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# Uncountable $\gamma$-sets under axiom $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$

by

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**Abstract.** We formulate a Covering Property Axiom $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$, which holds in the iterated perfect set model, and show that it implies the existence of uncountable strong $\gamma$-sets in $\mathbb{R}$ (which are strongly meager) as well as uncountable $\gamma$-sets in $\mathbb{R}$ which are not strongly meager. These sets must be of cardinality $\omega_1 < \mathfrak{c}$, since every $\gamma$-set is universally null, while $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ implies that every universally null has cardinality less than $\mathfrak{c} = \omega_2$. We also show that $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ implies the existence of a partition of $\mathbb{R}$ into $\omega_1$ null compact sets.

**1. Axiom $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ and other preliminaries.** Our set theoretic terminology is standard and follows that of [3]. In particular, $|X|$ stands for the cardinality of a set $X$ and $\mathfrak{c} = |\mathbb{R}|$. The Cantor set $2^\omega$ will be denoted by $\mathfrak{C}$. We use the term *Polish space* for a complete separable metric space `without isolated points`. For a Polish space $X$, the symbol $\mathrm{Perf}(X)$ will denote the collection of all subsets of $X$ homeomorphic to $\mathfrak{C}$. We will consider $\mathrm{Perf}(X)$ to be ordered by inclusion.

Axiom $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ was first formulated by Ciesielski and Pawlikowski in [4]. (See also [6].) It is a simpler version of a Covering Property Axiom CPA which holds in the iterated perfect set model. (See [4] or [6].) In order to formulate $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ we need the following terminology and notation. A subset $C$ of a product $\mathfrak{C}^\omega$ of the Cantor set is said to be a *perfect cube* if $C = \prod_{n\in\omega} C_n$, where $C_n \in \mathrm{Perf}(\mathfrak{C})$ for each $n$. For a fixed Polish space $X$ let $\mathcal{F}_{\mathrm{cube}}$ stand

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for the family of all continuous injections from a perfect cube $C \subset \mathfrak{C}^\omega$ onto a set $P$ from $\mathrm{Perf}(X)$. We consider each function $f \in \mathcal{F}_{\mathrm{cube}}$ from $C$ onto $P$ as a coordinate system imposed on $P$. We say that $P \in \mathrm{Perf}(X)$ is a *cube* if we consider it with an (implicitly given) witness function $f \in \mathcal{F}_{\mathrm{cube}}$ onto $P$, and $Q$ is a *subcube* of a cube $P \in \mathrm{Perf}(X)$ provided $Q = f[C]$, where $f \in \mathcal{F}_{\mathrm{cube}}$ is the witness function for $P$ and $C \subset \mathrm{dom}(f) \subset \mathfrak{C}^\omega$ is a perfect cube. Here and in what follows, $\mathrm{dom}(f)$ stands for the domain of $f$.

We say that a family $\mathcal{E} \subset \mathrm{Perf}(X)$ is *cube dense* in $\mathrm{Perf}(X)$ provided every cube $P \in \mathrm{Perf}(X)$ contains a subcube $Q \in \mathcal{E}$. More formally, $\mathcal{E} \subset \mathrm{Perf}(X)$ is cube dense provided

$$\forall f \in \mathcal{F}_{\mathrm{cube}}\ \exists g \in \mathcal{F}_{\mathrm{cube}}\ (g \subset f\ \&\ \mathrm{range}(g) \in \mathcal{E}). \tag{1}$$

It is easy to see that the notion of cube density is a generalization of the notion of density with respect to $\langle \mathrm{Perf}(X), \subseteq \rangle$, that is, if $\mathcal{E}$ is cube dense in $\mathrm{Perf}(X)$ then $\mathcal{E}$ is dense in $\mathrm{Perf}(X)$. On the other hand, the converse implication is not true, as shown by the following simple example.

Example 1.1 ([5, 6]). Let $X = \mathfrak{C} \times \mathfrak{C}$ and let $\mathcal{E}$ be the family of all $P \in \mathrm{Perf}(X)$ such that either all vertical sections of $P$ are countable, or else all horizontal sections of $P$ are countable. Then $\mathcal{E}$ is dense in $\mathrm{Perf}(X)$, but it is not cube dense in $\mathrm{Perf}(X)$.

It is also worth noticing that in order to check that $\mathcal{E}$ is cube dense it is enough to consider in condition (1) only functions $f$ defined on the entire space $\mathfrak{C}^\omega$, that is:

Fact 1.2 ([4, 5, 6]). *$\mathcal{E} \subset \mathrm{Perf}(X)$ is cube dense if and only if*

$$\forall f \in \mathcal{F}_{\mathrm{cube}},\ \mathrm{dom}(f) = \mathfrak{C}^\omega,\ \exists g \in \mathcal{F}_{\mathrm{cube}}\ (g \subset f\ \&\ \mathrm{range}(g) \in \mathcal{E}). \tag{2}$$

Let $\mathrm{Perf}^*(X)$ stand for the family of all sets $P$ such that either $P \in \mathrm{Perf}(X)$ or $P$ is a singleton in $X$. In what follows we will consider singletons as *constant cubes*, that is, with the constant coordinate function from $\mathfrak{C}^\omega$ onto the singleton. In particular, a subcube of a constant cube is the same singleton.

Consider the following game $\mathrm{GAME}_{\mathrm{cube}}(X)$ of length $\omega_1$. The game has two players, Player I and Player II. At each stage $\xi < \omega_1$ of the game Player I can play an arbitrary cube $P_\xi \in \mathrm{Perf}^*(X)$ and Player II must respond with a subcube $Q_\xi$ of $P_\xi$. The game $\langle \langle P_\xi, Q_\xi \rangle \colon \xi < \omega_1 \rangle$ is won by Player I provided

$$\bigcup_{\xi < \omega_1} Q_\xi = X;$$

otherwise the game is won by Player II.

