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The diameter of a Lascar strong type

by

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Abstract. We prove that a type-definable Lascar strong type has finite diameter. We also answer some other questions from [1] on Lascar strong types. We give some applications on subgroups of type-definable groups.

In this paper T is a complete theory in language L and we work within a monster model $\mathfrak{C}$ of T. For $a_0, a_1 \in \mathfrak{C}$ let $a_0 \Theta a_1$ iff $\langle a_0, a_1 \rangle$ extends to an indiscernible sequence $\langle a_n, n < \omega \rangle$. We define a distance function d on $\mathfrak{C}$ by letting $d(a,b)$ be the minimal natural number n such that for some $a_0 = a, a_1, \ldots, a_{n-1}, a_n = b$ we have $a_0 \Theta a_1 \Theta \ldots a_{n-1} \Theta a_n$. If no such n exists, we set $d(a,b) = \infty$.

The transitive closure $\overset{\mathrm{Ls}}{\equiv}$ of Θ (denoted also by E_{L}) is the finest bounded invariant equivalence relation on $\mathfrak{C}$; its classes are called *Lascar strong types*. So $a \overset{\mathrm{Ls}}{\equiv} b \Leftrightarrow d(a,b) < \infty$. Moreover, $\overset{\mathrm{bd}}{\equiv}$ (denoted also by E_{KP}) is the finest bounded type-definable equivalence relation on $\mathfrak{C}$. For details see e.g. [1]. So $\overset{\mathrm{bd}}{\equiv}$ is coarser than $\overset{\mathrm{Ls}}{\equiv}$ and each $\overset{\mathrm{bd}}{\equiv}$-class is a union of a number of Lascar strong types.

1. Assume $a \in \mathfrak{C}$ and let X be the Lascar strong type of a. We define the diameter $\mathrm{diam}(X)$ as the supremum of $d(a,b)$, $b \in X$. In [1] the authors ask whether X being type-definable implies that X has finite diameter. (Strictly speaking, this is an equivalent version of the question from [1].) Also they ask how many Lascar strong types may be contained in a given $\overset{\mathrm{bd}}{\equiv}$-class. We answer both questions in Corollary 1.8. Before we approach them it is convenient to consider a more general problem: how many Lascar strong types are needed to make a type-definable set. We answer this question in the next theorem. For a type or formula $s(x)$, $[s(x)]$ denotes the set of types containing $s(x)$.

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THEOREM 1.1. *Assume that $p^* \in S(\emptyset)$ and $X \subseteq p^*(\mathfrak{C})$ is a type-definable set which is a union of a number of Lascar strong types of infinite diameter. Then $|X/\overset{\mathrm{Ls}}{\equiv}| \geq 2^{\aleph_0}$.*

In the proof of Theorem 1.1 we will need a topological lemma related to the Baire category theorem. Assume K is a compact space and $\mathcal{A}$ is a covering of K. We define an increasing sequence Z_α, $\alpha \in \mathrm{Ord} \cup \{-1\}$, of open subsets of K. We let $Z_{-1} = \emptyset$, for limit α we put $Z_\alpha = \bigcup_{\beta<\alpha} Z_\beta$, and for $\alpha = \beta + 1$ we define

$$Z_\alpha = \bigcup_{A \in \mathcal{A}} \mathrm{int}(Z_\beta \cup A).$$

We call $\langle Z_\alpha \rangle_{\alpha \in \mathrm{Ord} \cup \{-1\}}$ the *open analysis of K with respect to $\mathcal{A}$*. There is a minimal β such that $Z_\beta = Z_{\beta+1}$. We call this β the *height* of K with respect to $\mathcal{A}$. If $Z_\beta = K$, we say that K is *analyzable with respect to $\mathcal{A}$*, or *$\mathcal{A}$-analyzable*. The closed set $K \setminus Z_\beta$ is called the *core* of K with respect to $\mathcal{A}$, or the *$\mathcal{A}$-core* of K.

The *Cantor–Bendixson analysis* of K is the open analysis with respect to $\mathcal{A} = \{\{x\} : x \in K\}$. Also Morley rank may be defined in terms of open analyses of some compact spaces.

If $\mathcal{A}'$ is another covering of K, we say that $\mathcal{A}'$ refines $\mathcal{A}$ if every member of $\mathcal{A}'$ is contained in some member of $\mathcal{A}$.

REMARK 1.2. (1) If K is $\mathcal{A}$-analyzable and $Z_\alpha \neq K$, then $Z_{\alpha+1} \setminus Z_\alpha$ is relatively open and dense in $K \setminus Z_\alpha$ and the height of K with respect to $\mathcal{A}$ is a successor ordinal.

(2) If $\mathcal{A}'$ refines $\mathcal{A}$ and K is $\mathcal{A}'$-analyzable, then K is $\mathcal{A}$-analyzable.

LEMMA 1.3. *Assume $f : K' \to K$ is a continuous surjection of compact spaces, $\mathcal{A}$ is a covering of K and $\mathcal{A}'$ is a covering of K'.*

(1) *Assume $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$, $\bigcup \mathcal{A}_0 \cap \bigcup \mathcal{A}_1 = \emptyset$ and $S = \bigcup \mathcal{A}_0$. If K is $\mathcal{A}$-analyzable, then the set $\bigcup_{A \in \mathcal{A}_0} \mathrm{int}_S(A)$ is relatively open and dense in S.*

(2) *Assume $\mathcal{A}' = \{f^{-1}[A] : A \in \mathcal{A}\}$. Let C' be the $\mathcal{A}'$-core of K'. Then $f[C']$ is the $\mathcal{A}$-core of K. In particular, K' is $\mathcal{A}'$-analyzable iff K is $\mathcal{A}$-analyzable.*

(3) *Assume $\mathcal{A} = \{f[A'] : A' \in \mathcal{A}'\}$. If K' is $\mathcal{A}'$-analyzable, then K is $\mathcal{A}$-analyzable.*

Proof. Let $\langle Z_\alpha \rangle$ be the open analysis of K with respect to $\mathcal{A}$.

(1) Assume U is an open subset of K meeting S. We have $Z_0 \cap U \cap S \subseteq \bigcup_{A \in \mathcal{A}_0} \mathrm{int}_S(A)$. If $Z_0 \cap U \cap S = \emptyset$, then S is dense in $U \setminus Z_0$. Indeed, otherwise there is some open set $V \subseteq U$ with $V \setminus Z_0$ non-empty and disjoint from S. But then $V \subseteq K \setminus S \in \mathcal{A}_1$, hence $V \subseteq Z_0$, a contradiction.

Consequently, $Z_1 \cap U \cap S \neq \emptyset$ and $Z_1 \cap U \cap S \subseteq \bigcup_{A \in \mathcal{A}_0} \mathrm{int}_S(A)$.

(2) Let C be the $\mathcal{A}$-core of K. Clearly $f[C'] \subseteq C$. We will show the reverse inclusion: $C \subseteq f[C']$.

