FUNDAMENTA
MATHEMATICAE
214 (2011)

Some (non-)elimination results for curves in geometric structures

by

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Abstract. We show that the first order structure whose underlying universe is $\mathbb{C}$ and whose basic relations are all algebraic subsets of $\mathbb{C}^2$ does not have quantifier elimination. Since an algebraic subset of $\mathbb{C}^2$ is either of dimension ≤ 1 or has a complement of dimension ≤ 1, one can restate the former result as a failure of quantifier elimination for planar complex algebraic curves. We then prove that removing the planarity hypothesis suffices to recover quantifier elimination: the structure with the universe $\mathbb{C}$ and a predicate for each algebraic subset of $\mathbb{C}^n$ of dimension ≤ 1 has quantifier elimination.

1. Introduction. The theory of structures generated by binary relations definable in an o-minimal structure was studied in [6]. In particular, Theorem 3.2 of [6] implies the following proposition:

PROPOSITION 1.1. *Let $\mathcal{M}$ be an o-minimal structure with universe M, and let $\mathcal{B}(\mathcal{M})$ be the first order structure whose underlying set is M and whose basic relations are all subsets of M^2 which are $\emptyset$-definable in $\mathcal{M}$. The theory of $\mathcal{B}(\mathcal{M})$ has quantifier elimination.*

As an immediate consequence of quantifier elimination, the structure $\mathcal{B}(\mathcal{M})$ has trivial geometry (Lemma 1.8 in [6]).

Also, partially motivated by a restricted version of Zil'ber's Conjecture, various reducts of the field of complex numbers have been investigated (see for example [3, 5, 8]), and it is natural to ask whether a complex analogue of the previous proposition holds: does the structure $\mathcal{B}(\overline{\mathbb{C}}_{\mathbb{C}})$, obtained by equipping the universe $\mathbb{C}$ with a predicate for each complex algebraic constructible subset of $\mathbb{C}^2$, also eliminates quantifiers? Note that arity two is the only arity where this question occurs: for arity three and more, we recover

2010 *Mathematics Subject Classification*: 03C10, 03C60, 14H50.
Key words and phrases: quantifier elimination, definable set, geometric structure, constructible curve, definable closure.

DOI: 10.4064/fm214-2-5

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the full structure of a field on $\mathbb{C}$ and thus get quantifier elimination; as for arity zero and one, elimination of quantifiers is clear.

Section 2 will answer this question negatively: it provides a counterexample to the elimination of quantifiers for $\mathcal{B}(\overline{\mathbb{C}}_{\mathbb{C}})$. Still, one may ask what sets are definable in $\mathcal{B}(\overline{\mathbb{C}}_{\mathbb{C}})$, and what is its (combinatorial) geometry. These questions are answered in Section 3. Let $\mathcal{C}(\overline{\mathbb{C}}_{\mathbb{C}})$ denote the first order structure whose underlying universe is $\mathbb{C}$ and whose basic relations are all the subsets of cartesian products of $\mathbb{C}$, definable in the field structure, of dimension ≤ 1 (the constructible curves). We first note that $\mathcal{B}(\overline{\mathbb{C}}_{\mathbb{C}})$ is a reduct (in the sense of definability) of $\mathcal{C}(\overline{\mathbb{C}}_{\mathbb{C}})$. We then show that $\mathcal{C}(\overline{\mathbb{C}}_{\mathbb{C}})$ eliminates its quantifiers, and deduce that it has a trivial geometry. In particular $\mathcal{B}(\overline{\mathbb{C}}_{\mathbb{C}})$ is a proper reduct (in the sense of definability) of the field of complex numbers. The latter results are proven in the more general setting of *geometric structures*. In Section 4 we discuss the role of algebraic closure versus definable closure in this quantifier elimination. In Section 5, we generalize the construction of Example 2.1 to higher arity: for any fixed natural number n we consider the structure $\mathcal{C}_n(\overline{\mathbb{C}}_{\mathbb{C}})$ on $\mathbb{C}$ whose basic relations are the subsets of $\mathbb{C}^n$, definable in the field of complex numbers, of dimension ≤ 1. We show that none of the $\mathcal{C}_n(\overline{\mathbb{C}}_{\mathbb{C}})$ has quantifier elimination (Example 2.1 shows this fact for $n = 2$). Finally in Section 6, we discuss which sets are definable in those structures (allowing quantifications).

Remark. After the paper had been submitted we discovered that a result similar to our Theorem 3.5 was also proved recently by M. C. Laskowski [2].

2. Non-elimination for binary relations. Recall from the Introduction that $\mathcal{B}(\overline{\mathbb{C}}_{\mathbb{C}})$ denotes the first order structure whose universe is $\mathbb{C}$ and whose basic relations are all subsets of $\mathbb{C}^2$ which are $\mathbb{C}$-definable in $\overline{\mathbb{C}}$, the field of complex numbers. We show that $\mathcal{B}(\overline{\mathbb{C}}_{\mathbb{C}})$ does not eliminate its quantifiers:

Example 2.1. Let R be the binary definable relation

$$R(y,s) \iff \exists z\ (z \neq y \wedge y^4 + y = z^4 + z \wedge s = y + z)$$

and consider the ternary relation

$$T(s_1,s_2,s_3) \iff \exists y\ (R(y,s_1) \wedge R(y,s_2) \wedge R(y,s_3)).$$

The subset of $\mathbb{C}^3$ defined by the relation T is definable in $\mathcal{B}(\overline{\mathbb{C}}_{\mathbb{C}})$ but is *not quantifier-free definable* in $\mathcal{B}(\overline{\mathbb{C}}_{\mathbb{C}})$.

Proof. Let $\mathcal{M}$ be a proper elementary extension of $\overline{\mathbb{C}}$ and M its universe. We fix (any) $a \in M \setminus \mathbb{C}$, and let $\{\zeta_1,\zeta_2,\zeta_3,\zeta_4\}$ be the four roots of the polynomial $X^4 + X + a$ in M. Note that a is transcendental over $\mathbb{C}$.

We first claim that $\mathrm{Aut}(\mathcal{M}/\mathbb{C})$ (the group of automorphisms of $\mathcal{M}$ fixing $\mathbb{C}$) acts totally transitively on the set $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ (which really has four distinct elements). Since $\mathcal{M}$ is an algebraically closed field, every element of $\mathrm{Gal}\big(X^4+X+a\,/\,\mathbb{C}(a)\big)$ extends to an automorphism of $\mathcal{M}$, hence it is sufficient to show that $\mathrm{Gal}\big(X^4+X+a\,/\,\mathbb{C}(a)\big)$ is the symmetric group $\mathfrak{S}_4$.

Galois theory (see for instance Theorem 13.4 in [7]) tells us that this happens if and only if both the polynomial $X^4 + X + a$ and its resolvent $X^3 - 4aX - 1$ are irreducible over $\mathbb{C}(a)$ and the discriminant $256a^3 - 27$ is not a square in $\mathbb{C}(a)$.

We show that $X^3 - 4aX - 1$ is irreducible over $\mathbb{C}(a)$. Assume not; then $X^3 - 4aX - 1$ has a root in $\mathbb{C}(a)$, say $\alpha \in \mathbb{C}(a)$. Let $p(X), q(X) \in \mathbb{C}[X]$ be relatively prime polynomials such that $\alpha = p(a)/q(a)$. We have $p^3(a) - 4ap(a)q^2(a) - q^3(a) = 0$, and since a is transcendental over $\mathbb{C}$, the equality $p^3(X) - 4Xp(X)q^2(X) - q^3(X) = 0$ holds in $\mathbb{C}[X]$. If $\gamma \in \mathbb{C}$ is a root of $q(X)$ then it follows from the above equation that $p(\gamma) = 0$. Since p, q are relatively prime, they have no common roots, hence q must be a constant polynomial. But then a would be algebraic over $\mathbb{C}$, a contradiction.

