly all the points of the trajectory is not possible because their coordinates are roots of polynomials of degree 10. We do it in an implicit way.

We have also heard from P. Maličký [6] that it could be possible to prove that there are periodic points of periods greater than 5 (concretely, of periods 6, 7 and 8) for the system (2) defined on the whole space $\mathbb{R}^2$.

2. Notation and preliminary results. Given $(x,y) \in \Delta$, we define $F^n(x,y) = F(F^{n-1}(x,y))$ and F^0 as the identity map on Δ. The sequence $\{F^n(x,y)\}_{n=0}^{\infty}$ is called the *trajectory* of (x,y) under the action of the system (Δ, F). To know the dynamics of the system (Δ, F) is to have information on the asymptotic behavior of the trajectories of all points of Δ under F. Obviously, to reach this completely is very difficult and in most cases almost impossible, but there exist some classes of points such that from their study some information about the global behavior of the system is obtained. The most important of them is the class of periodic points.

DEFINITION 1. A point $(x,y) \in \varDelta$ is called *periodic* for F if there exists a positive integer m such that $F^m(x,y) = (x,y)$. The smallest such $m = m_{(x,y)}$ is called the *period* of x. When $m = 1$ we have *fixed points*. The trajectory of a periodic point is called a *periodic trajectory*.

DEFINITION 2. A point $(x,y) \in \varDelta$ is called *homoclinic* to a periodic point (p_1,p_2) of F if the following conditions are satisfied:

(1) $(x,y) \neq (p_1,p_2)$,
(2) for every neighborhood $\mathcal{U}$ of (p_1,p_2) there exists a positive integer k such that $(x,y) \in F^{m\cdot k}(\mathcal{U})$ where m is the period of (p_1,p_2),
(3) $F^{m\cdot l}(x,y) = (p_1,p_2)$ for some positive integer l.

The trajectory of a homoclinic point is called a *homoclinic trajectory*.

DEFINITION 3. Let (Y,G) be a discrete dynamical system. The systems $(\varDelta,F)$ and (Y,G) are called *topologically conjugate* (respectively *topologically semi-conjugate*) if there exists a homeomorphism (respectively an onto continuous map) $C : \varDelta \to Y$ such that $C \circ F(x,y) = G \circ C(x,y)$ for every $(x,y) \in \varDelta$.

Now, after the introduction of the main notions that we need, let us state some properties of the system $(\varDelta,F)$. First of all, as mentioned in the previous section, the system F restricted to I is the full parabola (i.e., $F(x,0) = (x(4-x),0)$). The dynamics of this unimodal map is well known (see for instance [2]).

The system $(\varDelta,F)$ has three fixed points: $(0,0)$, $(3,0)$ and $(1,2)$. The second and third are repellors (see [10]).

It is interesting to split the triangle $\varDelta$ into two sets,

$$\varDelta = \varDelta_l \cup \varDelta_r$$

where $\varDelta_l = \{(x,y) \in \varDelta : 0 \le x \le 2\}$ and $\varDelta_r = \{(x,y) \in \varDelta : 2 < x \le 4\}$. Since each point from $\mathrm{Int}(\varDelta)$ has two preimages, one in $\mathrm{Int}(\varDelta_r)$ and the other in $\mathrm{Int}(\varDelta_l)$, the map F is not invertible but F restricted to $\mathrm{Int}(\varDelta_l)$ or to $\mathrm{Int}(\varDelta_r)$ is. The inverse maps of these restrictions are given by:

$$F_l^{-1} : \mathrm{Int}(\varDelta_l) \to \mathrm{Int}(\varDelta_l), \qquad (x,y) \mapsto \left(2 - \sqrt{4-x-y}, \frac{y}{2-\sqrt{4-x-y}}\right),$$

$$F_r^{-1} : \mathrm{Int}(\varDelta_r) \to \mathrm{Int}(\varDelta_r), \qquad (x,y) \mapsto \left(2 + \sqrt{4-x-y}, \frac{y}{2+\sqrt{4-x-y}}\right).$$

From another point of view, it is easy to see that $\varDelta$ can be decomposed into five pairwise disjoint sets:

$$\Delta = \Big(\bigcup_{n=0}^{\infty} F^{-n}(0,0)\Big) \cup \Big(\bigcup_{n=0}^{\infty} F^{-n}(1,2)\Big) \cup \Big(\bigcup_{n=0}^{\infty} F^{-n}(3,0)\Big)$$
$$\cup \Big(I \setminus \Big(\bigcup_{n=0}^{\infty} F^{-n}(3,0) \cup \bigcup_{n=0}^{\infty} F^{-n}(0,0)\Big)\Big) \cup R.$$

From this decomposition it follows that if there are periodic points then they must be contained in the set $R \subset \mathrm{Int}(\Delta)$.