By a strategy for Player II we will understand any function $S$ such that $S(\langle \langle P_\eta, Q_\eta \rangle \colon \eta < \xi \rangle, P_\xi)$ is a subcube of $P_\xi$, where $\langle \langle P_\eta, Q_\eta \rangle \colon \eta < \xi \rangle$ is any partial game. (We abuse here slightly the notation, since the function $S$ depends also on the implicitly given coordinate functions $f_\eta\colon \mathfrak{C}^\omega \to P_\eta$ making each $P_\eta$ a cube.) A game $\langle\langle P_\xi, Q_\xi\rangle\colon \xi<\omega_1\rangle$ is played according to a strategy $S$ for Player II provided $Q_\xi = S(\langle\langle P_\eta, Q_\eta\rangle\colon \eta<\xi\rangle, P_\xi)$ for every $\xi<\omega_1$. A strategy $S$ is a *winning strategy* for Player II if he wins any game played according to $S$.

Here is the axiom:

$\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$: $\mathfrak{c}=\omega_2$ and for any Polish space $X$ Player II has no winning strategy in the game $\mathrm{GAME}_{\mathrm{cube}}(X)$.

PROPOSITION 1.3 ([4, 6]). *Axiom* $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ *implies*

$\mathrm{CPA}_{\mathrm{cube}}$: $\mathfrak{c}=\omega_2$ *and for every Polish space $X$ and every cube dense family $\mathcal{E}\subset \mathrm{Perf}(X)$ there is an $\mathcal{E}_0\subset\mathcal{E}$ such that $|\mathcal{E}_0|\le\omega_1$ and $|X\setminus\bigcup\mathcal{E}_0|\le\omega_1$.*

In [4] (see also [6]) it was proved that $\mathrm{CPA}_{\mathrm{cube}}$ (hence also $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$) implies that $\mathrm{cof}(\mathcal{N})=\omega_1$ and that all perfectly meager sets and all universally null sets have cardinality at most $\omega_1$.

In what follows we will also use the following simple fact. Its proof can be found in [5] and [6].

CLAIM 1.4. *Consider $\mathfrak{C}^\omega$ with the standard topology and standard product measure. If $G$ is a Borel subset of $\mathfrak{C}^\omega$ which is either of second category or of positive measure then $G$ contains a perfect cube $\prod_{i<\omega} P_i$.*

## 2. Disjoint coverings by $\omega_1$ null compacts

THEOREM 2.1. *Assume that $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ holds and let $X$ be a Polish space. If $\mathcal{D}\subset\mathrm{Perf}(X)$ is $\mathcal{F}_{\mathrm{cube}}$-dense and closed under perfect subsets then there exists a partition of $X$ into $\omega_1$ disjoint sets from $\mathcal{D}\cup\{\{x\}\colon x\in X\}$.*

In the proof we will use the following easy lemma.

LEMMA 2.2. *Let $X$ be a Polish space and let $\mathcal{P}=\{P_i\colon i<\omega\}\subset\mathrm{Perf}^*(X)$. For every cube $P\in\mathrm{Perf}(X)$ there exists a subcube $Q$ of $P$ such that either $Q\cap\bigcup_{i<\omega}P_i=\emptyset$ or $Q\subset P_i$ for some $i<\omega$.*

*Proof.* Let $f\in\mathcal{F}_{\mathrm{cube}}$ be such that $f[\mathfrak{C}^\omega]=P$.

If $P\cap\bigcup_{i<\omega}P_i$ is meager in $P$ then, by Claim 1.4, we can find a subcube $Q$ of $P$ such that $Q\subset P\setminus\bigcup_{i<\omega}P_i$.

If $P\cap\bigcup_{i<\omega}P_i$ is not meager in $P$ then there exists an $i<\omega$ such that $P\cap P_i$ has a non-empty interior in $P$. Thus, there exists a basic clopen set $C$ in $\mathfrak{C}^\omega$, which is a perfect cube, such that $f[C]\subset P_i$. So, $Q=f[C]$ is the desired subcube of $P$. ∎

*Proof of Theorem 2.1.* For a cube $P\in\mathrm{Perf}(X)$ and a countable family $\mathcal{P}\subset\mathrm{Perf}^*(X)$ let $D(P)\in\mathcal{D}$ be a subcube of $P$ and $Q(\mathcal{P},P)\in\mathcal{D}$ be as in

Lemma 2.2 applied to $D(P)$ in place of $P$. For a singleton $P \in \mathrm{Perf}^*(X)$ we just put $Q(\mathcal{P}, P) = P$.

Consider the following strategy $S$ for Player II:

$$S(\langle\langle P_\eta, Q_\eta\rangle \colon \eta < \xi\rangle, P_\xi) = Q(\{Q_\eta \colon \eta < \xi\}, P_\xi).$$

By $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ it is not a winning strategy for Player II. So there exists a game $\langle\langle P_\xi, Q_\xi\rangle \colon \xi < \omega_1\rangle$ played according to $S$ in which Player II loses, that is, $X = \bigcup_{\xi<\omega_1} Q_\xi$.

Notice that for every $\xi < \omega_1$ either $Q_\xi \cap \bigcup_{\eta<\xi} Q_\eta = \emptyset$ or there is an $\eta < \omega_1$ such that $Q_\xi \subset Q_\eta$. Let

$$\mathcal{F} = \Big\{Q_\xi \colon \xi < \omega_1 \ \&\ Q_\xi \cap \bigcup_{\eta<\xi} Q_\eta = \emptyset\Big\}.$$

Then $\mathcal{F}$ is as desired. ■

Since the family of all measure zero perfect subsets of $\mathbb{R}^n$ is $\mathcal{F}_{\mathrm{cube}}$-dense we get the following corollary.

COROLLARY 2.3. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ *implies that there exists a partition of* $\mathbb{R}^n$ *into* $\omega_1$ *disjoint closed nowhere dense measure zero sets.*

Note that the conclusion of Corollary 2.3 does not follow from the fact that $\mathbb{R}^n$ can be covered by $\omega_1$ perfect measure zero subsets (see [10, Thm. 6]).