Using the same arguments it is not hard to see that $X^2 - 256a^3 - 27$ is irreducible and that $X^4 + X + a$ has no root in $\mathbb{C}(a)$.

To show that $X^4 + X + a$ is irreducible over $\mathbb{C}(a)$, it remains to prove that it cannot be written as a product of two quadratic polynomials. Assume $X^4+X+a = (X^2+\alpha_1X+\beta_1)(X^2+\alpha_2X+\beta_2)$ with $\alpha_i, \beta_i \in \mathbb{C}(a)$. Expanding the right side we obtain the equations

$$\text{(i) } \alpha_1 + \alpha_2 = 0, \qquad \text{(ii) } \beta_2 + \alpha_1\alpha_2 + \beta_1 = 0,$$
$$\text{(iii) } \alpha_1\beta_2 + \alpha_2\beta_1 = 1, \qquad \text{(iv) } \beta_1\beta_2 = a.$$

Combining (i) with (ii) and (iii), we get

$$\beta_2 + \beta_1 = -\alpha_1^2, \qquad \beta_2 - \beta_1 = 1/\alpha_1,$$

and therefore

$$4\beta_1\beta_2 = (\beta_2 + \beta_1)^2 - (\beta_2 - \beta_1)^2 = \alpha_1^4 - 1/\alpha_1^2.$$

By (iv), we have

$$4a = \alpha_1^4 - 1/\alpha_1^2.$$

If for a contradiction α_1 belonged to $\mathbb{C}(a)$ then $t = \alpha_1^2$ would belong to $\mathbb{C}(a)$. But by the previous equation we have $t^3 - 4at - 1 = 0$; this would contradict the irreducibility over $\mathbb{C}(a)$ of $X^3 - 4aX - 1$ proved earlier.

Thus the group $\mathrm{Gal}\big(X^4 + X + a\,/\,\mathbb{C}(a)\big)$ is $\mathfrak{S}_4$.

Consider the two triplets (s_1, s_2, s_3) and (s'_1, s'_2, s'_3) of elements of M defined by $s_i = \zeta_i + \zeta_4$ for $i = 1, 2, 3$ and $s'_{\tau(1)} = \zeta_{\tau(2)} + \zeta_{\tau(3)}$ for all $\tau \in \mathfrak{S}_3$ (see figure below).

Any two triplets of distinct ζ_m's are $\mathrm{Aut}(\mathcal{M}/\mathbb{C})$-conjugate, and we get the elementary equivalence $(s_i, s_j) \equiv_{\mathbb{C}} (s'_k, s'_l)$ (in the sense of $\overline{\mathbb{C}}$) for any $1 \le i \ne j \le 3$ and any $1 \le k \ne l \le 3$.

In particular the elementary equivalence for $(i,j) = (k,l)$ ensures that if T were quantifier-free definable in $\mathcal{B}(\overline{\mathbb{C}}_{\mathbb{C}})$ then we would have $T(s_1, s_2, s_3)$ if and only if $T(s'_1, s'_2, s'_3)$, so T would be equivalent to a boolean combination of formulæ in $\overline{\mathbb{C}}_{\mathbb{C}}$, each involving only two of the three possible variables (say indexed by (i,j)); such a formula would be satisfied by the corresponding subtuple of (s_i, s_j) if and only if it was satisfied by the subtuple (s'_i, s'_j).

But (s_1, s_2, s_3) does satisfy T whereas we will show that (s'_1, s'_2, s'_3) does not. Suppose for a contradiction that $T(s'_1, s'_2, s'_3)$ holds; then there are $\{\zeta'_1, \zeta'_2, \zeta'_3, \zeta'_4\}$ such that ${\zeta'_i}^4 + \zeta'_i = {\zeta'_j}^4 + \zeta'_j$ and $s'_i = \zeta'_i + \zeta'_4$. Thus

$$-2\zeta_4 = 2(\zeta_1 + \zeta_2 + \zeta_3) = s'_1 + s'_2 + s'_3 = \zeta'_1 + \zeta'_2 + \zeta'_3 + 3\zeta'_4 = 2\zeta'_4$$

and

$$\zeta'_i = s'_i - \zeta'_4 = s'_i + \zeta_4 = \zeta_1 + \zeta_2 + \zeta_3 - \zeta_i + \zeta_4 = -\zeta_i.$$

Therefore

$$-a - 2\zeta_i = \zeta_i^4 - \zeta_i = {\zeta'_i}^4 + \zeta'_i = {\zeta'_j}^4 + \zeta'_j = \zeta_j^4 - \zeta_j = -a - 2\zeta_j$$

for all $i \ne j$; this contradicts the fact that the ζ_i's are distinct. ∎

We get a slightly stronger result than announced: Let $\mathcal{B}(\overline{\mathbb{C}}_{\emptyset})$ denote the first order structure whose universe is $\mathbb{C}$ and whose basic relations are all subsets of $\mathbb{C}^2$ which are *$\emptyset$-definable* in $\overline{\mathbb{C}}$. Then $\mathcal{B}(\overline{\mathbb{C}}_{\emptyset})$ defines subsets of $\mathbb{C}^3$ which are not quantifier-free definable in $\mathcal{B}(\overline{\mathbb{C}}_{\mathbb{C}})$.

3. Elimination for curves. We have seen in the previous section that existential quantifiers can be used to bind variables together and define *essentially* non-binary algebraic relations from binary ones. Still one can ask how complicated a set defined using only binary relations can be. Could it be, for instance, that $\mathcal{B}(\overline{\mathbb{C}}_{\mathbb{C}})$ and the full field structure $\overline{\mathbb{C}}$ are interdefinable? We will show that it is not the case.

First note that each subset of $\mathbb{C}^2$ definable in $\overline{\mathbb{C}}$ (with parameters) is a boolean combination of subsets of $\mathbb{C}^2$ definable in $\overline{\mathbb{C}}$ (with parameters) of *dimension smaller than or equal to* 1 and vice versa (where "dimension" refers to the acl-dimension in the sense of $\overline{\mathbb{C}}$):

FACT 3.1. *Let $X \subseteq \mathbb{C}^2$ be definable in $\overline{\mathbb{C}}$, with parameters. Then either* $\dim X \leq 1$ *or* $\dim(\mathbb{C}^2 \setminus X) \leq 1$.

By Fact 3.1, we can view Example 2.1 as showing that the theory of $\mathbb{C}$ equipped with a predicate for each planar algebraic curve does not have quantifier elimination.

However we will show that if we remove the requirement that the curves are *planar*, the quantifier elimination holds. As a consequence, $\mathcal{B}(\overline{\mathbb{C}}_{\mathbb{C}})$ will be shown to have trivial geometry and thus to be a proper reduct of $\overline{\mathbb{C}}$.

The results of this section hold not only in $\overline{\mathbb{C}}$ but also in the more general setting of *geometric structures*. O-minimal structures, strongly minimal structures (such as algebraically closed fields), p-adic fields or algebraically closed valued fields with a predicate for their valuation ring are all geometric structures.

DEFINITION 3.2. Recall that the structure $\mathcal{M}$ is said to be a *geometric structure* if it satisfies

(1) Exchange Property: $a \in \operatorname{acl}(bC) \setminus \operatorname{acl}(C) \Rightarrow b \in \operatorname{acl}(aC)$;
(2) Uniform Finiteness Property: given a formula ψ, there is an integer k such that for each tuple $\mathbf{a}$ the set $\{b \mid \mathcal{M} \models \psi(b, \mathbf{a})\}$ is either infinite or of size $\leq k$.

Note that this property is a property of the theory of $\mathcal{M}$.