3. Construction of an invariant curve. In this section we construct an invariant curve joining the points $(0,0)$ and $(1,2)$. This curve is piecewise $\mathcal{C}^1$ and it is a homoclinic trajectory to $(0,0)$. It can be seen as an unstable manifold for $(0,0)$ and a stable one for $(1,2)$. This curve contains important information on the dynamics of the system.

Let us construct a set S with the following procedure:

$$\begin{aligned} K_0 &= \{(0,0)\},\\ K_1 &= \{(0,0)\} \times [0,4],\\ K_2 &= \{(x,y) \in \Delta : 0 \le x \le 2 \text{ and } x+y=4\},\\ K_{n+3} &= \{(x,y) \in \Delta_l : F(x,y) \in K_{n+2}\} \quad \text{for any } n \ge 0. \end{aligned}$$

It is easy to see that

$$F(K_{n+1}) = K_n \quad \text{for } n \ge 0, \tag{3}$$
$$F^n(K_n) = (0,0) \quad \text{for } n \ge 0. \tag{4}$$

Now set

$$S = \bigcup_{n=0}^{\infty} K_n. \tag{5}$$

By the injectivity of F on Δ_l, S is a piecewise $\mathcal{C}^1$ curve with vertices V_i ($i \in \{0,1,\dots\}$), where

$$\begin{aligned} V_0 &= (0,0),\\ K_1 \cap K_2 &= \{V_1\} = \{(0,4)\},\\ K_2 \cap K_3 &= \{V_2\} = \{(2,2)\},\\ K_3 \cap K_4 &= \{V_3\} = \{(2,1)\},\\ K_4 \cap K_5 &= \{V_4\} = \{(1,1)\},\\ &\vdots\\ K_{n+5} \cap K_{n+6} &= \{V_{n+5}\} = \{F_l^{-(n+1)}(1,1)\} \quad \text{for } n \ge 0. \end{aligned}$$

After proving the properties of S given in Lemmas 4–6 we will see that S is a piecewise $\mathcal{C}^1$ curve of spiral type. This can be done using the conjugacy given by the homeomorphism C and a map f conjugate to F (originally stated in [10]).

Let

$$\begin{aligned} D &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 4\}, \\ D_l &= \{(x,y) \in \mathbb{R}^2 : -2 < x < 0 \text{ and } x^2 + y^2 < 4\}, \\ D_r &= \{(x,y) \in \mathbb{R}^2 : 0 < x < 2 \text{ and } x^2 + y^2 < 4\}. \end{aligned}$$

Define the maps:

$$\begin{aligned} C &: \Delta \to D, \quad (x,y) \mapsto ((x-2)\sqrt{x(4-x-y)}, x(4-x-y)-2), \\ f &: D \to D, \quad (x,y) \mapsto (y|x|, x^2 - 2). \end{aligned}$$

Again, the map f is not invertible but f restricted to $\mathrm{Int}(D_l)$ or to $\mathrm{Int}(D_r)$ is and the inverse maps are

$$\begin{aligned} f_l^{-1} &: \mathrm{Int}(D_l) \to \mathrm{Int}(D_l), \quad (x,y) \mapsto (-\sqrt{2+y}, x/\sqrt{2+y}), \\ f_r^{-1} &: \mathrm{Int}(D_r) \to \mathrm{Int}(D_r), \quad (x,y) \mapsto (\sqrt{2+y}, x/\sqrt{2+y}). \end{aligned}$$