Replace K by C and K' by $f^{-1}[C]$, and then replace $\mathcal{A}$ by $\{A \cap C : A \in \mathcal{A}\}$ and $\mathcal{A}'$ by $\{A \cap f^{-1}[C] : A \in \mathcal{A}'\}$. So now the sets Z_α (the analysis of the new K) are all empty, and the $\mathcal{A}'$-core of K' is still C' (because $C' \subseteq f^{-1}[C]$). Let $\langle Z'_\alpha \rangle$ be the open analysis of K' with respect to $\mathcal{A}'$.

Suppose for a contradiction that $f[C'] \neq K$. We have $Z_0 = \emptyset$. This means that the sets from $\mathcal{A}$ have empty interior. We construct recursively non-empty open subsets U_l of K and numbers $\alpha_l \in \mathrm{Ord} \cup \{-1\}$, $l < \omega$, such that the sequence $\langle \alpha_l \rangle_{l<\omega}$ is strictly decreasing (hence we will reach a contradiction) and

$$(*) \qquad \alpha_l \text{ is minimal such that } f^{-1}[\mathrm{cl}(U_l)] \subseteq Z'_{\alpha_l+1}.$$

We define U_0 as a non-empty open subset of K with $\mathrm{cl}(U_0) \cap f[C'] = \emptyset$. Then for some β we have $f^{-1}[\mathrm{cl}(U_0)] \subseteq Z'_\beta$. Since $f^{-1}[\mathrm{cl}(U_0)]$ is compact, we can choose α_0 as in $(*)$.

Suppose we have defined U_l and α_l; we will define U_{l+1} and α_{l+1}. Since $f^{-1}[\mathrm{cl}(U_l)]$ is compact, by $(*)$ there are finitely many sets $A_0, \ldots, A_{k-1} \in \mathcal{A}$ (for some $k < \omega$) and open sets $V_i \subseteq K', i < k$, with $\mathrm{cl}(V_i) \subseteq Z'_{\alpha_l} \cup A'_i$ (where $A'_i = f^{-1}[A_i]$), such that $f^{-1}[\mathrm{cl}(U_l)] \subseteq \bigcup_{i<k} V_i$. Let $V = f[\bigcup_{i<k} \mathrm{cl}(V_i) \setminus Z'_{\alpha_l}]$. So V is a closed subset of K. There are two cases to consider:

Case 1: *V has non-empty interior.* Then one of the sets $f[\mathrm{cl}(V_i) \setminus Z'_{\alpha_l}]$ has non-empty interior, but $f[\mathrm{cl}(V_i) \setminus Z'_{\alpha_l}] \subseteq A_i$, and A_i has empty interior, a contradiction.

Case 2: *V has empty interior.* Choose a non-empty open set $U_{l+1} \subseteq U_l$ with $\mathrm{cl}(U_{l+1}) \cap V = \emptyset$. So $f^{-1}[\mathrm{cl}(U_{l+1})] \subseteq Z'_{\alpha_l}$. Hence $\alpha_l \geq 0$ and we may choose α_{l+1} so that $(*)$ holds.

In this way we have finished the construction and the proof of (2).

(3) Let $\mathcal{A}'' = \{f^{-1}[A] : A \in \mathcal{A}\}$. Then $\mathcal{A}'$ refines $\mathcal{A}''$, hence by Remark 2, K' is $\mathcal{A}''$-analyzable. By (2), K is $\mathcal{A}$-analyzable. ■

Let us consider the case where in Lemma 1.3(1), $\mathcal{A}_0$ is a countable family of closed sets, $S = \bigcup \mathcal{A}_0$ is a G_δ-set and $\mathcal{A}_1 = \{K \setminus S\}$. Then the remaining assumption of Lemma 1.3(1) holds: K is $\mathcal{A}$-analyzable.

Indeed, it is enough to show that $Z_0 \neq \emptyset$. By the Baire category theorem the conclusion of Lemma 1.3(1) holds, hence there is a non-empty set U such that $U \cap S$ is contained in a single closed set $F \in \mathcal{A}_0$. If $U \subseteq F$, we get $U \subseteq Z_0$ and $Z_0 \neq \emptyset$. Otherwise, there is an open non-empty set $V \subseteq U \setminus F$. Then necessarily $V \subseteq K \setminus S \in \mathcal{A}_1$, hence $V \subseteq Z_0$ and $Z_0 \neq \emptyset$, too. In this way Lemma 1.3 is related to the Baire category theorem.

From now on until the end of the proof of Theorem 1.1 we assume that $X \subseteq p^*(\mathfrak{C})$ is a type-definable union of a number of Lascar strong types of infinite diameter and $\overline{a} = \langle a_\alpha \rangle_{\alpha<\mu}$ is a tuple of representatives of the Lascar strong types contained in X. So X is definable by a type $\Phi_0(x)$ over some $C \subseteq \mathfrak{C}$. It follows that X is also type-definable over $\overline{a}$.

To see this, consider the restriction map $r : S(C\overline{a}) \to S(\overline{a})$. Since r is continuous, the image of the compact set $S(C\overline{a}) \cap [\Phi_0(x)]$ via r is closed in $S(\overline{a})$, hence $r[S(C\overline{a}) \cap [\Phi_0(x)]] = S(\overline{a}) \cap [\Phi(x, \overline{a})]$ for some type $\Phi(x, \overline{a})$ over $\overline{a}$. Since X is $\overline{a}$-invariant, $\Phi(\mathfrak{C}, \overline{a}) = X$.

Let $Y = S(\overline{a}) \cap [\Phi(x, \overline{a})] = \{\mathrm{tp}(b/\overline{a}) : b \in X\}$. So Y is a closed subset of $S(\overline{a})$. For $\alpha < \mu$ and $n < \omega$ let

$$Y_\alpha = \{\mathrm{tp}(b/\overline{a}) : b \overset{\mathrm{Ls}}{\equiv} a_\alpha\}, \qquad Y_\alpha^n = \{\mathrm{tp}(b/\overline{a}) : d(a_\alpha, b) \le n\}.$$

Then the sets Y_α^n are closed in $S(\overline{a})$, $Y_\alpha = \bigcup_n Y_\alpha^n$ and $Y = \bigcup_{\alpha,n} Y_\alpha^n$. Let $\langle Z_\alpha \rangle$ be the open analysis of Y with respect to $\mathcal{Y} = \{Y_\alpha^n : \alpha < \mu,\ n < \omega\}$ and let β^+ be the corresponding height of Y. The main part of the proof of Theorem 1.1 is the following proposition.

Proposition 1.4. *Y is not analyzable with respect to $\mathcal{Y}$, i.e., $Z_{\beta^+} \neq Y$.*

Proof. Suppose for a contradiction that $Z_{\beta^+} = Y$ and Y is $\mathcal{Y}$-analyzable. For every $b \in X$ and $n < \omega$ let $U_b = \{\mathrm{tp}(c/b) : c \in X\}$, $Y_b = \{\mathrm{tp}(c/b) : c \overset{\mathrm{Ls}}{\equiv} b\}$, $Y_b^n = \{\mathrm{tp}(c/b) : d(c, b) \le n\}$ and

$$Z_b^0 = \{r \in Y_b : Y_b \cap [\varphi(x)] \subseteq Y_b^n \text{ for some } \varphi(x) \in r \text{ and } n < \omega\}.$$

Claim 1.5. *Z_b^0 is a relatively open and dense subset of Y_b. Moreover there is no bound on $d(c, b)$ for $c \overset{\mathrm{Ls}}{\equiv} b$ with* $\mathrm{tp}(c/b) \in Z_b^0$.