In what follows, we will work in a fixed geometric structure $\mathcal{M}$ and use "dimension" for the acl-dimension of its definable sets: if Φ is a formula (with parameters B) defining such a set X and $\widetilde{\mathcal{M}}$ is a saturated extension of $\mathcal{M}$, the *dimension* of X is the maximal d for which there exists $(a_1, \ldots, a_n)$ satisfying Φ (in $\widetilde{\mathcal{M}}$) and a subtuple $(a_{i_1}, \ldots, a_{i_d})$ of $(a_1, \ldots, a_n)$ of length d such that $a_{i_{j+1}} \notin \operatorname{acl}(\{a_{i_1}, \ldots, a_{i_j}\} \cup B)$. (This quantity is independent of the choice of the formula Φ, the parameters B and the structure $\widetilde{\mathcal{M}}$.)

DEFINITION 3.3. Let $\mathcal{M}$ be a geometric structure with universe M.

A set $X \subseteq M^n$ definable with parameters from $A \subseteq M$, of dimension ≤ 1, will be called an (A-definable) *n-curve*.

An (A-definable) *curve* is an (A-definable) n-curve for some n.

A set $\tilde{C} \subseteq M^m$ is said to be an (A-definable) *cylinder based on an n-curve* if there are indices $1 \leq i_1 < \cdots < i_n \leq m$ and an A-definable n-curve C such that $\tilde{C} = \{(x_1, \ldots, x_m) \in M^m \mid (x_{i_1}, \ldots, x_{i_n}) \in C\}$.

A set is called an (A-definable) *curve-based cylinder* if it is an (A-definable) cylinder whose base is an n-curve for some $n \in \mathbb{N}$.

REMARK 3.4. These definitions reflect the fact that one needs to pay attention to the variables used: the formula $x_1 = x_2$ viewed as a formula in the variables x_1 and x_2 defines a 2-curve but if we add a dummy variable x_3 it defines a cylinder based on a 2-curve, of dimension 2.

Dummy variables and cylinders allow one to think about "boolean combinations of curves" involving different sets of variables, as shown in Example 2.1.

As previously announced, the aim of this section is to prove Theorem 3.5 which easily implies that the structure obtained by equipping $\mathbb{C}$ with a predicate for each algebraic subset of $\mathbb{C}^2$ is a proper reduct of the field structure.

THEOREM 3.5. *Let $\mathcal{M}$ be a geometric structure with universe M. The structure $\mathcal{C}(\mathcal{M}_{\mathrm{acl}(\emptyset)})$ obtained by equipping M with predicates for each* $\mathrm{acl}(\emptyset)$*-definable curve has quantifier elimination.*

Proof. Without loss of generality, we can assume that $\mathcal{M}$ is sufficiently saturated.

By syntactic arguments, we only need to consider formulæ of the form

$$\exists y \bigwedge_{i=1}^{r} C_i(\mathbf{x}^i, y) \wedge \bigwedge_{j=r+1}^{r+s} \neg C_j(\mathbf{x}^j, y) \tag{1}$$

where

- each $\mathbf{x}^k$ denotes an l_k-tuple of variables among $(x_1, \dots, x_n)$, and
- each C_k denotes a formula in the (l_k+1)-subtuple $\mathbf{x}^k y$ of free variables among $(x_1, \dots, x_n, y)$, defining an (l_k+1)-curve,

and prove that they define a boolean combination of $\mathrm{acl}(\emptyset)$-definable curve-based cylinders.

In what follows $C(\mathbf{w}, y)$, $C_i(\mathbf{w}, y)$ and $E(\mathbf{w}, y)$ will denote formulæ with distinguished last variable y, defining $\mathrm{acl}(\emptyset)$-definable $(|\mathbf{w}|+1)$-curves. Similarly $\phi_l(\mathbf{x})$ will denote a formula defining a boolean combination of $\mathrm{acl}(\emptyset)$-definable curve-based cylinders. (Note that the bases of these cylinders may involve different tuples of coordinates among those of $\mathbf{x}$.)

LEMMA 3.6. *Any formula $\chi(\mathbf{x}, y)$ of the form*

$$\bigwedge_{i=1}^{r} C_i(\mathbf{x}^i, y) \tag{2}$$

is equivalent to a disjunction

$$E(\mathbf{x}', y) \vee \bigvee_{l=1}^{L} (y = q_l \wedge \phi_l(\mathbf{x}')) \tag{3}$$

where $\mathbf{x}'$ is the subtuple of $\mathbf{x}$ of all those variables involved in some of *the tuples* $\mathbf{x}^i$ $(i = 1, \ldots, r)$, $q_1, \ldots, q_L$*'s are elements of* $\mathrm{acl}(\emptyset)$, *and* $\mathcal{M} \models E(\mathbf{x}', y) \to \bigwedge_{l=1}^{L} y \neq q_l$.

Proof. Consider $\boldsymbol{\xi}\gamma$ in $M^{|\mathbf{x}|+1}$ such that $\chi(\boldsymbol{\xi}, \gamma)$ holds and let the subtuples $\boldsymbol{\xi}^i$ and $\boldsymbol{\xi}'$ of $\boldsymbol{\xi}$ correspond, respectively, to the subtuples $\mathbf{x}^i$ and $\mathbf{x}'$ of $\mathbf{x}$.

By the Exchange Property, either γ belong to $\mathrm{acl}(\emptyset)$ or each coordinate of $\boldsymbol{\xi}'$ belongs to $\mathrm{acl}(\gamma)$. In the latter case, $\boldsymbol{\xi}'\gamma$ satisfies some formula defining an $\mathrm{acl}(\emptyset)$-definable curve.

Since $\mathcal{M}$ is saturated enough, we deduce by compactness that for some $\mathrm{acl}(\emptyset)$-definable curve $C(\mathbf{x}', y)$ and $q_1, \ldots, q_L \in \mathrm{acl}(\emptyset)$ we have

$$\mathcal{M} \models \chi(\mathbf{x}, y) \to \Big(\bigvee_{i=1}^{L} y = q_i \ \vee C(\mathbf{x}', y)\Big).$$

We can take

$$C(\mathbf{x}', y) \wedge \bigwedge_{i=1}^{r} C_i(\mathbf{x}^i, y) \wedge \bigwedge_{i=1}^{L} y \neq q_i \quad \text{for } E(\mathbf{x}', y)$$

and

$$\bigwedge_{i=1}^{r} C_i(\mathbf{x}^i, q_l) \quad \text{for } \phi_l(\mathbf{x}'),\ l = 1, \ldots, L. \ \blacksquare$$

LEMMA 3.7. *Let d be a natural number. Any formula of the form*

$$\exists^{\geq d} y \ \bigwedge_{i=1}^{r} C_i(\mathbf{x}^i, y) \tag{4}$$

defines a boolean combination of $\mathrm{acl}(\emptyset)$*-definable curve-based cylinders.*

Proof. By Lemma 3.6, the formula (4) is equivalent to some

$$\exists^{\geq d} y \ E(\mathbf{x}', y) \vee \bigvee_{l=1}^{L} (y = q_l \wedge \phi_l(\mathbf{x}'))$$

with $E(\mathbf{x}', y) \to \bigwedge_{l=1}^{L} y \neq q_l$ and the q_l's all distinct. It is thus also equivalent to the disjunction of the formulæ

$$(\exists^{\geq d - |\widetilde{L}|} y \ E(\mathbf{x}', y)) \wedge \Big(\bigwedge_{l \in \widetilde{L}} \phi_l(\mathbf{x}') \wedge \bigwedge_{l \notin \widetilde{L}} \neg\phi_l(\mathbf{x}')\Big)$$

as $\widetilde{L}$ ranges over the subsets of $\{1, \ldots, L\}$.