**3. Uncountable $\gamma$-sets.** In this section we will prove that $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ implies the existence of an uncountable $\gamma$-set. Recall that a subset $T$ of a Polish space $X$ is a *$\gamma$-set* provided for every open $\omega$-cover $\mathcal{U}$ of $T$ there is a sequence $\langle U_n \in \mathcal{U} \colon n < \omega\rangle$ such that $T \subset \bigcup_{n<\omega} \bigcap_{i>n} U_i$, where $\mathcal{U}$ is an *$\omega$-cover* of $T$ if for every finite set $A \subset T$ there is a $U \in \mathcal{U}$ with $A \subset U$.

$\gamma$-sets were introduced by Gerlits and Nagy [8]. They were studied by Galvin and Miller [7], Recław [12], Bartoszyński and Recław [2], and others. It is known that under Martin's axiom there are $\gamma$-sets of cardinality continuum [7]. On the other hand, every $\gamma$-set has strong measure zero [8], so it is consistent with ZFC that every $\gamma$-set is countable. Moreover, $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ implies that every $\gamma$-set has cardinality at most $\omega_1 < \mathfrak{c}$, since every strong measure zero set is universally null and under $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ every universally null set has cardinality $\le \omega_1$.

In what follows we will use the characterization of $\gamma$-sets due to Recław [12]. To formulate it we need some terminology. We will consider $\mathcal{P}(\omega)$ as a Polish space by identifying it with $2^\omega$ via characteristic functions. For $A, B \subset \omega$ we will write $A \subseteq^* B$ when $|A \setminus B| < \omega$. We say that a family $\mathcal{A} \subset \mathcal{P}(\omega)$ is *centered* provided $\bigcap \mathcal{A}_0$ is infinite for every finite $\mathcal{A}_0 \subset \mathcal{A}$; and $\mathcal{A}$ *has a pseudointersection* provided there exists a $B \in [\omega]^\omega$ such that $B \subseteq^* A$ for every $A \in \mathcal{A}$. In addition for the rest of this section $\mathcal{K}$ will stand for the family of all continuous functions from $\mathcal{P}(\omega)$ to $\mathcal{P}(\omega)$ and for $A \in \mathcal{P}(\omega)$ we put $A^* = \{B \in \mathcal{P}(\omega) \colon B \subseteq^* A\}$.

PROPOSITION 3.1 (Recław [12]). *For $T \subset \mathcal{P}(\omega)$ the following conditions are equivalent.*

(i) *$T$ is a $\gamma$-set.*

(ii) *For every $f \in \mathcal{K}$ if $f[T]$ is centered then $f[T]$ has a pseudointersection.*

In the proof that follows we will apply axiom $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ to the cubes from the space $\mathcal{K}$. The fact that the subcubes given by the axiom cover $\mathcal{K}$ will allow us to use the above characterization to conclude that the constructed set is indeed a $\gamma$-set. It is also possible to construct an uncountable $\gamma$-set by applying axiom $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ to the space $\mathcal{Y}$ of all $\omega$-covers of $\mathcal{P}(\omega)$ ([1]), as in Section 5. However, we believe that greater diversification of spaces to which we apply $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ makes the paper more interesting.

We will need the following two lemmas.

LEMMA 3.2. *For every countable set $Y \subset \mathcal{P}(\omega)$ the set*

$$\mathcal{K}_Y = \{f \in \mathcal{K} \colon f[Y] \text{ is centered}\}$$

*is Borel in $\mathcal{K}$.*

*Proof.* Let $Y = \{y_i \colon i < \omega\}$ and note that

$$\mathcal{K}_Y = \bigcap_{n,k<\omega} \bigcup_{m \ge k} \bigcap_{i<n} \{f \in \mathcal{K} \colon m \in f(y_i)\}.$$

So, $\mathcal{K}_Y$ is a $G_\delta$ set, since each set $\{f \in \mathcal{K} \colon m \in f(y_i)\}$ is open in $\mathcal{K}$. ∎

LEMMA 3.3. *Let $Y \subset \mathcal{P}(\omega)$ be countable and such that $[\omega]^{<\omega} \subset Y$. For every $W \in [\omega]^\omega$ and a compact set $Q \subset \mathcal{K}_Y$ there exist $V \in [W]^\omega$ and a continuous function $\varphi \colon Q \to [\omega]^\omega$ such that $\varphi(f)$ is a pseudointersection of $f[Y] \cup f[V^*]$ for every $f \in Q$.*

*Moreover, if $\mathcal{J}$ is an infinite family of non-empty pairwise disjoint finite subsets of $W$ then we can choose $V$ containing infinitely many $J$'s from $\mathcal{J}$.*

*Proof.* First notice that there exists a continuous $\psi \colon Q \to [\omega]^\omega$ such that $\psi(f)$ is a pseudointersection of $f[Y]$ for every $f \in Q$.

Indeed, let $Y = \{y_i \colon i < \omega\}$ and for every $f \in Q$ let $\psi(f) = \{n^f_i \colon i < \omega\}$, where $n^f_0 = \min f(y_0)$ and $n^f_{i+1} = \min\{n \in \bigcap_{j \le i} f(y_j) \colon n > n^f_i\}$. The set in the definition of $n^f_{i+1}$ is non-empty, since $f[Y]$ is centered, as $f \in Q \subset \mathcal{K}_Y$. It is easy to see that $\psi$ is continuous and that $\psi(f)$ is as desired.

([1]) More precisely, if $\mathcal{B}_0$ is a countable base for $\mathcal{P}(\omega)$ and $\mathcal{B}$ is the collection of all finite unions of elements from $\mathcal{B}_0$ then we can define $\mathcal{Y}$ as $\mathcal{B}^\omega$ considered with the product topology, where $\mathcal{B}$ is taken with discrete topology.