Proof. We could have chosen $\overline{a}$ so that $a_0 = b$. So we may assume $b = a_0$. The set U_b is closed as a continuous image (via the restriction map) of the closed set Y. If μ is countable, then one can show that the set Y_b is a G_δ-subset of U_b, and then the claim follows directly from the Baire category theorem (which holds in a G_δ-subset of a compact space), since $Y_b = \bigcup_n Y_b^n$.

In general μ may be uncountable, so we have to argue differently. Let $f : Y \to U_b$ be the restriction map and $Y_0^\omega = Y \setminus \bigcup_n Y_0^n$. Then $\mathcal{A}' = \{Y_0^n : n \le \omega\}$ is a covering of Y such that $\mathcal{Y}$ is finer than $\mathcal{A}'$. Since Y is $\mathcal{Y}$-analyzable, by Remark 2 it is also $\mathcal{A}'$-analyzable.

Let $\mathcal{A} = \{Y_b^n : n \le \omega\}$, where $Y_b^\omega = U_b \setminus \bigcup_{n<\omega} Y_b^n$. By Lemma 1.3 (for $K' := Y$ and $K := U_b$) we find that U_b is $\mathcal{A}$-analyzable and Z_b^0 is dense in Y_b. Let $\langle Z_\alpha^* \rangle$ be the open analysis of U_b with respect to $\mathcal{A}$.

For the last clause, suppose there is a bound k on $d(c, b)$ for $c \overset{\mathrm{Ls}}{\equiv} b$ with $\mathrm{tp}(c/b) \in Z_b^0$. We will prove that $Y_b = Z_b^0$.

Suppose otherwise. Choose the first α such that Z_α^* meets $Y_b \setminus Z_b^0$. It follows that Z_α^* contains an open subset W of U_b such that $\emptyset \neq$

$W \cap (Y_b \setminus Z^0_b) \subseteq Y^n_b$ for some $n < \omega$. But then for all c with $\mathrm{tp}(c/b) \in (W \cap Y_b) \cup Z^0_b$ we have $d(c,b) \le \max\{n,k\}$, hence $W \cap Y_b \subseteq Z^0_b$, a contradiction.

Now $Y_b = Z^0_b$ implies that the diameter of the Lascar strong type of b is $\le k$, contradicting the assumptions of Theorem 1.1. ■

For any $b \in X$ we define $\bar{d}(\overline{a}, b)$ as $d(a_\alpha, b)$ for the a_α with $a_\alpha \overset{\mathrm{Ls}}{\equiv} b$. We carry out an inductive analysis of X. For $n < \omega$ let

$$X^n = \{b \in X : \bar{d}(\overline{a}, b) \le n\}, \qquad Y^n = \{\mathrm{tp}(b/\overline{a}) : b \in X^n\}.$$

We see that $X = \bigcup_n X^n$, $Y = \bigcup_n Y^n$ and $Y^n, n < \omega$, are unions of the closed sets Y^n_α, $\alpha < \mu$. Let $\langle Z^\alpha \rangle$ be the open analysis of Y with respect to $\mathcal{Y}' = \{Y^n : n < \omega\}$. Since $\mathcal{Y}$ refines $\mathcal{Y}'$ and Y is $\mathcal{Y}$-analyzable, Remark 1.2 shows that Y is also $\mathcal{Y}'$-analyzable. Let β^* be the height of Y with respect to $\mathcal{Y}'$. By Remark 1.2, β^* is a successor, say $\beta^* = \alpha^* + 1$ for some $\alpha^* \in \mathrm{Ord} \cup \{-1\}$.

Lemma 1.6. (1) *If there is a finite bound on $\bar{d}(\overline{a}, b)$ for $b \in \varphi(\mathfrak{C}, \overline{a})$ with $\mathrm{tp}(b/\overline{a}) \in Z^{\alpha+1} \setminus Z^\alpha$, then $Y \cap [\varphi(x, \overline{a})] \subseteq Z^{\alpha+1}$.*

(2) *There is some $k > 0$ such that for all $b \in X$ with $\mathrm{tp}(b/\overline{a}) \in Y \setminus Z^{\alpha^*}$, we have $\bar{d}(\overline{a}, b) \le k$.*

(3) *$\beta^* = 0$ iff there is a finite bound on the diameters of the Lascar strong types contained in X.*

Proof. (1) By Remark 1.2, $Z^{\alpha+2} \setminus Z^{\alpha+1}$ is dense in $Y \cap [\varphi(x, \overline{a})] \setminus Z^{\alpha+1}$. On the other hand our assumptions imply that $Z^{\alpha+2} \cap [\varphi(x, \overline{a})] \subseteq Z^{\alpha+1}$ (apply directly the definition of $Z^{\alpha+2}$). Therefore the set

$$(Z^{\alpha+2} \setminus Z^{\alpha+1}) \cap Y \cap [\varphi(x, \overline{a})]$$

is empty, and so is $Y \cap [\varphi(x, \overline{a})]$ (because it has an empty dense subset). Hence $Y \cap [\varphi(x, \overline{a})] \subseteq Z^{\alpha+1}$.

(2) The set $Y \setminus Z^{\alpha^*}$ is covered by relatively open subsets of some Y^n, $n < \omega$. By compactness, a finite number of these sets cover $Y \setminus Z^{\alpha^*}$, hence the conclusion follows.

(3) Immediate. ■

Proof of Proposition 1.4 continued. We will define recursively elements $b_l \in X$, formulas $\varphi_l(x, \overline{a}), \psi_l(x, b_l)$ and numbers $\alpha_l, \beta_l \in \mathrm{Ord} \cup \{-1\}$ for $l < \omega$ so that $\alpha_l < \beta_l$, the sequences $\langle \alpha_l \rangle_{l<\omega}, \langle \beta_l \rangle_{l<\omega}$ are strictly decreasing (hence we will reach a contradiction) and the following hold:

(a) $\mathrm{tp}(b_l/\overline{a}) \in Z^{\beta_l+1} \setminus Z^{\beta_l}$.

(b) $\psi_l(x, b_l) \vdash \varphi_l(x, \overline{a})$.

(c) $\emptyset \ne Y_{b_l} \cap [\psi_l(x, b_l)] \subseteq Y^m_{b_l}$ for some $m < \omega$.

(d) $\alpha_l < \alpha^*$ is minimal such that $Y \cap [\varphi_l(x, \overline{a})] \subseteq Z^{\alpha_l} \cup Y^n$ for some $n < \omega$.

First we deal with the case $l=0$. Choose a $b_0 \in X$ with $\mathrm{tp}(b_0/\overline{a}) \in Y \setminus Z^{\alpha^*}$ and let $\beta_0 = \alpha^*$. Let $k > 0$ be as in Lemma 1.6. So $\bar{d}(\overline{a}, b_0) \le k$.