Since for every $e \in \mathbb{N}$ the set $\{\mathbf{x}' \mid \exists^{\geq e} y \ E(\mathbf{x}', y)\}$ has dimension ≤ 1, any formula of the form $\exists^{\geq e} y \ E(\mathbf{x}', y)$ defines a curve. $\blacksquare$

We proceed by induction on s (the number of negations involved) to show that any formula of the form

$$\exists y \bigwedge_{i=1}^{r} C_i(\mathbf{x}^i, y) \wedge \bigwedge_{j=r+1}^{r+s} \neg C_j(\mathbf{x}^j, y) \tag{1}$$

defines a boolean combination of $\operatorname{acl}(\emptyset)$-definable curve-based cylinders.

The result is proved for $s = 0$ in Lemma 3.7. Fix $s \geq 1$ and a formula of the form (1), and suppose that the induction hypothesis holds for any $s' < s$.

Note first that (1) is equivalent to the formula

$$\exists y \bigwedge_{i=1}^{r} C_i(\mathbf{x}^i, y) \wedge \bigwedge_{j=r+1}^{r+s} \neg\Big(C_j(\mathbf{x}^j, y) \wedge \bigwedge_{i=1}^{r} C_i(\mathbf{x}^i, y)\Big).$$

This formula says that there is a y satisfying $\bigwedge_{i=1}^{r} C_i(\mathbf{x}^i, y)$ but not satisfying $\bigvee_{j=r+1}^{r+s}(C_j(\mathbf{x}^j, y) \wedge \bigwedge_{i=1}^{r} C_i(\mathbf{x}^i, y))$.

Thus (1) is equivalent to the disjunction of (5) and (6) below:

$$\bigvee_{d\in\mathbb{N}} \Big(\Big(\exists^{\geq d+1} y \bigwedge_{i=1}^{r} C_i(\mathbf{x}^i, y)\Big) \wedge \Big(\exists^{=d} y \bigvee_{j=r+1}^{r+s} \Big(C_j(\mathbf{x}^j, y) \wedge \bigwedge_{i=1}^{r} C_i(\mathbf{x}^i, y)\Big)\Big)\Big) \tag{5}$$

(the case when there are only finitely many y's satisfying the condition $\bigvee_{j=r+1}^{r+s}(C_j(\mathbf{x}^j, y) \wedge \bigwedge_{j=i}^{r} C_i(\mathbf{x}^i, y))$) and the formula

$$\begin{aligned}\bigvee_{j=r+1}^{r+s} \Big(\Big(\exists^{\infty} y\ C_j(\mathbf{x}^j, y) \wedge \bigwedge_{i=1}^{r} C_i(\mathbf{x}^i, y)\Big)&\\ \wedge \Big(\exists y \bigwedge_{i=1}^{r} C_i(\mathbf{x}^i, y) \wedge \bigwedge_{k=r+1}^{r+s} \neg C_k(\mathbf{x}^k, y)\Big)\Big)&\end{aligned} \tag{6}$$

(the case where there is some $j > r$ for which there are infinitely many y's satisfying the $C_j(\mathbf{x}^j, y) \wedge \bigwedge_{i=1}^{r} C_i(\mathbf{x}^i, y)$).

Since the formula $C_j(\mathbf{x}^j, y)$ defines a subset of $M^{|\mathbf{x}^j|+1}$ of dimension ≤ 1, we get

$$\exists^{\infty} y\ C_j(\mathbf{x}^j, y) \leftrightarrow \bigvee_{p=1}^{P_j} \mathbf{x}^j = \mathbf{r}_p^j$$

for some finite collection of tuples $\mathbf{r}_p^j$ of elements of $\operatorname{acl}(\emptyset)$. Thus (6) is equivalent to a disjunction of formulæ of the form

$$(\mathbf{x}^{j_0} = \mathbf{r}_p^{j_0}) \wedge \exists y \bigwedge_{i=1}^{r} C_i(\mathbf{x}^i, y) \wedge \neg C_j(\mathbf{r}_p^{j_0}, y) \wedge \bigwedge_{\substack{j=r+1\\ j\neq j_0}}^{r+s} \neg C_j(\mathbf{x}^j, y). \tag{7}$$

Observe here that the formula $\neg C_{j_0}(\mathbf{r}_p^{j_0}, y)$ in the free variables $\mathbf{x}y$ is not only the *negation of* a formula defining a curve-based cylinder with parameters from $\mathrm{acl}(\emptyset)$: the formula $\neg C_{j_0}(\mathbf{r}_p^{j_0}, y)$ in the free variables $\mathbf{x}y$ also defines *itself* a curved-based cylinder definable over $\mathrm{acl}(\emptyset)$ (for it only concerns the distinguished variable y). Therefore the induction hypothesis implies that each formula of the form (7) defines a boolean combination of $\mathrm{acl}(\emptyset)$-definable curve-based cylinders.

It only remains to prove that the formula (5) is equivalent to a boolean combination of curve-based cylinders. By the Uniform Finiteness Property we can replace the infinite disjunction in (5) by a finite one.

Applying the Inclusion-Exclusion Formula for finite sets,

$$\Big|\bigcup_{h=1}^{t} X_h\Big| = \sum_{H\subseteq\{1,\ldots t\}} (-1)^{|H|+1}\Big|\bigcap_{h\in H} X_h\Big|,$$

we find that

$$\exists^{=d} y \bigvee_{j=r+1}^{r+s}\Big(C_j(\mathbf{x}^j, y)\wedge\bigwedge_{i=1}^{r} C_i(\mathbf{x}^i, y)\Big)$$

is equivalent to a boolean combination of formulæ of the form

$$\exists^{=e} y \bigwedge_{j\in J} C_j(\mathbf{x}^j, y)\wedge\bigwedge_{i=1}^{r} C_i(\mathbf{x}^i, y)$$

for some J's ranging over subsets of $\{r+1,\ldots,r+s\}$, and some natural number e not larger than d. By Lemma 3.7, each of these last formulæ is equivalent to a boolean combination of one-dimensional formulæ with parameters in $\mathrm{acl}(\emptyset)$. ■

Quantifier elimination for $\mathcal{C}(\overline{\mathbb{C}}_{\mathbb{C}})$ (the structure obtained by equipping $\mathbb{C}$ with a predicate for each curve $\mathbb{C}$-definable in $\overline{\mathbb{C}}$) easily implies:

Corollary 3.8. *The structure* $\mathcal{C}(\overline{\mathbb{C}}_{\mathbb{C}})$ *has trivial geometry, that is,* $\mathrm{acl}(A) = \bigcup_{a\in A}\mathrm{acl}(\{a\})$ *for all* $A\subseteq\mathbb{C}$.

In particular we get:

Corollary 3.9. *The structure* $\mathcal{C}(\overline{\mathbb{C}}_{\mathbb{C}})$ *is a proper reduct of* $\overline{\mathbb{C}}$.

Since Fact 3.1 ensures that $\mathcal{B}(\overline{\mathbb{C}}_{\mathbb{C}})$ (the structure obtained by equipping $\mathbb{C}$ with a predicate for each subset of $\mathbb{C}^2$ which is $\mathbb{C}$-definable in $\overline{\mathbb{C}}$; see Section 2) is a reduct (in the sense of definability) of $\mathcal{C}(\overline{\mathbb{C}}_{\mathbb{C}})$, the structure $\mathcal{B}(\overline{\mathbb{C}}_{\mathbb{C}})$ is a proper reduct (in the sense of definability) of the structure $\overline{\mathbb{C}}$ and has trivial geometry.