In fact the map C is a semiconjugacy, but on the interiors of the pieces it is one-to-one, hence a conjugacy. The conjugate images of V_n in D are (i.e., $C(V_n) = W_n$):

$$\begin{aligned} W_0 &= (0,-2), \\ W_1 &= (0,-2), \\ W_2 &= (0,-2), \\ W_3 &= (-\sqrt{2}, 0), \\ W_4 &= f_l^{-1}(W_3) = (-\sqrt{2}, -1), \\ W_5 &= f_l^{-2}(W_3) = (-1, -\sqrt{2}), \\ W_6 &= f_l^{-3}(W_3) = (-\sqrt{2-\sqrt{2}}, -1/\sqrt{2-\sqrt{2}}), \\ W_7 &= f_l^{-4}(W_3) = (-\sqrt{2-A_0^{-1}}, -A_0/\sqrt{2-A_0^{-1}}), \\ &\vdots \\ W_{n+7} &= f_l^{-n-4}(W_3) = (-A_{n+1}, -A_n/A_{n+1}) \quad \text{for } n \ge 0, \end{aligned}$$

where

$$\begin{aligned} A_0 &= \sqrt{2-\sqrt{2}}, \\ A_1 &= \sqrt{2 - A_0^{-1}}, \\ A_2 &= \sqrt{2 - A_0 A_1^{-1}}, \\ A_3 &= \sqrt{2 - A_1 A_2^{-1}}, \\ &\vdots \\ A_{n+2} &= \sqrt{2 - A_n A_{n+1}^{-1}} \quad \text{for } n \ge 0. \end{aligned}$$

LEMMA 4. $\lim_{n\to\infty} A_n = 1$.

Proof. We can prove by induction that each A_n $(n \geq 0)$ satisfies

$$b^-(n) < A_n < b^+(n) \quad \text{where}$$

$$b^-(n) = \sqrt{1 - \frac{1}{n+1}} \quad \text{and} \quad b^+(n) = \sqrt{1 + \frac{1}{n+1}}.$$

The assertion follows from the fact that $\lim_{n\to\infty} b^-(n) = \lim_{n\to\infty} b^+(n) = 1$. We prove by induction the right inequality (the left one can be shown similarly).

Step 1. We can see that $b^+(0) > A_0$, $b^+(1) > A_1$ etc. We can assume that $b^+(n) > A_n$ for each $0 \leq n \leq k$.

Step 2. It is easy to see that

$$b^+(k)^2 < \frac{(k+2)^2}{k(k+1)} = \frac{k+1}{k}\,\frac{(k+2)^2}{(k+1)^2}$$

and hence

$$A_k^2 < \frac{k+1}{k}\,\frac{(k+2)^2}{(k+1)^2}.$$

So

$$A_k\,\frac{k+1}{k+2} < A_{k-1} < b^+(k-1).$$

We have

$$1 + \frac{1}{k+2} > 2 - A_{k-1}A_k^{-1}$$

so $b^+(k+1)^2 > A_{k+1}^2$. ■

LEMMA 5. $\lim_{n\to\infty} V_n = (1,2)$.

Proof. This follows from Lemma 1 (i.e., $\lim_{n\to\infty} W_n = (-1,-1)$) and the fact that $C(1,2) = (-1,-1)$. ■

LEMMA 6. *Let $i = j + n$, where $n > 0$. Then $K_i \cap K_j = \emptyset$.*

Proof. Let $x \in K_i \cap K_j$. If $F^n(x) = x$ then x is periodic, which is a contradiction (each point of S is eventually fixed, which follows from (4)). So there is $y \in K_i$ such that $x \neq y$ and $F^n(y) = x$. Then $F^i(y) \neq (0,0)$, in contradiction with $F^i(K_i) = (0,0)$ for each $i \geq 0$, which follows from the construction of S. ■

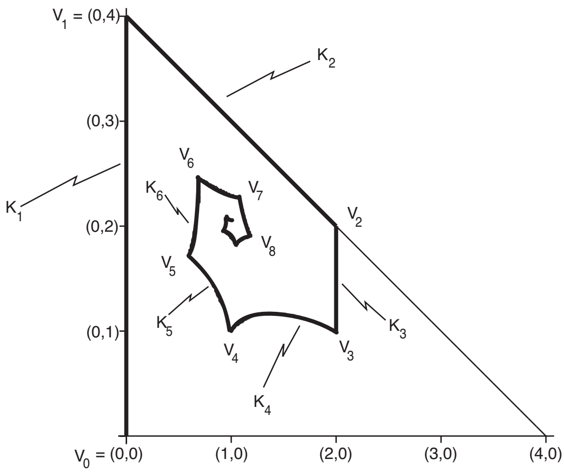
THEOREM 7. *There is a curve S in Δ with the following properties*:

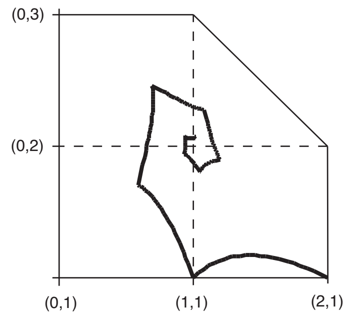
(i) *It is of spiral type with focus at the point* $(1,2)$.