By Claim 1.5 choose $c \stackrel{\mathrm{Ls}}{\equiv} b_0$ with $\mathrm{tp}(c/b_0) \in Z^0_{b_0}$ and $d(b_0, c) \ge 3k$. By the triangle inequality it follows that $\bar{d}(\overline{a}, c) \ge 2k$, hence by the choice of k, $\mathrm{tp}(c/\overline{a}) \in Z^{\alpha^*}$ and the same is true for any other $c' \models \mathrm{tp}(c/b_0)$.

The set $Y \setminus Z^{\alpha^*}$ is closed in $S(\overline{a})$, so we can regard it as a type over $\overline{a}$. We know that the type $(Y \setminus Z^{\alpha^*})(x) \cup \mathrm{tp}(c/b_0)(x)$ is inconsistent, hence there are formulas $\psi_0(x, b_0) \in \mathrm{tp}(c/b_0)$ and $\varphi_0(x, \overline{a})$ satisfying (b), (c) and $Y \cap [\varphi_0(x, \overline{a})] \subseteq Z^{\alpha^*}$. Then we choose $\alpha_0 < \alpha^*$ satisfying (d) by the definition of Z^{α^*}.

Next suppose we have found $b_l, \varphi_l, \psi_l, \alpha_l$ and β_l satisfying (a)–(d) and we will define $b_{l+1}, \varphi_{l+1}, \psi_{l+1}, \alpha_{l+1}$ and β_{l+1}.

Choose a formula $\theta(y, \overline{a}) \in \mathrm{tp}(b_l/\overline{a})$ with $\psi_l(x,y) \wedge \theta(y, \overline{a}) \vdash \varphi_l(x, \overline{a})$. Since $\mathrm{tp}(b_l/\overline{a}) \in Z^{\beta_l+1} \setminus Z^{\beta_l}$, by Lemma 6 for every $\gamma < \beta_l$ there is no finite bound on $\bar{d}(\overline{a}, b')$ for $b' \in \theta(\mathfrak{C}, \overline{a})$ with $\mathrm{tp}(b'/\overline{a}) \in Z^{\gamma+1} \setminus Z^{\gamma}$. If β_l is a successor, let β_{l+1} be the predecessor of β_l, while for limit β_l choose $\beta_{l+1} < \beta_l$ with $\alpha_l < \beta_{l+1}$. Then choose $b_{l+1} \in \theta(\mathfrak{C}, \overline{a})$ with $\mathrm{tp}(b_{l+1}/\overline{a}) \in Z^{\beta_{l+1}+1} \setminus Z^{\beta_{l+1}}$ and such that $\bar{d}(\overline{a}, b_{l+1}) > n + m$.

Since $\mathrm{tp}(b_l) = \mathrm{tp}(b_{l+1})$, $\psi_l(\mathfrak{C}, b_l) \cap Y_{b_l}$ being non-empty implies that also $\psi_l(\mathfrak{C}, b_{l+1}) \cap Y_{b_{l+1}} \ne \emptyset$. There are two cases.

Case 1: *There is some* $c' \in \psi_l(\mathfrak{C}, b_{l+1})$ *with* $c' \stackrel{\mathrm{Ls}}{\equiv} b_{l+1}$ *and* $\mathrm{tp}(c'/\overline{a}) \notin Z^{\alpha_l}$. For such a c' we have $\bar{d}(\overline{a}, c') \le n, d(b_{l+1}, c') \le m$ (by (c), (d)), while $\bar{d}(\overline{a}, b_{l+1}) > n + m$, which violates the triangle inequality.

This contradiction shows that $\alpha_l \ge 0$ and the following Case 2 holds.

Case 2: *For every* $c' \in \psi_l(\mathfrak{C}, b_{l+1})$ *with* $c' \stackrel{\mathrm{Ls}}{\equiv} b_{l+1}$ *we have* $\mathrm{tp}(c'/\overline{a}) \in Z^{\alpha_l}$. Choose such a c'. Again we see that the type $\mathrm{tp}(c'/b_{l+1})(x) \cup (Y \setminus Z^{\alpha_l})(x)$ is inconsistent, hence for some $\psi_{l+1}(x, b_{l+1}) \in \mathrm{tp}(c'/b_{l+1})$ implying $\psi_l(x, b_{l+1})$ and for some $\varphi_{l+1}(x, \overline{a})$ we have

(b′) $\psi_{l+1}(x, b_{l+1}) \vdash \varphi_{l+1}(x, \overline{a})$ and

(d′) $Y \cap [\varphi_{l+1}(x, \overline{a})] \subseteq Z^{\alpha_{l+1}} \cup Y^n$ for some minimal $\alpha_{l+1} \in \mathrm{Ord} \cup \{-1\}$ with $\alpha_{l+1} < \alpha_l$ and some $n < \omega$.

In this way we have completed the recursive construction and the proof of Proposition 1.4. ■

Lemma 1.7. (1) *There is a non-empty set* $X' \subseteq X$ *type-definable over* $\overline{a}$ *such that for every formula* $\varphi(x)$ *over* $\overline{a}$, *if* $X' \cap \varphi(\mathfrak{C}) \ne \emptyset$, *then*

$$|(X' \cap \varphi(\mathfrak{C}))/\stackrel{\mathrm{Ls}}{\equiv}| \ge 2.$$

(2) *Assume* $a, b \in X$ *and* $d(a, b) = \infty$. *Then there are formulas* $\varphi(x) \in \mathrm{tp}(a/\bar{a})$ *and* $\psi(x) \in \mathrm{tp}(b/\bar{a})$ *such that for all* $a' \in \varphi(\mathfrak{C})$ *and* $b' \in \psi(\mathfrak{C})$ *we have* $d(a', b') > n$.

Proof. (1) Let $X' = \{b \in X : \mathrm{tp}(b/\bar{a}) \in Y \setminus Z_{\beta+}\}$. By Proposition 1.4, X' is non-empty. We will prove that X' satisfies our demands.

Consider a formula $\varphi(x)$ over $\bar{a}$ with $X' \cap \varphi(\mathfrak{C}) \neq \emptyset$. Suppose for a contradiction that $X' \cap \varphi(\mathfrak{C})$ is contained in a single Lascar strong type, say $a_\gamma / \overset{\mathrm{Ls}}{\equiv}$. Then

$$(Y \setminus Z_{\beta+}) \cap [\varphi(x)] \subseteq Y_\gamma = \bigcup_n Y_\gamma^n,$$

hence by the Baire category theorem one of the sets Y_γ^n, $n < \omega$, has non-empty interior in $Y \setminus Z_{\beta+}$. This means that $Z_{\beta+ +1} \neq Z_{\beta+}$, a contradiction.

