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Small profinite m-stable groups

by

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Abstract. A small profinite m-stable group has an open abelian subgroup of finite $\mathcal{M}$-rank and finite exponent.

1. Introduction. In a series of papers [7]–[12], Ludomir Newelski has developed the theory of multiplicity in analogy to the theory of independence. The basic set-up is that of a profinite structure (which may be thought of as a hyperdefinable set of algebraic hyperimaginaries), where he defines the notion of *m-independence* similarly to forking independence. This notion is automorphism invariant, symmetric, and transitive; if the ambient theory is small (with only countably many pure types), it also satisfies extension over finite sets. The corresponding foundation rank $\mathcal{M}$ has similar properties to Lascar rank in stability theory; a structure is *m-stable* (really, this should be m-*super*stable) if every type has ordinal $\mathcal{M}$-rank. Newelski asked two questions:

(1) $\mathcal{M}$-GAP CONJECTURE: In a small profinite structure, $\mathcal{M}(o)$ is either finite or ∞ for any orbit o.

(2) Does any small profinite group have an open abelian subgroup?

In this paper we shall prove the $\mathcal{M}$-gap conjecture for groups, and answer question (2) affirmatively in the m-stable case. In fact, we show:

THEOREM 1. *A small m-stable profinite group has an open abelian subgroup*, *and is of finite $\mathcal{M}$-rank.*

The line of argument follows the ideas in [4], where it is shown that a supersimple ω-categorical group is finite-by-abelian-by-finite of finite SU-rank (which in turn was inspired by the ω-stable case [1]). It also borrows some techniques of the bad group analysis from [3, 6, 13].

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2. Profinite structures. We shall quickly review the basic definitions and properties we shall use. For a more detailed exposition, the reader may consult [11] or [12].

Definition 1. A *profinite topological space* is a compact Hausdorff topological space U together with a system $(E_i : i < \omega)$ of refining equivalence relations with finitely many classes, such that:

- each E_i is closed (as a subset of U^2 with the product topology),
- the E_i-classes form a basis of open sets for the topology.

(More generally, one should have a directed system of equivalence relations, but we shall restrict ourselves to the countable case.)

Let $\mathrm{Aut}_0^*(U)$ be the topological group of automorphisms of U preserving all equivalence relations $(E_i : i < \omega)$, whose basic open subgroups are the stabilizers of finite subsets of U. A *profinite structure* is a pair $\langle U, \mathrm{Aut}^*(U)\rangle$, where $\mathrm{Aut}^*(U)$ is a closed subgroup of $\mathrm{Aut}_0^*(U)$; the group $\mathrm{Aut}^*(U)$ is called the *structure group*.

For a finite set A of parameters, let $\mathrm{Aut}^*(U/A)$ be the group of automorphisms in $\mathrm{Aut}^*(U)$ fixing A pointwise. A subset X of U is *A-invariant* if it is invariant under $\mathrm{Aut}^*(U/A)$; it is *A-closed* if it is closed and A-invariant. If $A = \emptyset$, it is usually omitted. A set is **-closed* if it is A-closed for some finite A.

If a is a finite tuple of elements of U, the orbit of a under $\mathrm{Aut}^*(U/A)$ is denoted by $o(a/A)$.

Thus A-closed sets correspond to A-type-definable sets in ordinary model theory, and orbits correspond to types; moreover orbits are closed. Note that Newelski says *A-definable* instead of A-closed. As one really should say A-*type*-definable (the complement of a *-closed set need not be *-closed), we prefer our terminology.

Definition 2. A profinite structure is *small* if there are only countably many orbits on finite tuples over $\emptyset$.

Equivalently, we may ask that there are only countably many orbits on finite tuples, or just 1-orbits, over any finite set of parameters.

Remark 2. In a small profinite structure, every A-closed set contains an open orbit over A.