We will define a sequence $\langle J_i \in \mathcal{J}\colon i < \omega\rangle$ such that $\max J_i < \min J_{i+1}$ for every $i < \omega$. We are aiming for $V = \bigcup_{i<\omega} J_i$.

A set $J_0 \in \mathcal{J}$ is chosen arbitrarily. Now, if $J_i$ is already defined for some $i < \omega$ we define $J_{i+1}$ as follows. Let $w_i = 1 + \max J_i$. Thus $J_i \subset w_i$. For every $f \in Q$ define

$$m_i^f = \min\left(\psi(f) \cap \bigcap f[\mathcal{P}(w_i)]\right).$$

The set $\psi(f) \cap \bigcap f[\mathcal{P}(w_i)]$ is infinite, since $\psi(f)$ is a pseudointersection of $f[Y]$ while $\mathcal{P}(w_i) \subset Y$. Let $k_i^f = \min K_i^f$, where

$$K_i^f = \{k \geq w_i\colon m_i^f \in f(a) \text{ for all } a \subset \omega \text{ with } a \cap k \subset w_i\}.$$

That $K_i^f \neq \emptyset$ follows from the continuity of $f$ since $m_i^f \in f(a)$ for all $a \subset w_i$. Notice that, by the continuity of $\psi$ and the definition of $k_i^f$, for every $p < \omega$ the set $U_p = \{f \in Q\colon k_i^f < p\}$ is open in $Q$. Since the sets $\{U_p\colon p < \omega\}$ form an increasing cover of $Q$, compactness of $Q$ implies the existence of $p_i < \omega$ such that $Q \subset U_{p_i}$. Thus, $w_i \leq k_i^f < p_i$ for every $f \in Q$. We define $J_{i+1}$ to be an arbitrary element of $\mathcal{J}$ disjoint from $p_i$ and notice that

$$m_i^f \in f(a) \quad \text{for every } f \in Q \text{ and } a \subset \omega \text{ with } a \cap \min J_{i+1} \subset w_i.$$

This finishes the inductive construction.

Let $V = \bigcup_{i<\omega} J_i \subset W$ and $\varphi(f) = \{m_i^f\colon i < \omega\}$. It is easy to see that $\varphi$ is continuous (though we will not use this fact). To finish the proof it is enough to show that $\varphi(f)$ is a pseudointersection of $f[Y] \cup f[V^*]$ for every $f \in Q$.

So, fix an $f \in Q$. Clearly $\varphi(f) \subset \psi(f)$ is a pseudointersection of $f[Y]$ since so was $\psi(f)$. To see that $\varphi(f)$ is a pseudointersection of $f[V^*]$ take an $a \subseteq^* V$. Then for almost all $i < \omega$ we have $a \cap \min J_{i+1} \subset w_i$, so that $m_i^f \in f(a)$. Thus $\varphi(f) \subseteq^* f(a)$. ∎

Theorem 3.4. $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ *implies that there exists an uncountable $\gamma$-set in $\mathcal{P}(\omega)$.*

*Proof.* For $\alpha < \omega_1$ and a $\subseteq^*$-decreasing sequence $\mathcal{V} = \{V_\xi \in [\omega]^\omega\colon \xi < \alpha\}$ let $W(\mathcal{V}) \in [\omega]^\omega$ be such that $W(\mathcal{V}) \subsetneq^* V_\xi$ for all $\xi < \alpha$. Moreover, if $P \in \mathrm{Perf}^*(\mathcal{K})$ is a cube then we define a subcube $Q = Q(\mathcal{V}, P)$ of $P$ and an infinite subset $V = V(\mathcal{V}, P)$ of $W = W(\mathcal{V})$ as follows. Let $Y = \mathcal{V} \cup [\omega]^{<\omega}$ and choose a subcube $Q$ of $P$ such that either $Q \cap \mathcal{K}_Y = \emptyset$ or $Q \subset \mathcal{K}_Y$. This can be done by Claim 1.4 since $\mathcal{K}_Y$ is Borel. If $Q \cap \mathcal{K}_Y = \emptyset$ we put $V = W$. Otherwise we apply Lemma 3.3 to find $V$.

Consider the following strategy $S$ for Player II:

$$S(\langle\langle P_\eta, Q_\eta\rangle\colon \eta < \xi\rangle, P_\xi) = Q(\{V_\eta\colon \eta < \xi\}, P_\xi),$$

where the sets $V_\eta$ are defined inductively by $V_\eta = V(\{V_\zeta\colon \zeta < \eta\}, P_\eta)$. In other words, Player II remembers (recovers) the sets $V_\eta$ associated with the cubes $P_\eta$ played so far, and he uses them (and Lemma 3.3) to get the next answer $Q_\xi = Q(\{V_\eta : \eta < \xi\}, P_\xi)$, while remembering (or recovering each time) the set $V_\xi = V(\{V_\eta : \eta < \xi\}, P_\xi)$.

By $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ this is not a winning strategy for Player II. So there exists a game $\langle\langle P_\xi, Q_\xi\rangle : \xi < \omega_1\rangle$ played according to $S$ in which Player II loses, that is, $\mathcal{K} = \bigcup_{\xi<\omega_1} Q_\xi$. Let $\mathcal{V} = \{V_\xi : \xi < \omega_1\}$ be a sequence associated with this game, which is strictly $\subseteq^*$-decreasing, and let $T = \mathcal{V} \cup [\omega]^{<\omega}$. We claim that $T$ is a $\gamma$-set.

In the proof we use Lemma 3.2. So, let $f \in \mathcal{K}$ be such that $f[T]$ is centered. There exists an $\alpha < \omega_1$ such that $f \in Q_\alpha$. Since $f[\{V_\xi : \xi < \alpha\} \cup [\omega]^{<\omega}] \subset f[T]$ we must have applied Lemma 3.3 in the choice of $Q_\alpha$ and $V_\alpha$. Therefore, the family $f[\{V_\xi : \xi < \alpha\} \cup [\omega]^{<\omega} \cup V_\alpha^*]$ has a pseudointersection. Hence so does $f[T]$, since $T \subset \{V_\xi : \xi < \alpha\} \cup [\omega]^{<\omega} \cup V_\alpha^*$. ∎

Since $\mathcal{P}(\omega)$ embeds into any Polish space, we conclude that, under $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$, any Polish space contains an uncountable $\gamma$-set. In particular, there exists an uncountable $\gamma$-set $T \subset \mathbb{R}$.