(2) Let $p = \mathrm{tp}(a/\bar{a})$ and $q = \mathrm{tp}(b/\bar{a})$. The type

$$\{\text{“}d(x, y) \leq n\text{”}\} \cup p(x) \cup q(y)$$

is inconsistent. So there is a formula $\chi(x, y)$ such that “$d(x, y) \leq n$” $\vdash \chi(x, y)$, and there are formulas $\varphi(x) \in p(x)$ and $\psi(y) \in q(y)$ such that the formula $\chi(x, y) \wedge \varphi(x) \wedge \psi(y)$ is contradictory. Clearly the formulas $\varphi(x)$ and $\psi(x)$ satisfy our demands. ∎

Proof of Theorem 1.1. Choose X' as in Lemma 1.7(1). Using Lemma 1.7(2) we construct a tree $\varphi_\eta(x)$, $\eta \in 2^{<\omega}$, of formulas over $\bar{a}$ such that

(a) $\varphi_\eta(\mathfrak{C}) \cap X' \neq \emptyset$,
(b) $\varphi_{\eta ^\frown \langle i \rangle}(x) \vdash \varphi_\eta(x)$ for $i = 0, 1$, and
(c) if $\eta \neq \nu \in 2^n$, then for all $a \in \varphi_\eta(\mathfrak{C})$ and $b \in \varphi_\nu(\mathfrak{C})$ we have $d(a, b) \geq n$.

Since X' is type-definable over $\bar{a}$, for $\eta \in 2^\omega$ we can choose $a_\eta \in X' \cap \bigcap_{n<\omega} \varphi_{\eta \restriction n}(\mathfrak{C})$. We see that for $\eta \neq \nu \in 2^\omega$ we have $d(a_\eta, a_\nu) = \infty$. ∎

Corollary 1.8. (1) *A type-definable Lascar strong type has finite diameter.*

(2) *Assume* X *is a* $\overset{\mathrm{bd}}{\equiv}$*-class which is not a Lascar strong type. Then* $|X/\overset{\mathrm{Ls}}{\equiv}| \geq 2^{\aleph_0}$.

Proof. (1) Let X be a type-definable Lascar strong type. If $\mathrm{diam}(X)$ is infinite, then we get a contradiction with Theorem 1.1. (2) is immediate. ∎

Ziegler [1] has given an example of a theory where $\overset{\mathrm{Ls}}{\equiv}$ and $\overset{\mathrm{bd}}{\equiv}$ differ. This example is constructed from a sequence of definable Lascar strong types with growing finite diameters. Using Theorem 1.1 we can see that this is not accidental.

Corollary 1.9. (1) *Assume in* T *there is a sequence of type-definable Lascar strong types* X_n, $n < \omega$, *with growing finite diameters. Then in* T *there is a Lascar strong type which is not type-definable. In particular*, $\overset{\mathrm{Ls}}{\equiv}$ *and* $\overset{\mathrm{bd}}{\equiv}$ *differ.*

(2) $\overset{\mathrm{Ls}}{\equiv}$ *and* $\overset{\mathrm{bd}}{\equiv}$ *agree iff there is a finite bound on the diameters of Lascar strong types.*

Proof. (1) Let $a_n \in X_n$, $a = \langle a_n \rangle_{n<\omega}$ and let X be the Lascar strong type of a. Then X projects onto each X_n and for $a' = \langle a'_n \rangle_{n<\omega} \in X$, $d(a, a') \geq d(a_n, a'_n)$. So X has infinite diameter and is not type-definable.

(2) follows from (1). ∎

Related to $\overset{\mathrm{Ls}}{\equiv}$ and $\overset{\mathrm{bd}}{\equiv}$ are the groups

$$\mathrm{Autf}_{\mathrm{L}}(\mathfrak{C}) = \{f \in \mathrm{Aut}(\mathfrak{C}) : f \text{ preserves each } \overset{\mathrm{Ls}}{\equiv}\text{-class}\},$$

$$\mathrm{Autf}_{\mathrm{KP}}(\mathfrak{C}) = \{f \in \mathrm{Aut}(\mathfrak{C}) : f \text{ preserves each } \overset{\mathrm{bd}}{\equiv}\text{-class}\}.$$

Moreover, as a subgroup of $\mathrm{Aut}(\mathfrak{C})$, $\mathrm{Autf}_{\mathrm{L}}(\mathfrak{C})$ is generated by $\bigcup\{\mathrm{Aut}(\mathfrak{C}/M) : M \prec \mathfrak{C}\}$ (see [1]).

Corollary 1.10. $\mathrm{Autf}_{\mathrm{L}}(\mathfrak{C}) = \mathrm{Autf}_{\mathrm{KP}}(\mathfrak{C}) \Leftrightarrow \mathrm{Autf}_{\mathrm{L}}(\mathfrak{C})$ *is generated by* $\bigcup\{\mathrm{Aut}(\mathfrak{C}/M) : M \prec \mathfrak{C}\}$ *in finitely many steps.*

The fact that $\overset{\mathrm{Ls}}{\equiv}$ and $\overset{\mathrm{bd}}{\equiv}$ differ is equivalent to $\mathrm{Autf}_{\mathrm{L}}(\mathfrak{C}) \neq \mathrm{Autf}_{\mathrm{KP}}(\mathfrak{C})$. Hence we get the following corollary.

Corollary 1.11. *Assume* $\mathrm{Autf}_{\mathrm{L}}(\mathfrak{C}) \neq \mathrm{Autf}_{\mathrm{KP}}(\mathfrak{C})$. *Then*

$$|\mathrm{Autf}_{\mathrm{KP}}(\mathfrak{C})/\mathrm{Autf}_{\mathrm{L}}(\mathfrak{C})| \geq 2^{\aleph_0}.$$

Corollary 1.11 answers another question from [1]. When T is countable, then in the above results we can replace $\geq 2^{\aleph_0}$ by $= 2^{\aleph_0}$. This is because the objects in question are then Borel by nature. For example, as explained in [1], when X is a $\overset{\mathrm{bd}}{\equiv}$-class, then we can interpret $X/\overset{\mathrm{Ls}}{\equiv}$ as the set of equivalence classes of some Borel equivalence relation on a Polish space.

More generally, in the above results $\overset{\mathrm{Ls}}{\equiv}$ may be replaced by any equivalence relation E defined as the reflexive and transitive closure of some 0-type-definable symmetric binary relation $R(x, y)$ implying $\mathrm{tp}(x) = \mathrm{tp}(y)$. The corresponding distance function d_R on an E-class is given by:

$$d_R(a, b) = \text{the minimal number of steps needed to go from } a \text{ to } b \text{ via } R.$$

Let S be a 0-type-definable set (possibly of infinite tuples). Let $R(x, y)$ be the conjunction of all formulas $\varphi(x, y)$ such that on S, $x \overset{\mathrm{Ls}}{\equiv} y$ implies φ. In other words, R is the closure of $\overset{\mathrm{Ls}}{\equiv}$ in the Stone topology on $S \times S$. Let E be the transitive closure of R. [1, Corollary 2.6] proves that on S, E equals $\overset{\mathrm{bd}}{\equiv}$.

Hence by the extended version of Corollary 1.8(1), the d_R-diameter of each E-class is finite. In fact, [1, Corollary 2.6] proves further that this diameter is ≤ 2.

Let us consider an even more general situation. We say that an equivalence relation E is $\bigvee\bigwedge$-*definable* if $E = \bigcup_{n<\omega} \Phi_n$, where each Φ_n is type-definable. We can and will assume additionally that each Φ_n is reflexive, symmetric, and $\Phi_n(x,y) \wedge \Phi_n(y,z)$ implies $\Phi_{n+1}(x,z)$. In this case we say that $\bigvee_{n<\omega} \Phi_n$ is a *normal form* of E.