4. Algebraic and definable closure. In the construction of Example 2.1, a key fact is that one cannot distinguish the four roots of the polynomial $X^4 + X + a$, which is an illustration of the fact that algebraic closure and definable closure are two different notions in $\overline{\mathbb{C}}$. Proposition 4.1 below ensures that this condition is needed: if we consider a geometric structure $\mathcal{M}$ on the universe M for which $\mathrm{acl}() = \mathrm{dcl}()$ then the structure $\mathcal{C}_2(\mathcal{M}_{\mathrm{acl}(\emptyset)})$ (whose universe is M and basic relations are all the $\mathrm{acl}(\emptyset)$-definable subsets of M^2 of dimension ≤ 1) eliminates its quantifiers.

PROPOSITION 4.1. *Consider a geometric structure $\mathcal{M}$ on the universe M such that* $\mathrm{acl}(A) = \mathrm{dcl}(A)$ *for all* $A \subseteq M$.

(i) *Any subset of M^n of dimension ≤ 1 definable in $\mathcal{M}$ over* $\mathrm{acl}(\emptyset)$ *is a boolean combination of cylinders, each of whose base is either the graph of a function of one variable $\emptyset$-definable in $\mathcal{M}$ or an element of* $\mathrm{dcl}(\emptyset)$.

(ii) *In particular the structure $\mathcal{C}_2(\mathcal{M}_\emptyset)$ on M generated by all $\emptyset$-definable subsets of M^2 of dimension ≤ 1 and the structure $\mathcal{C}(\mathcal{M}_{\mathrm{acl}(\emptyset)})$ on M generated by all* $\mathrm{acl}(\emptyset)$*-definable subsets of a cartesian product of M of dimension ≤ 1 define the same sets and have quantifier elimination.*

Proof. In this setting, it is clear that a set is definable in $\mathcal{M}$ over $\mathrm{acl}(\emptyset)$ if and only if it is definable in $\mathcal{M}$ without parameters.

By the definition of dimension and the assumption that $\mathrm{acl}() = \mathrm{dcl}()$, a formula in n variables that defines a one-dimensional set in $\mathcal{M}$ over $\emptyset$ is equivalent to an infinite disjunction

$$\bigvee_{i=1}^{n} \bigvee_{F \in \mathcal{F}} \mathbf{x} = F(x_i) \tag{8}$$

where $\mathcal{F}$ is a set of $\emptyset$-definable 1-variable functions from M to M^n and $\mathbf{x} = (x_1, \ldots, x_n)$.

By compactness, we can extract an equivalent finite disjunction from (8), which gives the first part of the proposition.

The second part easily follows (either from a direct argument or from Theorem 3.5). ∎

In Proposition 4.1, we noted that in the case of a geometric structure $\mathcal{M}$ on the universe M with $\mathrm{acl}() = \mathrm{dcl}()$, the use of parameters in $\mathrm{acl}(\emptyset)$ is not needed in the statement of Theorem 3.5. These parameters are however essential for Theorem 3.5 to hold in general: the theory of curves $\emptyset$-*definable* in a geometric structure does not, in general, admit quantifier elimination.

EXAMPLE 4.2. Let $\mathcal{M} = (M; +, \cdot, \mathfrak{m})$ be a saturated algebraically closed valued field of characteristic $\neq 2$, in the language of fields with a unary predicate $\mathfrak{m}$ for its maximal ideal (the structure is known to be geometric; see for instance Section 4 in [4]).

Denote by i one of the two square roots of -1.

The formula ρ in the free variables (x, y) without parameters

$$\exists z\ (z^2 + 1 = 0 \wedge x - z \in \mathfrak{m} \wedge y - z \in \mathfrak{m})$$

is equivalent to

$$(x - i \in \mathfrak{m} \wedge y - i \in \mathfrak{m}) \vee (x + i \in \mathfrak{m} \wedge y + i \in \mathfrak{m}),$$

which defines a boolean combination of $\operatorname{acl}(\emptyset)$-definable curve-based cylinders in M^2.

But ρ does not define a boolean combination of $\emptyset$-definable curve-based cylinders.

Proof. Proceeding toward a contradiction suppose that such a boolean combination exists. We can suppose that it is in disjunctive normal form and each disjunctant is of the form

$$\phi_1(x) \wedge \phi_2(y) \wedge \phi_3(x, y) \wedge \neg\phi_4(x, y)$$

where

- $\phi_1(x)$ and $\phi_2(y)$ are formulæ without parameters in the language of valued fields, and
- $\phi_3(x, y)$ and $\phi_4(x, y)$ each define a (possibly empty) subset of M^2 definable in $\mathcal{M}$ without parameters, which is either the whole M^2 or of dimension ≤ 1.

Since the formula ρ defines a set of dimension 2, there is some disjunctant such that ϕ_3 is a tautology, the sets $\{x \in M \mid \mathcal{M} \models \phi_1(x)\}$ and $\{y \in M \mid \mathcal{M} \models \phi_2(y)\}$ have dimension one (precisely one; not zero or $-\infty$!), and the formula ϕ_4 is not a tautology. Fix such a disjunctant.

Consider $\sigma \in \operatorname{Aut}(\mathcal{M}/\emptyset)$ sending i to $-i$. Let $\beta \in M \setminus \operatorname{acl}(\emptyset)$ be such that $\phi_2(\beta)$ holds. We can find α such that

$$\phi_1(\alpha) \wedge \neg\phi_4(\alpha, \beta) \wedge \neg\phi_4(\alpha, \sigma(\beta))$$

holds: since ϕ_4 defines a set of dimension ≤ 1 and the elements β and $\sigma(\beta)$ are transcendental, the set $\{x \in M \mid \mathcal{M} \models \phi_4(x, \beta) \vee \phi_4(x, \sigma(\beta))\}$ is finite and cannot cover the infinite set $\{x \in M \mid \mathcal{M} \models \phi_1(x)\}$.

Since all ϕ_i's are $\emptyset$-definable and $\phi_2(\beta)$ holds, so does $\phi_2(\sigma(\beta))$. Therefore

$$\phi_1(\alpha) \wedge \phi_2(\beta) \wedge \neg\phi_4(\alpha, \beta) \quad \text{and} \quad \phi_1(\alpha) \wedge \phi_2(\sigma(\beta)) \wedge \neg\phi_4(\alpha, \sigma(\beta))$$

both hold.

The three points (α, β), $(\sigma(\alpha), \sigma(\beta))$ and $(\alpha, \sigma(\beta))$ would satisfy ρ, which cannot be the case. Indeed suppose for instance that α and β belong to $i + \mathfrak{m}$. Then $\sigma(\beta)$ belongs to both $-i + \mathfrak{m}$ and $\alpha + \mathfrak{m} = i + \mathfrak{m}$, which is impossible. (The case when α and β belong to $-i + \mathfrak{m}$ is similar.) ∎

Example 4.3. We get a similar result in $\mathcal{M} = (\mathbb{C}; +, \cdot\,)$ by considering

$$\exists z\ (z^2 + 1 = 0 \wedge u + v = z \wedge w + t = z)$$

in the free variables (u, v, w, t).

5. Higher arities. For each $n \in \mathbb{N}$, let $\mathcal{C}_n(\overline{\mathbb{C}}_{\mathbb{C}})$ (respectively $\mathcal{C}(\overline{\mathbb{C}}_{\mathbb{C}})$) denote the structure on $\mathbb{C}$ whose basic definable sets are the subsets of $\mathbb{C}^k$ for all $k \leq n$ (resp. for all $k \in \mathbb{N}$), $\mathbb{C}$-definable in $\overline{\mathbb{C}}$, of dimension ≤ 1. Similarly, $\mathcal{C}_n(\overline{\mathbb{C}}_{\mathrm{acl}(\emptyset)})$ (respectively $\mathcal{C}(\overline{\mathbb{C}}_{\mathrm{acl}(\emptyset)})$) denotes the structure on $\mathbb{C}$ whose basic definable sets are the subsets of $\mathbb{C}^k$ for all $k \leq n$ (resp. for all $k \in \mathbb{N}$), $\mathrm{acl}(\emptyset)$-definable in $\overline{\mathbb{C}}$, of dimension ≤ 1.