(ii) *It is strongly invariant. Moreover, each point of S is eventually fixed* (*i.e., for each $(x,y) \in S$ there is $n \geq 0$ such that $F^n(x,y) = (0,0)$*).

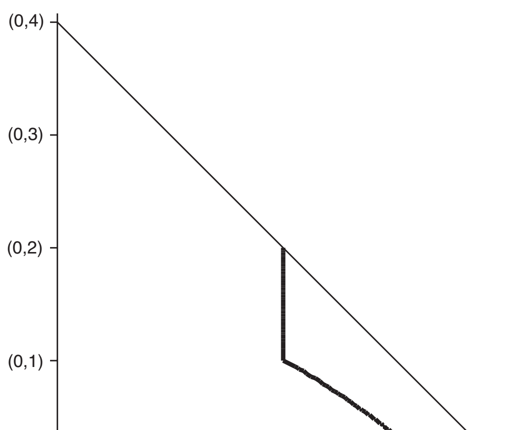
(iii) *There is a unique curve satisfying* (i) *and* (ii).

Proof. The fact that the curve S is of spiral type follows from Lemmas 5 and 6. Strong invariance follows from (5) and from the fact that each point is eventually fixed by (4). The proof of (iii) will be given in the next section. ■





REMARK 8. It is interesting to observe that if in the previous construction of the curve S we replace the map F_l^{-1} by F_r^{-1} (now the preimages are in Δ_r) we obtain a strongly invariant curve which is *not* of spiral type but is also strongly invariant and converges to the repulsive fixed point $(3,0)$ (the proof of the convergence is similar to the proof of Lemmas 4 and 5).



4. Decomposition of the triangle. To understand the periodic structure of the map F we decompose the triangle Δ into infinitely many pairwise disjoint regions. These regions will be defined in such a way that it will be obvious where the cycles should be.

We use the same notation as in the previous section. Let us start with the following modification. Let

$$L_0 = \{2\} \times [0,2], \quad L_{i+1} = \{(x,y) \in \Delta_l : F(x,y) \in L_i\} \quad \text{for } i \geq 0.$$

(Note that $K_{n+3} \subset L_n$ for $n \geq 0$.) Put

$$X_0 = (2,0), \quad X_{n+1} = F_l^{-n}(X_0) \quad \text{for } n \geq 0.$$

Thus, the points X_n and V_{n+2} are on the boundary of L_n for $n \geq 0$. The sequence $\{X_i\}_{i=0}^{\infty}$ converges to $(0,0)$ and the point X_0 is homoclinic to $(0,0)$ (see, e.g., [2] and Definition 2).

Let $\omega_0 = \mathrm{Int}(\Delta_r)$ and ω_{n+1} be the interior of the region in Δ_l bounded by L_n, L_{n+1} and $[X_{n+1}, X_n]$, for $n \geq 0$.

THEOREM 9. *Let ω_n be the regions in Δ defined above for all $n \geq 0$. Then*

(i) $F(\omega_{n+1}) = \omega_n$ *for* $n \geq 0$,
(ii) $F(\omega_0) = \mathrm{Int}(\Delta)$,
(iii) $\bigcup_{n=0}^{\infty} \overline{\omega}_n = \Delta$ *and* $\omega_n \cap \omega_{n+i} = \emptyset$ *for* $n \geq 0$, $i > 0$.

Proof. Statements (i) and (ii) are obtained directly from the construction.

To prove (iii) it suffices to use the property of the map F that each point from $\mathrm{Int}(\Delta)$ has two preimages, one in $\mathrm{Int}(\Delta_r)$ and the other in $\mathrm{Int}(\Delta_l)$. Assuming the existence of an invariant region having empty intersection with each ω_n leads to a contradiction with this property. ∎