COROLLARY 1.12. *Assume $E(x,y)$ is an $\bigvee\bigwedge$-definable equivalence relation implying* $\mathrm{tp}(x) = \mathrm{tp}(y)$, *with normal form $\bigvee_{n<\omega} \Phi_n$. Assume $p \in S(\emptyset)$ and $X \subseteq p(\mathfrak{C})$ is a type-definable set which is a union of some E-classes. Then either E is equivalent on X to some $\Phi_n(x,y)$ (and is type-definable on X) or $|X/E| \geq 2^{\aleph_0}$.*

Proof. For $a, b \in X$ let $d_E(a,b)$ be the minimal n such that $a\Phi_n b$. Then d_E satisfies the triangle inequality, hence we can repeat the proof of Theorem 1.1. ∎

2. Thus far we have not used the fact that $\overset{\mathrm{Ls}}{\equiv}$ is bounded. We shall take advantage of this property in the proofs of the next results.

Assume X is a Lascar strong type and $\bar{a} = \langle a_i \rangle_{i<k}$ is a non-empty (possibly infinite) tuple of elements of $\mathfrak{C}$ with $a_0 \in X$. For $a \in X$ let $X_a^n = \{b \in X : d(a,b) \leq n\}$.

We define subsets $Z_{\bar{a}}^{\alpha}$ of X, $\alpha \in \mathrm{Ord} \cup \{-1\}$, recursively relatively $\bigvee$-definable over $\bar{a}$. We put $Z_{\bar{a}}^{-1} = \emptyset$, $Z_{\bar{a}}^{\alpha} = \bigcup_{\beta<\alpha} Z_{\bar{a}}^{\beta}$ for limit α, and for $\alpha = \beta + 1$ we define

$$Z_{\bar{a}}^{\alpha} = \{b \in X : X \cap \varphi(\mathfrak{C}) \subseteq Z_{\bar{a}}^{\beta} \cup X_{a_0}^n \text{ for some } \varphi(x) \in \mathrm{tp}(b/\bar{a}) \text{ and } n < \omega\}.$$

The minimal α such that $Z_{\bar{a}}^{\alpha} = Z_{\bar{a}}^{\alpha+1}$ is called the *height* of X over $\bar{a}$. We say that X is *analyzable* (over $\bar{a}$) if $X = Z_{\bar{a}}^{\alpha}$ for some α. By Lemma 1.3, X is analyzable over $\bar{a}$ iff X is analyzable over a_0 iff X is analyzable over any $\bar{b}$ with $b_0 \in X$.

On the level of types, the sets $Z_{\bar{a}}^{\alpha}$ correspond to an open analysis of the set $Y_{\bar{a}} = \{\mathrm{tp}(b/\bar{a}) : b \in X\}$. If X is type-definable, then $Y_{\bar{a}}$ is a closed subset of $S(\bar{a})$. In general $Y_{\bar{a}}$ is only an F_σ-subset of $S(\bar{a})$, hence this analysis does not have properties as nice as in Section 1. However, choosing $\bar{a}$ suitably and using the boundedness of $\overset{\mathrm{Ls}}{\equiv}$ we can recover some of these properties in the present setting. This is done in the next lemma.

LEMMA 2.1. *Assume X is an analyzable Lascar strong type. Then for some $\bar{a} = \langle a_i \rangle_{i<k}$, the height of X over $\bar{a}$ is a successor $\gamma + 1$ for some $\gamma \in \mathrm{Ord} \cup \{-1\}$ and there is a finite bound on $d(a_0, b)$, $b \in X \setminus Z_{\bar{a}}^{\gamma}$.*

Proof. For $\bar{a} = \langle a_i \rangle_{i<k}$ with $a_0 \in X$ choose a minimal β such that $X^1_{a_0} \subseteq Z^\beta_{\bar{a}}$. Choose $\bar{a}$ so that β is minimal possible. Since $X^1_{a_0}$ is type-definable, β is a successor, say $\beta = \gamma + 1$. Let $\Phi(x, \bar{a})$ be a disjunction of formulas with $\Phi(\mathfrak{C}, \bar{a}) \cap X = Z^\gamma_{\bar{a}}$. By compactness choose $\varphi(x, \bar{a})$ such that $X^1_{a_0} \setminus Z^\gamma_{\bar{a}} \subseteq \varphi(\mathfrak{C}, \bar{a}) \cap X \subseteq Z^\beta_{\bar{a}}$. Using the definition of $Z^\alpha_{\bar{a}}$ we get a bound $m < \omega$ on $d(a_0, b)$ for $b \in X \cap \varphi(\mathfrak{C}, \bar{a}) \setminus Z^\gamma_{\bar{a}}$. We prove that

(∗) there are finitely many tuples $\bar{a}^j = \langle a^j_i \rangle_{i<k}$, $j < n$ (for some n), realizing $\mathrm{tp}(\bar{a})$ and such that $X \subseteq \bigcup_{j<n} (Z^\gamma_{\bar{a}^j} \cup \varphi(\mathfrak{C}, \bar{a}^j))$.

Suppose not. Then we find $\bar{a}^j$, $j < \omega$, such that $a^j_0 \in X$, $\mathrm{tp}(\bar{a}^j) = \mathrm{tp}(\bar{a})$ and $a^j_0 \notin \bigcup_{i<j} (\Phi(\mathfrak{C}, \bar{a}^i) \cup \varphi(\mathfrak{C}, \bar{a}^i))$. By Ramsey's theorem we may assume that the sequence $\langle \bar{a}^j \rangle_{j<\omega}$ is indiscernible. But then $d(a^0_0, a^1_0) = 1$, hence $a^1_0 \in \Phi(\mathfrak{C}, \bar{a}^0) \cup \varphi(\mathfrak{C}, \bar{a}^0)$, a contradiction.

Choose $\bar{a}^0, \ldots, \bar{a}^{n-1}$ as in (∗) and let $\bar{a}' = \langle a'_i \rangle_{i<kn}$ be the concatenation of $\bar{a}^0, \ldots, \bar{a}^{n-1}$. We see that $X \subseteq Z^\beta_{\bar{a}'}$. By the choice of $\bar{a}$, $X^1_{a'_0} \not\subseteq Z^\gamma_{\bar{a}'}$, hence β is the height of X over $\bar{a}'$. Also, $\bigcup_{j<n} Z^\gamma_{\bar{a}^j} \subseteq Z^\gamma_{\bar{a}'}$, hence $X \setminus Z^\gamma_{\bar{a}'} \subseteq \bigcup_{j<n} \varphi(\mathfrak{C}, \bar{a}^j)$. Let $l = \max\{d(a^0_0, a^j_0) : j < n\}$. By the triangle inequality, $m + l$ is a bound on $d(a'_0, b), b \in X \setminus Z^\gamma_{\bar{a}'}$. ■

Clearly any Lascar strong type of finite diameter is analyzable and has height 0.