**4. $\gamma$-sets in $\mathbb{R}$ which are not strongly meager.** Recall (see e.g. [1, p. 437]) that a subset $X$ of $\mathbb{R}$ is *strongly meager* provided $X + G \neq \mathbb{R}$ for every measure zero subset $G$ of $\mathbb{R}$. This is a notion which is dual to a strong measure zero subset of $\mathbb{R}$, since Galvin, Mycielski, and Solovay proved (see e.g. [1, p. 405]) that: $X \subset \mathbb{R}$ has strong measure zero if and only if $X + M \neq \mathbb{R}$ for every meager subset $M$ of $\mathbb{R}$.

Now, although every $\gamma$-set has strong measure zero, under Martin's axiom Bartoszyński and Recław [2] constructed a $\gamma$-set $T$ in $\mathbb{R}$ which is not strongly meager. We will show that the existence of such a set also follows from $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$. The construction is a generalization of that used in the proof of Theorem 3.4.

In the proof we will use the following notation. For $A, B \subset \omega$ we write $A + B$ for the symmetric difference of $A$ and $B$. Upon identification of a set $A \subset \omega$ with its characteristic function $\chi_A \in 2^\omega$ this definition is motivated by the fact that $\chi_{A+B}(n) = \chi_A(n) +_2 \chi_B(n)$, where $+_2$ is addition modulo 2. Also, let $\overline{\mathcal{J}} = \{J_n \in [\omega]^{2^n} : n < \omega\}$ be a family of pairwise disjoint sets and let $\widetilde{G}$ be the family of all $W \subset \omega$ which are disjoint from infinitely many $J \in \overline{\mathcal{J}}$. Notice that $\widetilde{G}$ has measure zero with respect to the standard measure on $\mathcal{P}(\omega)$ induced by the product measure on $2^\omega$.

Lemma 4.1. *If $\mathcal{J} \in [\overline{\mathcal{J}}]^\omega$ and $P$ is a cube in $\mathcal{P}(\omega)$ then there exists a subcube $Q$ of $P$ and a set $V \subset \bigcup \mathcal{J}$ containing infinitely many $J \in \mathcal{J}$ such that $V + Q \subset \widetilde{G}$.*

*Proof.* Let $D = \bigcup \mathcal{J}$ and

$$\begin{aligned} H &= \{\langle U, W\rangle \in \mathcal{P}(D)\times\mathcal{P}(\omega)\colon (U+W)\cap J = \emptyset \text{ for infinitely many } J \in \mathcal{J}\} \\ &\subseteq \{\langle U, W\rangle \in \mathcal{P}(D)\times\mathcal{P}(\omega)\colon U+W \in \widetilde{G}\}. \end{aligned}$$

Note that $H$ is a $G_\delta$ subset of $\mathcal{P}(D) \times \mathcal{P}(\omega)$ since the set $H_J = \{\langle U, W\rangle\colon (U+W)\cap J = \emptyset\}$ is open for every $J \in \mathcal{J}$. Moreover horizontal sections of $H$ are dense in $\mathcal{P}(D)$. So, $\overline{H} = H \cap (\mathcal{P}(D) \times P)$ is a dense $G_\delta$ subset of $\mathcal{P}(D) \times P$, as all its horizontal sections are dense. Thus, by the Kuratowski–Ulam theorem, there is a dense $G_\delta$ subset $\mathcal{K}_0$ of $\mathcal{P}(D)$ such that for every $U \in \mathcal{K}_0$ the vertical section $\overline{H}_U$ of $\overline{H}$ is dense in $P$. Now, since

$$\mathcal{K}_1 = \{U \in \mathcal{P}(D)\colon J \subset U \text{ for infinitely many } J \in \mathcal{J}\}$$

is a dense $G_\delta$ there is a $V \in \mathcal{K}_0 \cap \mathcal{K}_1$. In particular, $V$ contains infinitely many $J \in \mathcal{J}$ and $\overline{H}_V$ is a dense $G_\delta$ subset of $P$. So, by Claim 1.4, there exists a subcube $Q$ of $P$ contained in $\overline{H}_V$. Thus, $Q \subset \overline{H}_V \subset \{W \in P\colon V+W \in \widetilde{G}\}$ and so $V + Q \subset \widetilde{G}$. ∎

THEOREM 4.2. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ *implies that there exists a* $\gamma$*-set* $T \subset \mathcal{P}(\omega)$ *such that* $T + \widetilde{G} = \mathcal{P}(\omega)$.

*Proof.* We will use $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ for the space $X = \mathcal{K} \cup \mathcal{P}(\omega)$, a direct sum of $\mathcal{K}$ and $\mathcal{P}(\omega)$, where $\mathcal{K}$ is as in Proposition 3.1.

For $\alpha < \omega_1$ and a $\subseteq^*$-decreasing sequence $\mathcal{V} = \{V_\xi \in [\omega]^\omega\colon \xi < \alpha\}$ such that each $V_\xi$ contains infinitely many $J \in \overline{\mathcal{J}}$ let $W(\mathcal{V}) \in [\omega]^\omega$ be such that $\mathcal{J} = \{J \in \overline{\mathcal{J}}\colon J \subset W(\mathcal{V})\}$ is infinite and $W(\mathcal{V}) \subsetneq^* V_\xi$ for all $\xi < \alpha$. For a cube $P \in \mathrm{Perf}^*(\mathcal{K})$ we define a subcube $Q = Q(\mathcal{V}, P)$ of $P$ and an infinite subset $V = V(\mathcal{V}, P)$ of $W = W(\mathcal{V})$ as follows. By Claim 1.4 we can find a subcube $P'$ of $P$ such that either $P' \subset \mathcal{K}$ or $P' \subset \mathcal{P}(\omega)$.