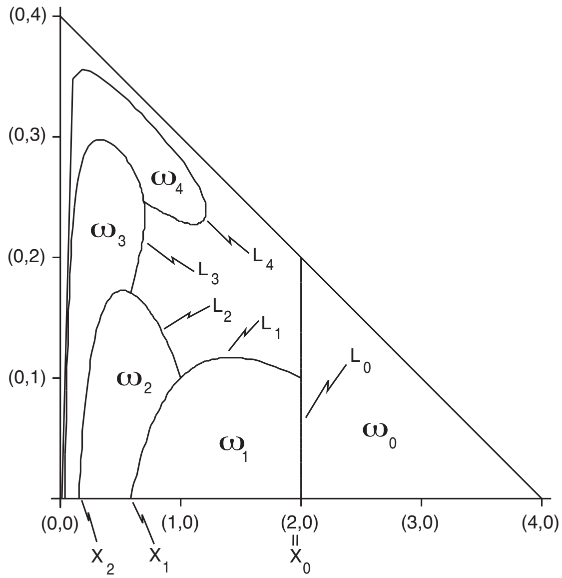
Proof of Theorem 7(iii). We have constructed an invariant curve S by adding pieces of the boundaries of the regions ω_n in Δ_r. The trajectory of a point $(x,y) \in S$ turns left around the point $(1,2)$ under the map F_l^{-1} and so the distance between the points $F^{-n}(x,y)$ and $(1,2)$ tends to zero as $n \to \infty$ by Lemma 5.

The edges of the curve S compose a homoclinic trajectory of the point $(1,2)$, i.e., $\lim_{n\to\infty} F^{-n}(V_n) = (1,2)$. In fact S is the unstable manifold of $(1,2)$ and the stable manifold of $(0,0)$ since for any point $(x,y) \in S$ we have $\lim_{n\to\infty} F^{n}(x,y) = (0,0)$ and $\lim_{n\to\infty} F^{-n}(x,y) = (1,2)$. In this situation S connects dynamically the points $(0,0)$ and $(1,2)$.

Also S is the unique curve with properties (i) and (ii) of Theorem 7. To see this, let S' be another curve different from S connecting the vertices V_i and let $(x,y) \in S'$ be not a vertex point. Then (x,y) would belong to some region ω_k and as a consequence $F^n(x,y)$ would not converge to $(0,0)$, nor $F^{-n}(x,y)$ to $(1,2)$, as $n \to \infty$. Therefore S is the unique curve having properties (i) and (ii) from Theorem 7. ∎

Corollary 10. *$F_l^{-1}|_{\Delta_l}$ has the point $(1,2)$ as its global attractor.*

From Theorem 9 it is straightforward that there are no cycles in $\mathrm{Int}(\Delta_l)$ and analogously it can also be proved that there are no cycles in $\mathrm{Int}(\Delta_r)$. We conclude that if there is a cycle in $\mathrm{Int}(\Delta)$ then it must have non-empty intersection with both $\mathrm{Int}(\Delta_l)$ and $\mathrm{Int}(\Delta_r)$.



5. Periodic trajectories of (Δ, F). We are interested in the periodic points in $\mathrm{Int}(\Delta)$.

The fixed points satisfy $F(x,y) = (x,y)$. It is easy to check that the unique fixed point in $\mathrm{Int}(\Delta)$ is $(1,2)$.

The 2-periodic points satisfy $F^2(x,y) = (x,y)$, or equivalently

$$(-(-2+x)^2 x(-4+x+y), -x^2 y(-4+x+y)) = (x,y).$$

It is not difficult to see that this system has no solution in $\mathrm{Int}(\Delta)$.

The 3-periodic points satisfy $F^3(x,y) = (x,y)$, or equivalently

$$\begin{aligned} -(-2+x)^2 x(-4+x+y)(2-4x+x^2+xy)^2 &= x, \\ (-2+x)^2 x^3 y(-4+x+y)^2 &= y. \end{aligned}$$

A straightforward computation shows that this system has no solution in $\mathrm{Int}(\Delta)$.

In short, F has no periodic trajectory of periods 2 and 3 in $\mathrm{Int}(\Delta)$. But, as proved in Theorem 11, it has a unique periodic trajectory of period 4 in

$\mathrm{Int}(\Delta)$ which is obtained explicitly using some properties of the resultant. This procedure allows us to prove the existence of a periodic point of period 5; its trajectory is given in an implicit way. Before proving this result we need to recall some properties of the resultant.

Let a_i, $i \in \{1, \ldots, n\}$, and b_j, $j \in \{1, \ldots, m\}$, be the roots of the polynomials $P(x)$ and $Q(x)$, respectively, both with leading coefficient 1. The *resultant* of P and Q, $\mathrm{Res}[P, Q]$, is the product of all the differences $a_i - b_j$, $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$. In order to see how to compute $\mathrm{Res}[P, Q]$, see for instance [5] and [7]. The main property of the resultant is that if P and Q have a common root then necessarily $\mathrm{Res}[P, Q] = 0$.