THEOREM 2.2. *No Lascar strong type of infinite diameter is analyzable.*

Proof. Suppose for a contradiction that X is an analyzable Lascar strong type of infinite diameter. By Lemma 2.1 choose $\bar{a}$ such that the height of X over $\bar{a}$ is a successor ordinal $\beta^* = \alpha^* + 1$ and there is a bound on $d(a_0, b)$ for $b \in X \setminus Z^{\alpha^*}_{\bar{a}}$.

Now essentially we may repeat the proof of Proposition 1.4, reaching a contradiction. For example, for $b \in X$ let $Y_b = \{\mathrm{tp}(c/b) : c \in X\}$. By analyzability, the set

$$Z^0_b = \{r \in Y_b : \varphi(\mathfrak{C}) \cap X \subseteq X^n_b \text{ for some } \varphi(x) \in r \text{ and } n < \omega\}$$

is open and dense in Y_b. We leave the details to the reader. ■

We say that a countable theory T is *small* if $S(A)$ is countable for every finite $A \subseteq \mathfrak{C}$.

COROLLARY 2.3. *Assume T is small. Then $\overset{\mathrm{Ls}}{\equiv}$ and $\overset{\mathrm{bd}}{\equiv}$ agree on finite tuples and* $\mathrm{Autf_L}(\mathfrak{C})$ *is dense in* $\mathrm{Autf_{KP}}(\mathfrak{C})$.

Proof. The first clause is equivalent to the second one. Choose a Lascar strong type X of a finite tuple a. Let $Y = \{\mathrm{tp}(b/a) : b \in X\}$ and $Y^n = \{\mathrm{tp}(b/a) : b \in X^n_a\}$. Then $Y = \bigcup_n Y^n$ is an F_σ-subset of $S(a)$. But since $S(a)$ is countable, every subset of $S(a)$ is also a G_δ-set. Hence as noticed after the proof of Lemma 1.3, $S(a)$ is analyzable with respect to $\{Y^n : n \leq \omega\}$, where $Y^\omega = S(a) \setminus Y$. It follows that X is analyzable, hence has finite diameter and is the $\overset{\text{bd}}{\equiv}$-class of a. □

In [1] there is an example of a small theory where $\overset{\text{Ls}}{\equiv}$ and $\overset{\text{bd}}{\equiv}$ differ (on infinite tuples; we mentioned it before Corollary 1.9), so Corollary 2.3 is sharp. In this example the height of the Lascar strong type with infinite diameter equals -1. Corollary 2.3 should be compared with a result of Kim [2], who proves that in a small theory $\overset{\text{bd}}{\equiv}$ equals $\equiv$ (equality of types; another proof is given in [3]). A. Ivanov has found an $\aleph_0$-categorical theory where $\overset{\text{Ls}}{\equiv}$ and $\overset{\text{bd}}{\equiv}$ differ. Still, no theory is known where $\overset{\text{Ls}}{\equiv}$ and $\overset{\text{bd}}{\equiv}$ differ and there is a finite bound on the diameter of all type-definable Lascar strong types.

In [1] the authors conjecture that if $\overset{\text{bd}}{\equiv}$ and $\overset{\text{Ls}}{\equiv}$ differ, then $\overset{\text{Ls}}{\equiv}$ should be complicated from the Borel point of view. Theorem 2.2 supports this conjecture. For example, assume X is a Lascar strong type with infinite diameter. Then by the proof of Corollary 2.3, $S(a)$ is not analyzable with respect to $\{Y^n : n \leq \omega\}$, where $a \in X$. In particular, Y is not a G_δ-subset of $S(a)$.

More generally, let M be any model of T and let $g : \mathfrak{C} \to S(M)$ be the function defined by $g(a) = \mathrm{tp}(a/M)$. If $\mathrm{tp}(a/M) = \mathrm{tp}(b/M)$, then $d(a,b) \leq 1$, hence each Lascar strong type is type-definable over M. For $p, q \in S(M)$ let $d(p,q) = \inf\{d(a,b) : a \models p,\ b \models q\}$. Define $\overset{\text{Ls}}{\equiv}$ on $S(M)$ by $p \overset{\text{Ls}}{\equiv} q \Leftrightarrow d(p,q) < \infty$. For each $p \in S(M)$, the set $Y^n_p = \{q \in S(M) : d(p,q) \leq n\}$ is closed (and equals $g(X^n_a)$ for every $a \models p$), hence $\overset{\text{Ls}}{\equiv}$ is an F_σ-equivalence relation on $S(M)$, and for every $a, b \in \mathfrak{C}$, $a \overset{\text{Ls}}{\equiv} b \Leftrightarrow \mathrm{tp}(a/M) \overset{\text{Ls}}{\equiv} \mathrm{tp}(b/M)$.

Let $Y = \{\mathrm{tp}(a/M) : a \in X\}$ and let $p \in Y$. Then by Lemma 1.3 (using g), $S(M)$ is not analyzable with respect to $\{Y^n_p : n \leq \omega\}$, where $Y^\omega_p = S(M) \setminus \bigcup_{n<\omega} Y^n_p$. In particular, Y is not a G_δ-subset of $S(M)$.

The last results may be generalized to an arbitrary bounded $\bigvee\!\bigwedge$-definable equivalence relation E refining $\equiv$, but the assumption of boundedness is essential. For example, in an algebraically closed field K consider the relation $x \sim y \Leftrightarrow x$ and y are interalgebraic. The equivalence classes of $\sim$ are analyzable and of infinite diameter.

3. The methods developed in this paper apply to yet another context. Assume $G \subseteq \mathfrak{C}$ is a 0-type-definable group and H is a subgroup of G generated (as a group) by countably many 0-type-definable sets $V_n, n < \omega$. For $x, y \in G$ let $x \equiv_H y \Leftrightarrow xH = yH$. So $\equiv_H$ is an equivalence relation on G whose classes are the right cosets of H.

When G is definable, our methods apply to $\equiv_H$ almost directly. Namely, let G^* be an auxiliary copy of G on which G acts by right translation, denoted by $*$. Consider the 2-sorted structure $\mathfrak{C}^* = (G, G^*, *)$, where G is equipped with the structure induced from $\mathfrak{C}$ and there is no structure on G^*, except for the action $*$. Then in $\mathfrak{C}^*$, G^* is the set of realizations of a complete isolated type p^*, and the orbit relation on G^* defined by $x\,E\,y \Leftrightarrow (\exists g \in H)(x * g = y)$ is an $\bigvee\bigwedge$-relation. So our previous results apply.

In general we cannot associate with G its affine copy so smoothly. Still, G acts transitively on itself by right translation, and this makes it similar to the set of realizations of a complete type (on which $\mathrm{Aut}(\mathfrak{C})$ acts transitively). So we have the following result.