If $P' \subset \mathcal{K}$ we proceed as in the proof of Theorem 3.4. We put $Y = \mathcal{V} \cup [\omega]^{<\omega}$ and we use Claim 1.4 to find a subcube $Q$ of $P'$ such that either $Q \cap \mathcal{K}_Y = \emptyset$ or $Q \subset \mathcal{K}_Y$. If $Q \cap \mathcal{K}_Y = \emptyset$ we put $V = W$. Otherwise we apply Lemma 3.3 to find $V$. If $P' \subset \mathcal{P}(\omega)$ we use Lemma 4.1 to find $Q$ and $V$.

Consider the following strategy $S$ for Player II:

$$S(\langle\langle P_\eta, Q_\eta\rangle\colon \eta < \xi\rangle, P_\xi) = Q(\{V_\eta\colon \eta < \xi\}, P_\xi),$$

where the sets $V_\eta$ are defined inductively by $V_\eta = V(\{V_\zeta\colon \zeta < \eta\}, P_\eta)$. By $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ this is not a winning strategy for Player II. So there exists a game $\langle\langle P_\xi, Q_\xi\rangle\colon \xi < \omega_1\rangle$ played according to $S$ in which Player II loses, that is, $X = \bigcup_{\xi<\omega_1} Q_\xi$. Let $\mathcal{V} = \{V_\xi\colon \xi < \omega_1\}$ be a sequence associated with this game, which is strictly $\subseteq^*$-decreasing, and let $T = \mathcal{V} \cup [\omega]^{<\omega}$. We claim that $T$ is as desired.

The argument that $T$ is a $\gamma$-set is the same as in the proof of Theorem 3.4. To see that $\mathcal{P}(\omega) \subset T + \widetilde{G}$ notice that for every $A \in \mathcal{P}(\omega)$ there is an $\alpha < \omega_1$

such that $A \in Q_\alpha$. But then at step $\alpha$ we used Lemma 4.1 to find $Q_\alpha$ and $V_\alpha$. In particular, $V_\alpha + Q_\alpha \subset \widetilde{G}$. So, $A \in Q_\alpha \subset V_\alpha + \widetilde{G} \subset T + \widetilde{G}$. ∎

COROLLARY 4.3. $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ *implies that there exists a* $\gamma$*-set* $X \subset \mathbb{R}$ *which is not strongly meager.*

*Proof.* This is the argument from [2]. Let $T$ be as in Theorem 4.2 and let $f\colon \mathcal{P}(\omega) \to [0,1]$, $f(A) = \sum_{i<\omega} 2^{-(i+1)}\chi_A(i)$. Then $f$ is continuous, so $X = f[T]$ is a $\gamma$-set. Let $H = \bigcap_{m<\omega}\bigcup_{n>m} f[J_n]$. Then $H$ has measure zero and it is easy to see that $[0,1] = f[\mathcal{P}(\omega)] \subset f[T] + H = X + H$. Then $\overline{G} = H + \mathbb{Q}$ has measure zero and $X + \overline{G} = \mathbb{R}$. ∎

**5. Uncountable strongly meager $\gamma$-sets in $\mathbb{R}$.** Let $X$ be a Polish space with topology $\tau$. We say that $\mathcal{U} \subset \tau$ is a cover of $Z \subset [X]^{<\omega}$ provided for every $A \in Z$ there is a $U \in \mathcal{U}$ with $A \subset U$. Following [7] we say that a subset $S$ of $X$ is a *strong* $\gamma$*-set* provided there exists an increasing sequence $\langle k_n < \omega\colon n < \omega\rangle$ such that for every sequence $\langle J_n \subset \tau\colon n < \omega\rangle$, where each $J_n$ is a cover of $[X]^{k_n}$, there exists a sequence $\langle D_n \in J_n\colon n < \omega\rangle$ with $X \subset \bigcup_{n<\omega}\bigcap_{m>n} D_m$. It is proved in [7] that every strong $\gamma$-set $X \subset \mathbb{R}$ is strongly meager. The goal of this section is to construct, under $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$, an uncountable strong $\gamma$-set in $\mathcal{P}(\omega)$. So, after identifying $\mathcal{P}(\omega)$ with its homeomorphic copy in $\mathbb{R}$, this will become an uncountable $\gamma$-set in $\mathbb{R}$ which is strongly meager. Under Martin's axiom a strong $\gamma$-set in $\mathcal{P}(\omega)$ of cardinality continuum exists (see [7]).

Let $\mathcal{B}_0$ be a countable basis for the topology of $\mathcal{P}(\omega)$ and let $\mathcal{B}$ be the collection of all finite unions of elements from $\mathcal{B}_0$. Since every open cover of $[\mathcal{P}(\omega)]^k$, $k < \omega$, contains a refinement from $\mathcal{B}$, in the definition of strong $\gamma$-set it is enough to consider only sequences $\langle J_n\colon n < \omega\rangle$ with $J_n \subset \mathcal{B}$.

Now, consider $\mathcal{B}$ with the discrete topology. Since $\mathcal{B}$ is countable, the space $\mathcal{B}^\omega$, considered with the product topology, is a Polish space and so is $\mathcal{X} = (\mathcal{B}^\omega)^\omega$. For $J \in \mathcal{X}$ we will write $J_n$ in place of $J(n)$. It is easy to see that a subbasis for the topology of $\mathcal{X}$ is given by the clopen sets

$$\{J \in \mathcal{X}\colon J_n(m) = B\},$$

where $n, m < \omega$ and $B \in \mathcal{B}$.

For the remainder of this section we fix an increasing sequence $\langle k_n < \omega\colon n < \omega\rangle$ such that $k_n \geq n2^n + n$ for every $n < \omega$. Then we have the following lemma.