Consider now polynomials in two variables, say $P(U, V)$ and $Q(U, V)$. These polynomials can be seen as polynomials in $X = U$ with polynomial coefficients in $Y = V$. Then the resultant with respect to U, $\mathrm{Res}[P, Q, U]$, is a polynomial in Y with the following property. If $P(U, V)$ and $Q(U, V)$ have a common root (U_0, V_0), then

$$\mathrm{Res}[P, Q, U](V_0) = 0,$$

and similarly for the variable V. In particular, if we compute all the roots of the following two polynomials in one variable $p(U) = \mathrm{Res}[P, Q, V]$, $q(V) = \mathrm{Res}[P, Q, U]$, and denote them by U_0 and V_0, respectively, then we can check when (U_0, V_0) is a solution of the system $P(U, V) = Q(U, V) = 0$.

In short, using the resultant we reduce the problem of finding solutions of a polynomial system in two variables to finding roots of two polynomials in one variable.

THEOREM 11. *The map F has a unique periodic trajectory of period* 4 *in* $\mathrm{Int}(\Delta)$. *This trajectory is*

$$\left\{\left(2-\sqrt{2}, \frac{1}{2}\right), \left(1+\frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}\right), \left(2+\sqrt{2}, \frac{1}{2}\right), \left(1-\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right)\right\}.$$

Proof. We must find the points (x, y) in $\mathrm{Int}(\Delta)$ such that $F^4(x, y) = (x, y)$, or equivalently

$$(f(x, y), g(x, y)) = (0, 0), \tag{6}$$

where

$$\begin{aligned} f(x, y) &= (-2+x)^2 x(-4+x+y)(2-4x+x^2+xy)^2 \\ &\quad \times (2-16x+20x^2-8x^3+x^4+4xy-4x^2y+x^3y)^2 + x, \\ g(x, y) &= (-2+x)^4 x^4 y(-4+x+y)^3 (2-4x+x^2+xy)^2 + y. \end{aligned}$$

Therefore to find the 4-periodic trajectories of the map F in $\mathrm{Int}(\Delta)$ is equivalent to finding the solutions of the polynomial system (6) in $\mathrm{Int}(\Delta)$.

Now the resultant $\mathrm{Res}[f, g, x]$ is the polynomial in y given by

$$q(y) = -8589934592(-2+y)y^{20}(-1+2y)^2(1-4y+2y^2).$$

Its roots without taking into account their multiplicity are 0, 2, 1/2, $1+1/\sqrt{2}$ and $1-1/\sqrt{2}$. Hence, a fixed point (x, y) of F^4 (i.e. a periodic point of F with period 1, 2 or 4) must have its y coordinate equal to one of these five possible values.

We are not interested in the fixed points (x, y) of F^4 with $v = 0$ because all these points are on the boundary of the triangle Δ.

Now we study the fixed points (x, y) of F^4 with $v = 2$. From easy computations we get

$$f(x, 2) = (1-x)xF(x), \qquad g(x, 2) = 2(x-1)G(x),$$

where

$$\begin{aligned} F(x) = (&127 - 1345x + 6559x^2 - 19505x^3 + 39311x^4 \\ &- 56801x^5 + 60807x^6 - 49101x^7 + 30099x^8 - 13949x^9 \\ &+ 4811x^{10} - 1197x^{11} + 203x^{12} - 21x^{13} + x^{14}, \\ G(x) = &-1 - x - x^2 - x^3 + 511x^4 - 2305x^5 + 4991x^6 \\ &- 6721x^7 + 6175x^8 - 4017x^9 + 1863x^{10} - 605x^{11} \\ &+ 131x^{12} - 17x^{13} + x^{14}. \end{aligned}$$

The common factor $x - 1$ of $f(x, 2)$ and $g(x, 2)$ provides the fixed point $(1, 2)$ of F, which is also fixed for F^4. Since the factor x of $f(x, 2)$ does not appear in $g(x, 2)$, the point $(0, 2)$ is not a solution of system (6). Finally, since $\mathrm{Res}[F(x), G(x), x] = 154618822656 \neq 0$, there are no additional periodic points (x, y) with $y = 2$.