Theorem 3.1. *Assume G is a 0-type-definable group and H is a subgroup of G generated by countably many 0-type-definable sets V_n, $n < \omega$.*

(1) *If H is type-definable, then H is generated by finitely many of the sets V_n, in finitely many steps.*

(2) *If H is not type-definable, then $[G : H] \geq 2^{\aleph_0}$. If moreover T is small and G consists of finite tuples, then $[G : H]$ is unbounded.*

Proof. Let W_n, $n < \omega$, be an increasing sequence of 0-type-definable subsets of G such that $H = \bigcup_n W_n$, $W_0 = \{e\}$, $W_n = W_n^{-1}$ and $W_n \cdot W_n \subseteq W_{n+1}$. For $x, y \in G$ define $d(x, y)$ as the minimal n such that $x^{-1}y \in W_n$. If no such n exists, we put $d(x, y) = \infty$. So d is a distance function on G, which is invariant under left translation. The theorem may be restated as follows.

(a) If the diameter of H is infinite, then H is not type-definable and $[G : H] \geq 2^{\aleph_0}$.

(b) If moreover T is small, then $[G : H]$ is unbounded.

(a) corresponds to Theorem 1.1 and Proposition 1.4, while (b) corresponds to Theorem 2.2 and Corollary 2.3. We will sketch the proof.

For (a) we prove first that $[G : H] \geq 2^{\aleph_0}$. Here we may assume $[G : H]$ is bounded. Let $\bar{a} = \langle a_\alpha \rangle_{\alpha < \mu}$ be a tuple of representatives of the right cosets of H in G such that $a_0 = e$, the neutral element of G (notice that $e \in \mathrm{dcl}(\emptyset)$). We proceed as in the proof of Proposition 1.4, with $X = G$. Claim 1.5 is still true in our present setting: when $b = e$, the proof is the same, and this case implies the general case of an arbitrary $b \in X$ (since left translation by b maps Z^0_e into a subset of Z^0_b).

For the remaining part of (a) suppose that H is type-definable. Then we can replace G by H, getting $[G : H] = 1$ and contradicting $[G : H] \geq 2^{\aleph_0}$.

To prove (b), suppose for a contradiction that $[G : H]$ is bounded. It follows that every infinite indiscernible sequence of elements of G is contained in a single coset of H. So we may assume that if $a, b \in G$ and $\langle b, ba\rangle$ extends to an infinite indiscernible sequence, then $a \in W_1$.

We proceed as in the proofs of Lemma 2.1, Theorem 2.2 and Corollary 2.3, with the following modifications. Let $X = H$. We define subsets X_a^n and $Z_{\bar{a}}^\alpha$ of X for $a \in X$ and finite non-empty tuples $\bar{a} \subset X$ as in Section 2. Notice however the new meaning of d. Also we have:

(c) If $\bar{a} \subseteq \mathrm{dcl}(\bar{a}')$, then $Z_{\bar{a}}^\alpha \subseteq Z_{\bar{a}'}^\alpha$.
(d) For $b \in X,\ b \cdot Z_{\bar{a}}^\alpha \subseteq Z_{\bar{a}^\frown\langle b\rangle}^\alpha$.

We define the height and analyzability of X over $\bar{a}$ as before. The following lemma corresponds to Lemma 2.1. The proof is also similar.

LEMMA 3.2. *Assume X is analyzable. Then for some $\bar{a} \subset X$, the height of X over $\bar{a}$ is a successor $\gamma + 1$ for some $\gamma \in \mathrm{Ord} \cup \{-1\}$ and there is a finite bound on $d(a_0, b)$, $b \in X \setminus Z_{\bar{a}}^\gamma$.*

Proof. For $\bar{a} = \langle a_i\rangle_{i<k} \subset X$ choose a minimal β such that $X_e^1 \subseteq Z_{\bar{a}}^\beta$. Choose $\bar{a}$ so that β is minimal possible. β is a successor, say $\beta = \gamma + 1$. Choose $\Phi(x, \bar{a})$ and $\varphi(x, \bar{a})$ such that $\Phi(G, \bar{a}) \cap X = Z_{\bar{a}}^\gamma$ and $X_e^1 \setminus Z_{\bar{a}}^\gamma \subseteq \varphi(G, \bar{a}) \cap X \subseteq Z_{\bar{a}}^\beta$ (as in Lemma 2.1). Using the definition of $Z_{\bar{a}}^\alpha$, we get a bound $m < \omega$ on $d(a_0, b)$ for $b \in X \cap \varphi(G, \bar{a}) \setminus Z_{\bar{a}}^\gamma$. Notice that if $b \in X$, then by (d) we have

$$b \cdot Z_{\bar{a}}^\gamma = b \cdot \Phi(G, \bar{a}) \cap X \subseteq Z_{\bar{a}^\frown\langle b\rangle}^\gamma,$$

and by the left invariance of d,

$$X_b^1 = b \cdot X_e^1 \subseteq b \cdot (\Phi(G, \bar{a}) \cup \varphi(G, \bar{a})) \cap X \subseteq Z_{\bar{a}^\frown\langle b\rangle}^\beta.$$

We prove that

(∗) there are finitely many elements $b_j \in X$, $j < n$ (for some n), such that $X \subseteq \bigcup_{j<n} (Z_{\bar{a}^\frown\langle b_j\rangle}^\gamma \cup b_j \cdot \varphi(G, \bar{a}))$.

Suppose not. Then we find $b_j \in X,\ j < \omega$, such that

(e) $b_j \notin \bigcup_{i<j} b_i \cdot (\Phi(G, \bar{a}) \cup \varphi(G, \bar{a}))$.

By Ramsey's theorem we may assume (allowing $b_j \in G$) that, in addition to (e), the sequence $\langle b_j\rangle_{j<\omega}$ is indiscernible. But then by the choice of W_1, $d(b_0, b_1) = 1$, hence

$$b_1 \in X_{b_0}^1 \subseteq b_0 \cdot (\Phi(G, \bar{a}) \cup \varphi(G, \bar{a})),$$

a contradiction.

Choose $b_0, \ldots, b_{n-1}$ as in (∗) and let $\bar{a}' = \bar{a}^\frown\langle b_i\rangle_{i<n}$. We see that $X \subseteq Z_{\bar{a}'}^\beta$. By the choice of $\bar{a}$, $X_e^1 \not\subseteq Z_{\bar{a}'}^\gamma$, hence β is the height of X over $\bar{a}'$. Also, $\bigcup_{j<n} Z_{\bar{a}^\frown\langle b_j\rangle}^\gamma \subseteq Z_{\bar{a}'}^\gamma$, hence $X \setminus Z_{\bar{a}'}^\gamma \subseteq \bigcup_{j<n} b_j \cdot \varphi(G, \bar{a})$. The rest is as in the proof of Lemma 2.1. ∎

Using Lemma 3.2 we conclude the proof of (b) as in Theorem 2.2 and Corollary 2.3. ■

Just as in Theorem 1.1, under the assumptions of Theorem 3.1, if $X \subseteq G$ is a type-definable union of a number of right cosets of H, and H is not type-definable, then $|X/H| \geq 2^{\aleph_0}$.

There is a topological counterpart of Theorem 3.1(1). Assume G is a compact topological group and H is a closed subgroup of G generated by closed sets $V_n, n < \omega$. Then by the Baire category theorem H is generated by finitely many of the sets V_n, in finitely many steps.

Theorem 3.1 suggests the possibility of defining a "generic type" in an arbitrary type-definable group.

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