LEMMA 5.1. *Let* $X \in [\omega]^\omega$ *and let* $F$ *be a countable subset of* $\mathcal{P}(\omega)$ *such that* $[\omega]^{<\omega} \subset F$. *Assume that* $P$ *is a compact subset of* $\mathcal{X}$ *such that for every* $J \in P$ *and* $n < \omega$ *the family* $J_n[\omega] = \{J_n(m)\colon m < \omega\}$ *covers* $[F]^{k_n}$. *Then there exists a set* $Y \in [X]^\omega$ *and for each* $J \in P$ *a sequence* $\langle D_n^J \in J_n\colon n < \omega\rangle$ *such that* $F \cup Y^* \subset \bigcup_{n<\omega}\bigcap_{m>n} D_m^J$.

*Proof.* Let $\{F_n\colon n<\omega\}$ be an enumeration of $[\omega]^{<\omega}$ such that $F_n\subset n$ for all $n<\omega$ and let $F=\{f_n\colon n<\omega\}$. We will construct inductively sequences $\langle s_n\in X\colon n<\omega\rangle$ and $\langle\{D^J_n\in J_n[\omega]\colon J\in P\}\colon n<\omega\rangle$ such that for every $n<\omega$, $J\in P$, and $A\subset\omega$ we have

(i) $\{f_i\colon i<n\}\subset D^J_n$ and $s_n<s_{n+1}$;
(ii) if $i<j\le n+1$ and $(A\cap s_{n+1})\setminus\{s_0,\ldots,s_n\}=F_i$ then $A\in D^J_j$.

We choose $s_0\in X$ and $\{D^J_n\in J_n[\omega]\colon J\in P\}$ arbitrarily. Then conditions (i) and (ii) are trivially satisfied. Next, assume that the sequence $\{s_i\colon i\le n\}$ is already constructed. We will construct $s_{n+1}$ and sets $D^J_{n+1}$ as follows.

Let

$$Q=\{q\in[\omega]^{<\omega}\colon q\setminus\{s_0,\ldots,s_n\}=F_i\ \text{ for some }\ i\le n\}.$$

Then $|Q|\le(n+1)2^{n+1}$ and $|Q\cup\{f_0,\ldots,f_n\}|\le k_{n+1}$.

Fix $J\in P$. Since $J_{n+1}[\omega]$ covers $[F]^{\le k_{n+1}}$, there exists a $\overline{D}^J_{n+1}\in J_{n+1}[\omega]$ containing $Q\cup\{f_0,\ldots,f_n\}$. Since $\overline{D}^J_{n+1}$ is open and covers the finite set $Q$, there is an $s^J_{n+1}>s_n$ in $X$ such that for every $q\in Q$,

$$\{x\subset\omega\colon x\cap s^J_{n+1}=q\cap s^J_{n+1}\}\subset\overline{D}^J_{n+1}.$$

Notice that

$(*)$ for every $A\subset\omega$ and $\overline{s}_{n+1}\ge s^J_{n+1}$ condition (ii) holds.

Indeed, assume that $(A\cap\overline{s}_{n+1})\setminus\{s_0,\ldots,s_n\}=F_i$ for some $i<j\le n+1$. If $j\le n$ then $n\ge1$ and since $F_i\subset i\subset s_{n-1}$ we have

$$(A\cap s_n)\setminus\{s_0,\ldots,s_{n-1}\}=(A\cap\overline{s}_{n+1})\setminus\{s_0,\ldots,s_n\}=F_i.$$

So, by the inductive assumption, $A\in D^J_j$. If $j=n+1$ then $q=A\cap\overline{s}_{n+1}\in Q$. So $A\in\{x\subset\omega\colon x\cap\overline{s}_{n+1}=q\cap\overline{s}_{n+1}\}\subset\{x\subset\omega\colon x\cap s^J_{n+1}=q\cap s^J_{n+1}\}\subset\overline{D}^J_{n+1}$, finishing the proof of $(*)$.

For each $J\in P$ let $m^J<\omega$ be such that $J_{n+1}(m^J)=\overline{D}^J_{n+1}$ and define $U_J=\{K\in\mathcal{X}\colon K_{n+1}(m^J)=\overline{D}^J_{n+1}\}$. Then $U_J$ is an open neighborhood of $J$. In particular, $\{U_J\colon J\in P\}$ is an open cover of the compact set $P$, so there exists a finite $P_0\subset P$ such that $P\subset\bigcup\{U_{\bar J}\colon\bar J\in P_0\}$. Choose $s_{n+1}\in X$ such that $s_{n+1}\ge\max\{s^{\bar J}_{n+1}\colon\bar J\in P_0\}$. Moreover, for every $J\in P$ choose $\bar J\in P_0$ such that $J\in U_{\bar J}$ and define $D^J_{n+1}=\overline{D}^{\bar J}_{n+1}$. It is easy to see that, by $(*)$, conditions (i) and (ii) are preserved. This completes the inductive construction.

Put $Y=\{s_n\colon n<\omega\}$. To see that it satisfies the conclusion pick an arbitrary $J\in P$. We will show that $F\cup Y^*\subset\bigcup_{n<\omega}\bigcap_{m>n}D^J_m$.