Definition 3. The structure U^{eq} is obtained from U in the following way. For any $\emptyset$-closed equivalence relation on some U^n we adjoin a new (imaginary) sort $U_E = U^n/E$, and a new function $\pi_E : U^n \to U_E$ mapping a tuple to its E-class. U is identified with $U_=$. Then $\mathrm{Aut}^*(U)$ acts continuously on every sort, and hence on U^{eq} (with the disjoint union topology). Every sort (with the induced structure group) is again a profinite structure, and U^{eq} is a many-sorted profinite structure.

FACT 3 [11, Proposition 1.4]. *Let G be a group interpretable in a profinite structure U^{eq}, i.e. its domain and the graphs of multiplication and inversion are A-closed for some finite A. Then G is a profinite group, i.e. there are A-invariant open normal subgroups G_i with $\bigcap_{i<\omega} G_i = \{1\}$ whose cosets form a basis of open sets for a compact Hausdorff topology.*

EXAMPLE. Let G be an ω-saturated ω-homogeneous group (possibly with additional structure), and $(G_i : i < \omega)$ a system of $\emptyset$-definable normal subgroups of finite index. Put $G^0 = \bigcap_{i<\omega} G_i$. Then G/G^0 (with the induced structure group) is a profinite group; if G is small, so is G/G^0. A subset of G/G^0 is A-closed iff it is induced by an A-type-definable subset of G.

From now on, U will denote an infinite small profinite structure, and G an infinite small profinite group. $A, B, \ldots$ will be finite sets of parameters, and $a, b, \ldots$ finite tuples (from U^{eq} or G^{eq}, respectively).

FACT 4 [11, Lemma 2.2 and Proposition 2.3]. *An A-invariant subgroup of G is A-closed. The group generated by any family of A-invariant sets is A-closed, and generated in finitely many steps from finitely many sets. There is no infinite increasing chain of A-invariant subgroups of G. In particular all characteristic subgroups of G are $\emptyset$-closed, and the ascending (upper) central series of G becomes stationary after finitely many steps.*

COROLLARY 5 [11, Proposition 2.4]. *The intersection $G\cap \mathrm{acl}(A)$ is finite for all (finite) A. In particular, G is locally finite.*

Proof. $G \cap \mathrm{acl}(A)$ is an A-invariant subgroup, hence A-closed, and generated in finitely many steps from finitely many finite sets in $G \cap \mathrm{acl}(A)$. ∎

DEFINITION 4. A tuple $a \in U$ is *m-independent* of B over A, denoted by $a \mathop{\smile\!\!\!\!\!|}^{m}_{A} B$, if $o(a/AB)$ is open in $o(a/A)$. The *$\mathcal{M}$-rank* $\mathcal{M}$ is the least function from the collection of all orbits to the ordinals together with ∞ satisfying

$$\mathcal{M}(a/A) \geq \alpha + 1 \text{ if there is } B \supseteq A \text{ with } a \mathop{\not\smile\!\!\!\!\!|}^{m}_{A} B \text{ and } \mathcal{M}(a/B) \geq \alpha.$$

A theory is *m-stable* if every type has ordinal $\mathcal{M}$-rank.

FACT 6 ([11, Fact 1.10], [12, Lemma 1.5]). *In a small profinite structure U,*

(1) *m-independence is symmetric and transitive,*
(2) *if $a \in \mathrm{acl}(A)$, then $a \mathop{\smile\!\!\!\!\!|}^{m}_{A} B$ for all B,*
(3) *for any a, A, B there is some $a' \in o(a/A)$ with $a' \mathop{\smile\!\!\!\!\!|}^{m}_{A} B$,*
(4) *$\mathcal{M}(a/A, b) + \mathcal{M}(b/A) \leq \mathcal{M}(a, b/A) \leq \mathcal{M}(a/A, b) \oplus \mathcal{M}(b/A)$.*

Definition 5. Let H be a *-closed subgroup of G. A *-closed subset X of H is *generic* (for H) if it is open in H. In particular, an orbit is *generic* (for H) if it is open in H.