In Section 2, we showed that the structure $\mathcal{C}_2(\overline{\mathbb{C}}_{\mathbb{C}})$ does not have quantifier elimination and therefore we obtained the existence of a constructible curve in $\mathbb{C}^3$ that is not equivalent to a boolean combination of cylinders whose bases are constructible curves in $\mathbb{C}^2$. Here we show:

Proposition 5.1. *Given any natural number $n \geq 3$ there exists an $(n+1)$-ary relation $\emptyset$-definable in $\mathcal{C}_2(\overline{\mathbb{C}}_{\mathrm{acl}(\emptyset)})$ which is not quantifier-free definable in $\mathcal{C}_{n-1}(\overline{\mathbb{C}}_{\mathbb{C}})$.*

Proposition 5.1 and Example 2.1 show in particular that none of the structures $\mathcal{C}_n(\overline{\mathbb{C}}_{\mathbb{C}})$ has quantifier elimination, for $n \geq 2$.

Since by Theorem 3.5 any set definable in $\mathcal{C}_2(\overline{\mathbb{C}}_{\mathrm{acl}(\emptyset)})$ is equivalent to a boolean combination of cylinders whose bases are $\mathrm{acl}(\emptyset)$-definable curves of $\mathbb{C}^k$ for some $k \leq n+1$, we get:

Corollary 5.2. *For any natural number $n \geq 2$ there is a subset of $\mathbb{C}^{n+1}$, $\mathrm{acl}(\emptyset)$-definable in $\overline{\mathbb{C}}$, of dimension 1 which is not equivalent to any boolean combination of cylinders whose bases are k-curves with $k \leq \max\{2, n-1\}$, $\mathbb{C}$-definable in $\overline{\mathbb{C}}$.*

Let $\mathcal{M}$ be a sufficiently saturated extension of $\overline{\mathbb{C}}$ with universe M.

Claim 5.3. *There are two relations $S(s_1, \ldots, s_n, u)$ and $T(t_1, \ldots, t_n, u)$, both $\emptyset$-definable in $\mathcal{C}_2(\overline{\mathbb{C}}_{\mathrm{acl}(\emptyset)})$, such that:*

(**A**) *if $a \in M \setminus \mathbb{C}$, $(s_1, \ldots, s_n, a)$ satisfies S and $(t_1, \ldots, t_n, a)$ satisfies T, then*

$$(s_{\sigma(1)}, \ldots, s_{\sigma(n-1)}) \equiv_{\mathbb{C} \cup \{a\}} (t_{\sigma(1)}, \ldots, t_{\sigma(n-1)})$$

for all injections σ from $\{1, \ldots, n-1\}$ to $\{1, \ldots, n\}$ (the elementary equivalence being in the sense of ACF_0),

(**B**) *if $a \in M \setminus \mathbb{C}$, then the sets*

$$\{(u_1 \ldots, u_n) \in M^n \mid \mathcal{M} \models S(u_1, \ldots, u_n, a) \wedge \neg T(u_1, \ldots, u_n, a)\}$$

and

$$\{(u_1 \ldots, u_n) \in M^n \mid \mathcal{M} \models T(u_1, \ldots, u_n, a)\}$$

are non-empty.

The construction of such S and T and the proof that they satisfy these requirements will be the object of Lemmata 5.7–5.9. Let us admit for the moment their existence and prove Proposition 5.1.

Proof of Proposition 5.1. Suppose for a contradiction that T is equivalent to a boolean combination of formulæ $\mathbb{C}$-definable in $\overline{\mathbb{C}}$, each involving at most $n-1$ of the possible variables. Let U be one of these $(n-1)$-ary relations. Then there is some $k \le n$ such that U does not involve the kth variable (U should also either not involve the last variable or not involve the lth variable for some $l \neq k \le n$).

Fix $a \in M \setminus \mathbb{C}$, $(\mathbf{s}, a) \models S$ and $(\mathbf{t}, a) \models T$.

Since any subtuple of $(\mathbf{s}, a)$ of length $\le n-1$ is elementary equivalent to the corresponding subtuple of $(\mathbf{t}, a)$, since the relation U involves at most $n-1$ variables, and since $(\mathbf{t}, a) \models U$, we get $(\mathbf{s}, a) \models U$.

The same being true for all such U, we get the implication

$$S(\mathbf{s}, a) \to T(\mathbf{s}, a),$$

a contradiction to $(\mathbf{B})$. ■

Fix a natural number N. Given $a \in M$ we denote by $\Theta(a)$ the set of roots of the polynomial $Z^N + Z^{N-1} + a$. The following lemma tells us that the collection of sums of distinct elements of $\Theta(a)$ is in bijection with the power set $\mathcal{P}(\Theta(a))$. This will allow us to encode some finite combinatorics in $\mathcal{M}$.

LEMMA 5.4. *Let $a \in M \setminus \mathbb{C}$. For each natural number $1 \le k \le N$, let $[\Theta(a)]^k$ be the collection of all subsets of $\Theta(a)$ of size k. Then the mapping from $[\Theta(a)]^k$ to M sending A to $\sum_{z \in A} z$ is injective.*

Proof. It follows from Galois theory (see the proof of Theorem 9 in [1]) that

$$\mathrm{Gal}(Z^N + Z^{N-1} + a \,/\, \mathbb{C}(a)) = \mathfrak{S}_N.$$

Suppose for a contradiction that we have subsets $A \neq A'$ of $\Theta(a)$ such that $|A| = |A'|$ and $\sum_{z \in A} z = \sum_{z \in A'} z$. Without loss of generality we can assume that $|A| = |A'|$ is minimal; in particular this implies $A \cap A' = \emptyset$.

If $|A| = 1$, we clearly have a contradiction. Thus we must have $|A| > 1$.

Assume first $\Theta(a) = A \cup A'$. Then $-1 = \sum_{z \in A} z + \sum_{z \in A'} z$ so $\sum_{z \in A} z = -1/2 \in \mathbb{C}$. Let $\zeta \in A$ and $\zeta' \in A'$ be arbitrary and let σ be the element of $\mathrm{Gal}(Z^N + Z^{N-1} + a \,/\, \mathbb{C}(a))$ interchanging ζ and ζ' and fixing the other roots. We have

$$\sum_{z \in A} z = -1/2 = \sigma(-1/2) = \sum_{z \in A} \sigma(z),$$

hence $\zeta = \zeta'$, contradicting the fact that $A \cap A' = \emptyset$.

We can thus assume $A \cup A' \neq \Theta(a)$. Let $\zeta \in A$, $\zeta' \in \Theta(a) \setminus (A \cup A')$ and let σ be the permutation interchanging ζ and ζ' and fixing the other roots. We have

$$\sum_{z \in A} \sigma(z) = \sum_{\alpha \in A'} \sigma(z) = \sum_{z \in A'} z = \sum_{z \in A} z,$$

which gives $\zeta = \zeta'$, contradicting $\zeta' \notin A$. ∎

We now generalize the combinatorial configuration "triangle versus star" appearing in the figure of Example 2.1.

Definition 5.5. For $n \in \mathbb{N}$ we will denote by $\mathcal{L}_n$ the first order language $\{P_1, \dots, P_n\}$ where each P_i is a unary predicate ($\mathcal{L}_0$ being the language of pure equality).

Let $n > 1$ and $\mathcal{F} = \langle F; F_1, \dots F_n \rangle$ be an $\mathcal{L}_n$-structure (i.e. F_i is an interpretation of P_i in $\mathcal{F}$).