We study the fixed points (x, y) of F^4 with $y = 1/2$. From easy computations we get

$$\begin{aligned} f(x, 1/2) &= -\frac{1}{32}\, x(2 - 4x + x^2)F(x), \\ g(x, 1/2) &= -\frac{1}{64}(2 - 4x + x^2)G(x), \end{aligned}$$

where

$$\begin{aligned} F(x) = &-3568 + 60192x - 390696x^2 + 1304640x^3 - 2617156x^4 \\ &+ 3435080x^5 - 3096854x^6 + 1968868x^7 - 891687x^8 \\ &+ 286314x^9 - 63736x^{10} + 9360x^{11} - 816x^{12} + 32x^{13}, \\ G(x) = &\ 16 + 32x + 56x^2 + 96x^3 - 43740x^4 + 191576x^5 \\ &- 364250x^6 + 394624x^7 - 269087x^8 + 120010x^9 \\ &- 35064x^{10} + 6480x^{11} - 688x^{12} + 32x^{13}. \end{aligned}$$

Since the factor x of $f(x,1/2)$ does not appear in $g(x,1/2)$, the point $(0,1/2)$ is not a solution of system (6). Moreover, since $\mathrm{Res}[F(x),G(x),x]$ is

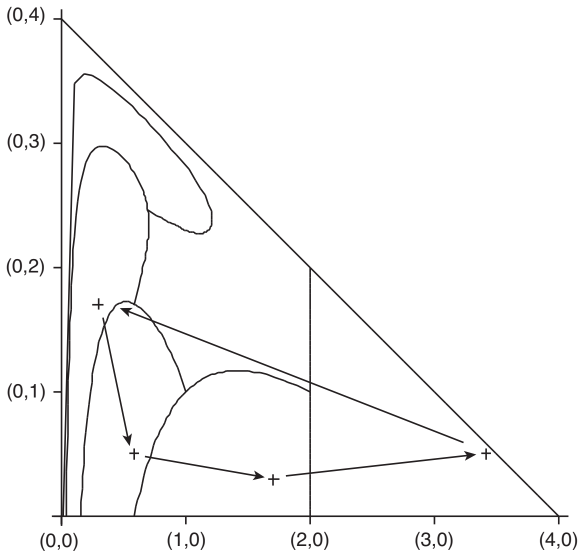
$$-137015778499772148581595453067151533092743675904 \neq 0,$$

there are no additional fixed points (x,y) of F^4 with $y=1/2$ coming from the factors $F(x)$ and $G(x)$. Finally, the common factor $2-4x+x^2$ to $F(x)$ and $G(x)$ provides two solutions to system (6), namely

$$\left(2-\sqrt{2},\frac{1}{2}\right) \quad \text{and} \quad \left(2+\sqrt{2},\frac{1}{2}\right).$$

So these two points are fixed points of F^4. It is easy to check that they belong to the periodic trajectory described in the statement of the proposition.

Lastly, if we study in a similar way the fixed points (x,y) of F^4 with $y=1\pm 1/\sqrt{2}$, we only get the same 4-periodic trajectory. ■



Concerning the trajectories of (Δ,F) of period 5, arguments similar to those for Theorem 11 can be used for the conjugate system on the disc $f: D\to D$ given by $(x,y)\mapsto (y|x|,x^2-2)$ to simplify calculations. The result is the following.

THEOREM 12. *The map f in* $\mathrm{Int}(D)$ *has a unique periodic trajectory of period* 5 *associated to the point*

$$(x,y)=(-0.7873282213706032,-1.5245690977552053).$$

The five second coordinates of the points of the trajectory are roots of the *following polynomial*:

$$1-98y-461y^2-560y^3+353y^4+1255y^5+903y^6+144y^7-76y^8-21y^9+y^{10}.$$

Analogously, the first coordinates of these points are roots of the following two polynomials:

$$1+20x-7x^2-74x^3+5x^4+89x^5+9x^6-40x^7-8x^8+5x^9+x^{10},$$
$$1-20x-7x^2+74x^3+5x^4-89x^5+9x^6+40x^7-8x^8-5x^9+x^{10}.$$

Obviously, for computational reasons it is not possible to analyze the existence of periodic trajectories of periods $n > 5$ using this procedure. There are no reasons for the non-existence of such trajectories. On the other hand, if we consider the system $(\mathbb{R}^2, F)$ other new periodic trajectories could appear and the uniqueness of the 4,5-periodic trajectories could disappear.

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