Generic orbits exist by smallness (Remark 2); it is easy to see that if o and o' are generic orbits for H, then $\mathcal{M}(o) = \mathcal{M}(o')$. We define $\mathcal{M}(H) = \mathcal{M}(o)$, where o is any generic orbit for H. In fact, the same reasoning works for coset spaces G/H, and $\mathcal{M}(G/H) = \mathcal{M}(o)$, where o is any orbit open in G/H.

Remark 7. For two m-independent generic elements g, h of H the inverse g^{-1} and the product gh are both generic, and gh is m-independent of g and of h (over any parameter set A).

Fact 6(4) immediately implies part (1) of Fact 8 below:

Fact 8 [11, Lemma 2.6]. *Let H be a *-closed subgroup of G.*

(1) $\mathcal{M}(H) + \mathcal{M}(G/H) \le \mathcal{M}(G) \le \mathcal{M}(H) \oplus \mathcal{M}(G/H)$.
(2) *H is open in G iff H has finite index in G.*
(3) *If G is m-stable, then H is open in G iff $\mathcal{M}(H) = \mathcal{M}(G)$.*

Hence if G is m-stable, there is no infinite descending chain of *-closed subgroups, each of infinite index in its predecessor.

Here are two results whose proofs are more involved.

Fact 9 [11, Corollary 3.2]. *If G is m-stable, then G has an infinite *-closed abelian subgroup.*

Fact 10 [11, Theorem 3.3]. *If G is m-stable and soluble, then G has an open nilpotent subgroup.*

Recall that two groups are *commensurable* if their intersection has finite index in either of them.

Lemma 11. *Let H_a be an a-closed subgroup of G, and suppose there is $a' \in o(a)$ with $a' \overset{m}{\perp} a$ such that H_a and $H_{a'}$ are commensurable. Let E be the equivalence relation on $o(a)$ given by $E(a', a'')$ if $H_{a'}$ and $H_{a''}$ are commensurable. Then E is closed, with finitely many classes, all of which are open; moreover, there is $n < \omega$ such that if $E(a', a'')$ holds, then $|H_{a'} : H_{a'} \cap H_{a''}| \le n$.*

Proof. Put $Y = o(a'/a)$. By homogeneity, $o(a)$ is covered by $\emptyset$-conjugates of Y; by compactness finitely many conjugates suffice. This shows that E has finitely many classes, which are all open, so E is closed.

Moreover, if $a_1, a_2 \in Y$, then the index of $H_{a_i} \cap H_a$ in H_{a_i} and in H_a equals the index of $H_{a'} \cap H_a$ in H_a and in $H_{a'}$, for $i = 1, 2$. It follows that the index of $H_{a_1} \cap H_{a_2}$ in H_{a_1} and in H_{a_2} is bounded independently of the choice of a_1, a_2. Since the same bound holds for all conjugates of Y, the lemma follows. ∎

Note that the E-class a_E of a is a canonical parameter for the conjugacy class of H_a in G^{eq}.

We finish this section with two purely group-theoretic theorems.

FACT 12 [5, Hauptsatz 7.6]. *Let G be a finite group, and H a proper nontrivial subgroup such that $H \cap H^g = \{1\}$ for all $g \in G - H$. Then $N := G - \bigcup_{g\in G}(H - \{1\})^g$ is a normal subgroup of G with $G = NH$ and $N \cap H = \{1\}$.*

FACT 13 [14, 2, 16, Theorem 4.2.4]. *Let G be any group, and $\mathfrak{H}$ a family of uniformly commensurable subgroups. Then there is a subgroup N of G, a finite extension of a finite intersection of groups in $\mathfrak{H}$ (and hence commensurable with them), such that N is invariant under all automorphisms of G fixing $\mathfrak{H}$ setwise.*

3. Small profinite groups of finite $\mathcal{M}$-rank. Let G be a profinite group.

DEFINITION 6. A subgroup H of G is *minimal* if it is infinite, *-closed, and every *-closed subgroup of infinite index in H is finite.