We say that $\mathcal{F}$ is *symmetric* if for any permutation $\sigma \in \mathfrak{S}_n$ the structure $\mathcal{F}$ is isomorphic to the $\mathcal{L}_n$-structure $\langle F; F_{\sigma(1)}, \dots, F_{\sigma}(n) \rangle$ (i.e. there is a bijection $\tilde{\sigma}\colon F \to F$ such that $\gamma \in F_i$ if and only if $\tilde{\sigma}(\gamma) \in F_{\sigma(i)}$).

Lemma 5.6. *For any $n > 1$ there are finite symmetric $\mathcal{L}_n$-structures $\mathcal{X} = \langle X; X_1, \dots, X_n \rangle$ and $\mathcal{Y} = \langle Y; Y_1, \dots, Y_n \rangle$ such that $\mathcal{X}$ and $\mathcal{Y}$ are not isomorphic, but their reducts to $\mathcal{L}_{n-1}$ are isomorphic.*

Proof. Set

$$X := \{\alpha \subseteq \{1, \dots, n\} \mid |\alpha| \text{ is odd}\},$$
$$Y := \{\beta \subseteq \{1, \dots, n\} \mid |\beta| \text{ is even}\}.$$

For each $i \in \{1, \dots, n\}$, let

$$X_i := \{\alpha \in X \mid i \in \alpha\} \quad \text{and} \quad Y_j := \{\beta \in Y \mid j \in \beta\}.$$

One can easily verify that $|X| = |Y| = 2^{n-1}$ and that the $\mathcal{L}_n$-structures $\mathcal{X} = \langle X; X_1, \dots, X_n \rangle$ and $\mathcal{Y} = \langle Y, Y_1, \dots, Y_n \rangle$ are symmetric.

Consider the mapping $\Phi : X \to Y$ given by

$$\Phi(\alpha) = \begin{cases} \alpha \setminus \{n\} & \text{if } n \in \alpha, \\ \alpha \cup \{n\} & \text{else.} \end{cases}$$

Clearly Φ is a bijection between X and Y and for $1 \leq i \leq n-1$, we have

$$(\alpha \in X_i) \Leftrightarrow (i \in \alpha) \Leftrightarrow (i \in \Phi(\alpha)) \Leftrightarrow (\Phi(\alpha) \in Y_i).$$

That is, Φ is an isomorphism between the $\mathcal{L}_{n-1}$-structures $\langle X; X_1, \dots, X_{n-1} \rangle$ and $\langle Y; Y_1, \dots, Y_{n-1} \rangle$.

Finally, to see that $\mathcal{X}$ and $\mathcal{Y}$ are not isomorphic (as $\mathcal{L}_n$-structures), note that one and only one of the two sets $\bigcap_{1 \leq i \leq n} X_i$ and $\bigcap_{1 \leq j \leq n} Y_j$ is non-empty:

- if n is even then $\bigcap_{1\le i\le n} X_i = \emptyset$ and $\bigcap_{1\le i\le n} Y_i = \{1,\dots,n\}$,
- if n is odd then $\bigcap_{1\le i\le n} Y_i = \emptyset$ and $\bigcap_{1\le i\le n} X_i = \{1\dots,n\}$.

Therefore there is no bijection between X and Y sending each X_i to Y_i. ∎

For the rest of this section, we let $\mathcal{X} = \langle X; X_1,\dots,X_n\rangle$ and $\mathcal{Y} = \langle Y; Y_1,\dots,Y_n\rangle$ be two symmetric $\mathcal{L}_n$-structures satisfying the conclusion of Lemma 5.6. We let $N = |X| = |Y|$ and, as in Lemma 5.4, we let $\Theta(a)$ denote the set of roots of the polynomial $Z^N + Z^{N-1} + a$. Note that if $a \in M\setminus\mathbb{C}$ then $Z^N + Z^{N-1} + a$ has N distinct roots, and $|\Theta(a)| = |X| = |Y|$.

Consider now the relations S' and T' given by:

- $S'(s_1,\dots,s_n,a)$ holds if and only if there is a bijection ϕ between X and $\Theta(a)$ such that
$$\text{for all } 1\le i\le n,\quad s_i = \sum_{\alpha\in X_i}\phi(\alpha),$$
- $T'(t_1,\dots,t_n,a)$ holds if and only if there is a bijection ψ between Y and $\Theta(a)$ such that
$$\text{for all } 1\le i\le n,\quad t_i = \sum_{\beta\in Y_i}\psi(\beta).$$

Using Lemma 5.4 to transfer the combinatorial properties of $\mathcal{X}$ and $\mathcal{Y}$, we will show that these relations, definable in $\mathcal{M}$, satisfy conditions (**A**) and (**B**) of Claim 5.3.

Lemma 5.7. *Fix $a\in M\setminus\mathbb{C}$. Let ϕ be a bijection between X and $\Theta(a)$, and ψ be a bijection between Y and $\Theta(a)$. For $i\in\{1,\dots,n\}$, let*
$$s_i = \sum_{\alpha\in X_i}\phi(\alpha)\quad\textit{and}\quad t_i = \sum_{\beta\in Y_i}\psi(\beta).$$
Then the tuples $(s_{\sigma(1)},\dots,s_{\sigma(n-1)})$ and $(t_{\tau(1)},\dots,t_{\tau(n-1)})$ are elementary equivalent over $\mathbb{C}\cup\{a\}$ (in the theory of $\overline{\mathbb{C}}$) for all injections σ and τ from $\{1,\dots,n-1\}$ to $\{1,\dots,n\}$.

Proof. By the choice of $\mathcal{X}$ and $\mathcal{Y}$, there is a bijection λ between X and Y that sends each set $X_{\sigma(i)}$ to the corresponding set $Y_{\tau(i)}$ for $i=1,\dots,n-1$.

But as noted in Lemma 5.4, the Galois group of $Z^N + Z^{N-1} + a$ over $\mathbb{C}(a)$ is $\mathfrak{S}_N$. Therefore the bijection $\phi(\alpha)\mapsto\psi(\lambda(\alpha))$ of $\Theta(a)$ extends to an $\mathcal{M}$-automorphism Λ of M fixing $\mathbb{C}\cup\{a\}$. We now have
$$t_{\tau(i)} = \sum_{\beta\in Y_{\tau(i)}}\psi(\beta) = \sum_{\alpha\in X_{\sigma(i)}}\psi(\lambda(\alpha)) = \Lambda\Big(\sum_{\alpha\in X_{\sigma(i)}}\phi(\alpha)\Big) = \Lambda(s_{\sigma(i)}),$$
so Λ sends $(s_{\sigma(1)},\dots,s_{\sigma(n-1)})$ to $(t_{\tau(1)},\dots,t_{\tau(n-1)})$; in particular these two tuples are elementary equivalent over $\mathbb{C}\cup\{a\}$ modulo the theory of $\overline{\mathbb{C}}$. ∎

LEMMA 5.8. *Let $a \in M \setminus \mathbb{C}$. Then the sets*

$$\{(u_1, \dots, u_n) \in M^n \mid \mathcal{M} \models S'(u_1, \dots, u_n, a) \wedge \neg T'(u_1, \dots, u_n, a)\}$$

and

$$\{(u_1, \dots, u_n) \in M^n \mid \mathcal{M} \models T'(u_1, \dots, u_n, a)\}$$

are non-empty.