Clearly $F\subset\bigcup_{n<\omega}\bigcap_{m>n}D^J_m$ since, by (i), $f_n\in D^J_m$ for every $m>n$. So, fix an $A\in Y^*$. Then $A\setminus Y=F_i$ for some $i<\omega$. Let $n<\omega$ be such that $i < n$ and $s_n > \max F_i$. Then for every $m > n$ we have $i < m \le m+1$ and $(A \cap s_{m+1}) \setminus \{s_0, \ldots, s_m\} = F_i$. So, by (ii), we have $A \in D^J_m$ for every $m > n$. Thus, $A \in \bigcap_{m>n} D^J_m$. ∎

LEMMA 5.2. *If $F \subset \mathcal{P}(\omega)$ is countable then the set*

$$\mathcal{X}_F = \{J \in \mathcal{X} \colon J_n[\omega] \text{ covers } [F]^{k_n} \text{ for every } n < \omega\}$$

*is Borel in $\mathcal{X}$.*

*Proof.* This follows from the fact that

$$\mathcal{X}_F = \bigcap_{n<\omega} \bigcap_{A \in [F]^{k_n}} \bigcup_{m<\omega} \bigcup_{A \subset B \in \mathcal{B}} \{J \in \mathcal{X} \colon J_n(m) = B\}$$

since each set $\{J \in \mathcal{X} \colon J_n(m) = B\}$ is clopen in $\mathcal{X}$. Thus, $\mathcal{X}_F$ is a $G_\delta$ set. ∎

THEOREM 5.3. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ *implies that there exists an uncountable strong $\gamma$-set in $\mathcal{P}(\omega)$.*

*Proof.* For $\alpha < \omega_1$ and a $\subseteq^*$-decreasing sequence $\mathcal{V} = \{V_\xi \in [\omega]^\omega \colon \xi < \alpha\}$ let $W(\mathcal{V}) \in [\omega]^\omega$ be such that $W(\mathcal{V}) \subsetneq^* V_\xi$ for all $\xi < \alpha$. Moreover, if $P \in \mathrm{Perf}^*(\mathcal{X})$ is a cube then we define a subcube $Q = Q(\mathcal{V}, P)$ of $P$ and an infinite subset $Y = V(\mathcal{V}, P)$ of $X = W(\mathcal{V})$ as follows. Let $F = \mathcal{V} \cup [\omega]^{<\omega}$ and choose a subcube $Q$ of $P$ such that either $Q \cap \mathcal{X}_F = \emptyset$ or $Q \subset \mathcal{X}_F$. This can be done by Claim 1.4 since $\mathcal{X}_F$ is Borel. If $Q \cap \mathcal{X}_F = \emptyset$ we put $Y = X$. Otherwise we apply Lemma 5.1 to find $Y$.

Consider the following strategy $S$ for Player II:

$$S(\langle\langle P_\eta, Q_\eta\rangle \colon \eta < \xi\rangle, P_\xi) = Q(\{V_\eta \colon \eta < \xi\}, P_\xi),$$

where the sets $V_\eta$ are defined inductively by $V_\eta = V(\{V_\zeta \colon \zeta < \eta\}, P_\eta)$. By $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ this is not a winning strategy for Player II. So there exists a game $\langle\langle P_\xi, Q_\xi\rangle \colon \xi < \omega_1\rangle$ played according to $S$ in which Player II loses, that is, $\mathcal{X} = \bigcup_{\xi<\omega_1} Q_\xi$. Let $\mathcal{V} = \{V_\xi \colon \xi < \omega_1\}$ be a sequence associated with this game, which is strictly $\subseteq^*$-decreasing, and let $T = \mathcal{V} \cup [\omega]^{<\omega}$. We claim that $T$ is a strong $\gamma$-set.

Indeed, let $\langle \mathcal{U}_n \subset \mathcal{B} \colon n < \omega\rangle$ be such that $\mathcal{U}_n$ covers $[T]^{k_n}$ for every $n < \omega$. Then there is a $J \in \mathcal{X}$ such that $J_n[\omega] = \mathcal{U}_n$ for every $n < \omega$. Let $\alpha < \omega_1$ be such that $J \in Q_\alpha$. Then $J \in \mathcal{X}_{\{V_\eta \colon \eta < \alpha\} \cup [\omega]^{<\omega}}$, so we must have used Lemma 5.1 to get $Q_\alpha$. In particular, there is a sequence $\langle D^J_n \in J_n[\omega] = \mathcal{U}_n \colon n < \omega\rangle$ such that $([\omega]^{<\omega} \cup \{V_\eta \colon \eta < \alpha\}) \cup (V_\alpha)^* \subset \bigcup_{n<\omega} \bigcap_{m>n} D^J_m$. So, $T \subset \bigcup_{n<\omega} \bigcap_{m>n} D^J_m$, as $\{V_\eta \colon \alpha \le \eta < \omega_1\} \subset (V_\alpha)^*$. ∎

Since every homeomorphic image of a strong $\gamma$-set is evidently a strong $\gamma$-set, we obtain immediately the following conclusion.

COROLLARY 5.4. $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ *implies that there exists an uncountable $\gamma$-set in $\mathbb{R}$ which is strongly meager.*

It is worth mentioning that a construction of an uncountable strong $\gamma$-set in $\mathcal{P}(\omega)$ under $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ can also be done in a formalism similar to that used in Section 3. In order to do it, we need the following definitions and facts. For a fixed sequence $\overline{k} = \langle k_n < \omega: n < \omega\rangle$ we say that $\mathcal{A} \subset (\mathcal{P}(\omega))^\omega$ is *$\overline{k}$-centered* provided for every $n < \omega$ any $k_n$ sets from $\{A(n): A \in \mathcal{A}\}$ have a common point; $B \in \omega^\omega$ is a *quasi-intersection* of $\mathcal{A} \subset (\mathcal{P}(\omega))^\omega$ provided for every $A \in \mathcal{A}$ there are infinitely many $n < \omega$ with $B(n) \in A(n)$. Now, if $\mathcal{K}^*$ is the family of all continuous functions from $\mathcal{P}(\omega)$ to $(\mathcal{P}(\omega))^\omega$ then the following is true:

> A set $X \subset \mathcal{P}(\omega)$ is a strong $\gamma$-set if and only if there exists an increasing sequence $\overline{k} = \langle k_n < \omega: n < \omega\rangle$ such that for every $f \in \mathcal{K}^*$ if $f[X]$ is $\overline{k}$-centered then $f[X]$ has a quasi-intersection.

With this characterization in hand we can construct an uncountable strong $\gamma$-set in $\mathcal{P}(\omega)$ by applying $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ to the space $\mathcal{K}^*$.

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