Note that in an m-stable profinite group every *-closed infinite subgroup of minimal $\mathcal{M}$-rank is minimal, so every *-closed subgroup contains a minimal one. By Fact 9 a minimal group has an open abelian subgroup.

DEFINITION 7. Let A and B be abelian minimal subgroups of G. A *virtual isogeny* f between A and B is a *-closed isomorphism $f : D/K \to I/C$, where D is open in A, I is open in B, and K and C are both finite. Two virtual isogenies f_1 and f_2 are *equivalent*, denoted by $f_1 \sim f_2$, if the derived maps from $D_1 \cap D_2$ to $(I_1 + I_2)/(C_1 + C_2)$ agree on an open subgroup of A.

Note that f_1 and f_2 are equivalent iff their graphs are commensurable. Equivalence of virtual isogenies is a congruence with respect to addition and composition (whenever composition makes sense). Moreover, an open subgroup, or a finite extension of a virtual isogeny (i.e. of the graph, as a subgroup of $A \times B$), is again a virtual isogeny, which is equivalent to the original one.

It is standard that in a minimal group G, the family of virtual autogenies (isogenies from G to G) modulo equivalence, with addition and composition as operations, forms the set of invertible (nonzero) elements of a division ring R. (See [15] for this, and related results on virtual iso- and endogenies in small groups.)

Lemma 14. *If G is small, then R is locally finite; for every a-closed virtual autogeny f_a the equivalence relation $E(x,y)$ on $o(a)$ given by $f_x \sim f_y$ is *-closed and has finitely many classes, which are all open.*

Proof. Let $\overline{f}$ be a finite tuple of virtual autogenies of G, and $\overline{a}$ a finite set of parameters over which $\overline{f}$ is defined. As G is locally finite, we may replace every $f \in \overline{f}$ by a finite extension, and assume that it is defined on the whole of G (we may have to increase $\overline{a}$ to do this). Choose $g \in G$ with $g \mathop{\smile\!\!\!\!\!|}^{m} \overline{a}$. For any $f, f' \in \langle \overline{f} \rangle$ we have $f(g), f'(g) \in \mathrm{acl}(\overline{a}, g) \cap G$, which is finite. But if $f(g) = f'(g)$, then $g \in \ker(f - f')$; as $g \notin \mathrm{acl}(\overline{a})$, the kernel of $f - f'$ must be infinite, whence open by minimality, and $f \sim f'$.

It follows that R is locally finite, whence a (commutative) field. If f_a is an a-closed virtual autogeny, then every $\emptyset$-conjugate of f_a has the same order as f_a modulo equivalence; as there are only finitely many elements in R of that order, there must be $a' \mathop{\smile\!\!\!\!\!|}^{m} a$ with $f_a \sim f_{a'}$. The rest follows from Lemma 11. ∎

In particular, we can consider the equivalence class $(f_a)_\sim$ of a virtual autogeny as an imaginary element a_E.

Theorem 15. *Let G be a small profinite abelian group of finite $\mathcal{M}$-rank. Then any *-closed subgroup of G is commensurable with one invariant over some finite tuple in* $\mathrm{acl}(\emptyset)$.

Proof. Consider first a minimal subgroup A of G; say it is a-closed for some parameter a. By the finiteness of rank, there exist finitely many conjugates of A, say $(A_i : i < n)$, such that every conjugate of A intersects $A^0 := \sum_{i<n} A_i$ in a subgroup of finite index. We may choose the A_i almost linearly independent, i.e. $A_i \cap \sum_{j \neq i} A_j$ is finite for all $i < n$. Fix virtual isogenies f_{ij} from A_i to A_j (whenever they exist), and let $\overline{a}$ be a finite set of parameters over which all of this is invariant.