Proof. Fix $a \in M \setminus \mathbb{C}$. By definition of T', it is clear that there is some $(u_1 \dots, u_n) \in M^n$ for which $T'(u_1, \dots, u_n, a)$ holds. Similarly, we can find some $(s_1, \dots, s_n) \in M^n$ such that $S'(s_1, \dots, s_n, a)$ holds. Suppose $T'(s_1, \dots, s_n, a)$ also holds. Then we get some bijections $\phi : X \to \Theta(a)$ and $\psi : Y \to \Theta(a)$ such that

$$\text{for all } 1 \leq i \leq n, \qquad \sum_{\alpha \in X_i} \phi(\alpha) = \sum_{\beta \in Y_i} \psi(\beta).$$

By Lemma 5.4 we thus get $\phi(X_i) = \psi(Y_i)$ for all $1 \leq i \leq n$, so $\psi^{-1} \circ \phi$ would be an isomorphism between $\mathcal{X}$ and $\mathcal{Y}$. This cannot be the case, hence $T'(s_1, \dots, s_n, a)$ must fail. ∎

It now remains to replace the formulæ S' and T' by formulæ definable in $\mathcal{B}(\overline{\mathbb{C}}_\emptyset)$:

LEMMA 5.9. *There are relations $S(s_1, \dots, s_n, u)$ and $T(t_1, \dots, t_n, u)$ definable in $\mathcal{B}(\overline{\mathbb{C}}_\emptyset)$ such that for all $a \in M \setminus \mathbb{C}$ we have:*

(i) *$S(s_1, \dots, s_n, a)$ holds if and only $S'(s_1, \dots, s_n, a)$ holds,*
(ii) *$T(s_1, \dots, s_n, a)$ holds if and only $T'(s_1, \dots, s_n, a)$ holds.*

Proof. Let $R(u, v)$ be the binary relation that says that u is the sum of $N' = |X_1|$ distinct roots of the polynomial $Z^N + Z^{N-1} - (v^N + v^{N-1})$, one of which is v. That is, $R(u, v)$ holds if and only if

$$\exists (z_1, \dots, z_{N'-1}, z) \Big(u = v + \sum_{k=1}^{N'-1} z_k \wedge v \in \Theta(z) \wedge \bigwedge_{k=1}^{N'-1} z_k \in \Theta(z) \wedge \bigwedge_{k=1}^{N'-1} v \neq z_k \wedge \bigwedge_{k \neq l} z_k \neq z_l \Big).$$

The relation R is definable in $\overline{\mathbb{C}}$ without parameters.

Let $\mathbf{x}$ be an N-tuple $(x_\alpha)_{\alpha \in X}$ of variables indexed by X and consider the $(n+1)$-ary relation S defined by

$$S(s_1, \dots, s_n, u) \Leftrightarrow \Big(\exists \mathbf{x} \bigwedge_{\alpha \in X} x_\alpha \in \Theta(u) \wedge \bigwedge_{\substack{\alpha, \alpha' \in X \\ \alpha \neq \alpha'}} x_\alpha \neq x_{\alpha'} \wedge \bigwedge_{\alpha \in X_i} R(s_i, x_\alpha) \Big).$$

Let $\mathbf{y}$ be an N-tuple $(y_\beta)_{\beta \in Y}$ of variables indexed by Y and consider the $(n+1)$-ary relation T defined by

$$T(t_1, \ldots, t_n, u) \;\Leftrightarrow\; \Big(\exists \mathbf{y} \bigwedge_{\beta \in Y} y_\beta \in \Theta(u) \wedge \bigwedge_{\substack{\beta, \beta' \in Y^2 \\ \beta \neq \beta'}} y_\beta \neq y_{\beta'} \wedge \bigwedge_{\beta \in Y_j} R(t_j, y_\beta)\Big).$$

These relations are definable in $\mathcal{B}(\overline{\mathbb{C}}_\emptyset)$ (the relations R and "$z \in \Theta(u)$" being binary). We will show that they satisfy conditions (i) and (ii) of the lemma.

Fix $a \in M \setminus \mathbb{C}$. It is clear that if $S'(s_1, \ldots, s_n, a)$ holds then $S(s_1, \ldots, s_n, a)$ holds. Conversely, let $(s_1, \ldots, s_n) \in M^n$ be such that $S(s_1, \ldots, s_n, a)$ holds. By the definition of S, we can find a bijection $\phi : X \to \Theta(a)$ and, for each i and each $\alpha \ni i$, an injection $\phi_{i,\alpha} : X_i \to \Theta(a)$, such that for each $1 \le i \le n$,

$$s_i = \sum_{\alpha' \in X_i} \phi_{i,\alpha}(\alpha') \quad \text{and} \quad \phi_{i,\alpha}(\alpha) = \phi(\alpha).$$

Fix such an i. Consider α and α' in X_i. By Lemma 5.4, $\phi_{i,\alpha}$ and $\phi_{i,\alpha'}$ have the same range and therefore $\phi(\alpha')$ belongs to the range of $\phi_{i,\alpha}$ for all $\alpha' \in X_i$. Thus $\phi(X_i) = \phi_{i,\alpha}(X_i)$ for some (in fact all) $\alpha \in X_i$ and

$$s_i = \sum_{\alpha \in X_i} \phi(\alpha).$$

The proof of (ii) is similar. ∎

Putting Lemmata 5.9, 5.7 and 5.8 together, we get, as announced in Claim 5.3, two relations S and T definable in $\mathcal{B}(\overline{\mathbb{C}}_\emptyset)$ that satisfy conditions (**A**) and (**B**).

6. Definability. Since Example 2.1 and Proposition 5.1 show that for any fixed $n \ge 2$, there are sets definable in $\mathcal{C}(\overline{\mathbb{C}}_\mathbb{C})$ (the structure on $\mathbb{C}$ whose basic relations are all the algebraic curves, of any arity) which are not *quantifier-free* definable in $\mathcal{C}_n(\overline{\mathbb{C}}_\mathbb{C})$ (the structure on $\mathbb{C}$ whose basic relations are all the algebraic curves of $\mathbb{C}^n$), it is natural to ask if all the sets definable in $\mathcal{C}(\overline{\mathbb{C}}_\mathbb{C})$ are definable in some $\mathcal{C}_n(\overline{\mathbb{C}}_\mathbb{C})$ (allowing quantifiers, this time).

PROPOSITION 6.1. *The two structures $\mathcal{C}_3(\overline{\mathbb{C}}_\mathbb{C})$ and $\mathcal{C}(\overline{\mathbb{C}}_\mathbb{C})$ define the same sets.*

Proof. By quantifier elimination for $\mathcal{C}(\overline{\mathbb{C}}_\mathbb{C})$, it suffices to show that any algebraic curve is definable in $\mathcal{C}_3(\overline{\mathbb{C}}_\mathbb{C})$. But it is well known that any affine curve $Y \subset \mathbb{C}^n$ is bi-rational to a planar curve $X \subset \mathbb{C}^2$ (see for example Chapter I, Section 3.3, Theorem 5 of [9]). Let $\phi = (\phi_1, \ldots, \phi_n)$ be such an isomorphism. Each restriction of ϕ_i to X is a basic definable set in $\mathcal{C}_3(\overline{\mathbb{C}}_\mathbb{C})$, thus the graph Γ of the restriction of ϕ to X is (quantifier-free) definable in $\mathcal{C}_3(\overline{\mathbb{C}}_{\mathbb{C}})$, and Y, which is the union of the projection of Γ on the last n coordinates and finitely many points, is definable in $\mathcal{C}_3(\overline{\mathbb{C}}_{\mathbb{C}})$. ■

REMARK 6.2. From the proof, we see that the depth of alternation of quantifier for formulæ in the language with a symbol for each algebraic curve of $\mathbb{C}^3$ is at most 1. The lack of quantifier elimination implies that this maximal depth is realized.

QUESTION 6.3. *Is* $\mathcal{C}_2(\overline{\mathbb{C}}_{\mathbb{C}})$ *a proper reduct* (*in the sense of definability*) *of* $\mathcal{C}(\overline{\mathbb{C}}_{\mathbb{C}})$*?*

Acknowledgments. We thank the referee for the careful reading of the paper, providing constructive comments, and help in improving the content of this paper.

The second author was partially supported by NSF grant DMS-0701364.

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Received 9 May 2011;
in revised form 15 August 2011