Now consider another conjugate A' of A. Since $A' \cap A^0$ is infinite by maximality of n, there must be some minimal $i = i(A') < n$ such that $A_i \cap (A' + \sum_{k \neq i} A_k)$ is infinite; because A and therefore A_i are both minimal, $|A_i : A_i \cap (A' + \sum_{k \neq i} A_k)|$ is finite. For every $j \neq i$ with $A_j \cap (A' + \sum_{k \neq j} A_k)$ infinite we define a virtual isogeny $r(A', j)$ from A_i to A_j via: $r(A', j)(x) := \{y \in A_j : x - y \in A' + \sum_{k \neq i,j} A_k\} = A_j \cap (x + A' + \sum_{k \neq i,j} A_k)$ (it is easy to check from minimality that this is indeed a virtual isogeny). If $A_j \cap (A' + \sum_{k \neq j} A_k)$ is finite, we put $r(A', j) = 0$. Suppose now A'' is such that $i(A'') = i(A')$, and $r(A', j)$ and $r(A'', j)$ are equivalent virtual isogenies for all $j \neq i(A')$ with $r(A', j) \neq 0$ or $r(A'', j) \neq 0$. One can check that then A' and A'' must be commensurable.

By smallness we may choose A' such that $X := o(a'/\overline{a})$ is open in $o(a) = o(a')$ (where the lower case letter denotes the parameter of the upper case group); note that if $a'' \in X$, then $i(A'') = i(A') =: i$. Consider the equivalence relation $F_j(a', a'')$ on X given by $f_{ji} \circ r(A', j) \sim f_{ji} \circ r(A'', j)$ for a fixed j. Since $f_{ji} \circ r(A', j)$ defines a virtual autogeny of A_i for all $a' \in X$, by Lemma 14 there are only finitely many F_j-classes. Hence there is $a'' \in X$ with $a'' \mathop{\perp}^{m}_{\bar a} a'$ such that $F_j(a', a'')$ holds for all j, so A' and A'' are commensurable. But $a' \mathop{\perp}^{m} a''$; by Lemma 11 there are only finitely many commensurability classes among the $\emptyset$-conjugates of A, and each of them is uniformly commensurable.

By Fact 13 there is a *-closed subgroup A^c commensurable with A and invariant under all automorphisms of G fixing the commensurability class of A. In other words, if $e \in \mathrm{acl}(\emptyset)$ is the canonical parameter for the conjugacy class of A, then A^c is e-closed. This proves the assertion for minimal groups.

If $H \leq G$ is *-closed but not minimal, then by m-stability it contains a minimal subgroup A which is commensurable with some $\mathrm{acl}(\emptyset)$-definable A^c. But HA^c/A^c is a subgroup of G/A^c of smaller $\mathcal{M}$-rank; by induction it is commensurable with an e'-closed group H_c/A^c, for some $e' \in \mathrm{acl}(\emptyset)$, whose preimage H^c in G is as required. ∎

Lemma 16. *If G is small and all centralizers of elements have finite index, then G has an open abelian subgroup.*

Proof. As $G \cap \mathrm{acl}(\emptyset)$ is finite, we may replace G by an open subgroup and assume $G \cap \mathrm{acl}(\emptyset) = \{1\}$. For any $g \in G$, since $C_G(g)$ has finite index in G, we get $[g, G] \subseteq \mathrm{acl}(g)$. If $g \mathop{\perp}^{m} g'$, then $[g, g'] \in \mathrm{acl}(g) \cap \mathrm{acl}(g') = \mathrm{acl}(\emptyset) = \{1\}$. Since every element g' of G can be written as $g' = g_1 g_2$ with $g \mathop{\perp}^{m} g_1$ and $g \mathop{\perp}^{m} g_2$, we obtain $[g, g'] = [g, g_1 g_2] = [g, g_2][g, g_1]^{g_2} = 1$. ∎

Proposition 17. *A small profinite group of finite $\mathcal{M}$-rank has an open abelian subgroup.*

Proof. Suppose not, and let G be a counterexample of minimal $\mathcal{M}$-rank possible.

Claim. *G has an open soluble subgroup.*

Proof of Claim. Suppose not. Note that if H were an infinite *-closed subgroup of infinite index in G with open normalizer, then $N_G(H)/H$ and H would have open abelian normal subgroups by inductive hypothesis, and G would have a 2-soluble open subgroup, a contradiction. Let K be the subgroup of all elements g whose centralizer $C_G(g)$ has finite index in G; this subgroup is $\emptyset$-invariant and hence closed by Fact 4. Moreover, K contains all finite subgroups whose normalizer is open in G. As K is characteristic and cannot have finite index by Lemma 16, it is finite; after replacing G by an open subgroup intersecting K trivially, we may assume that G has no nontrivial closed subgroup of infinite index whose normalizer is open in G.

G contains a minimal subgroup, and hence a *-closed abelian subgroup B, say, which we may take of maximal $\mathcal{M}$-rank possible; adding finitely many parameters, we assume B is $\emptyset$-closed. Suppose B' is another *-closed abelian subgroup such that $B \cap B'$ has infinite index in B'. Now $C_G(b)$ is b-closed for any $b \in B \cap B'$; since it contains B and B', it has greater $\mathcal{M}$-rank than B by the $\mathcal{M}$-rank inequalities. It therefore has no open abelian subgroup, and must be of finite index in G by inductive hypothesis, whence $b = 1$.

Let N be the subgroup of all $g \in G$ such that B^g is commensurable with B. It is $\emptyset$-invariant, and hence $\emptyset$-closed by Fact 4; note that the commensurability is uniform by Lemma 11: just consider B^g and $B^{g'}$ for generic m-independent $g, g' \in N$. By Fact 13 there is a *-closed normal subgroup of N commensurable with B, so N cannot be open in G by the first paragraph of the proof of the claim. Hence N has an open abelian subgroup by inductive hypothesis, and B is open in N by maximality of $\mathcal{M}$-rank. It follows that there is an open $H \le G$ such that $N \cap H \le B$. Then $\mathcal{M}(B \cap H) = \mathcal{M}(B)$ and $B \cap H$ is commensurable with $(B \cap H)^g$ if and only if B is commensurable with B^g, i.e. for $g \in N$. As $N \cap H = B \cap H$, we may thus replace G by H and assume that $B \cap B^g = \{1\}$ for any $g \in G - B$.

If G_0 is a finite subgroup of G such that $B_0 := B \cap G_0$ is proper nontrivial, then $B_0 \cap B_0^g = \{1\}$ for all $g \in G_0 - B_0$. Suppose that there is a G-conjugate B^g such that $B_1 := G_0 \cap B^g$ is nontrivial, but not G_0-conjugate to B_0. As B_0 and B_1 are self-normalizing in G_0, and all G_0-conjugates of B_0 or B_1 intersect trivially, we get

$$\begin{aligned} |G_0| &\ge |G_0/B_0|(|B_0| - 1) + |G_0/B_1|(|B_1| - 1) + 1 \\ &\ge 2|G_0| - |G_0/B_0| - |G_0/B_1| + 1, \end{aligned}$$

whence $|G_0/B_0| + |G_0/B_1| > |G_0|$. We may assume $|B_0| \ge |B_1|$, and obtain

$$|G_0/B_1| \ge |G_0/B_0| > |G_0/B_1|(|B_1| - 1) \ge |G_0/B_1|,$$

a contradiction. Hence all B-conjugates intersecting G_0 nontrivially are already conjugate in G_0.

Consider $X := G - \bigcup_{g \in G} (B - \{1\})^g$. By the preceding paragraph, if G_0 is a finite subgroup of G with $B_0 := G_0 \cap B$ nontrivial, then $X \cap G_0 = G_0 - \bigcup_{g \in G_0} (B_0 - \{1\})^g$; by Fact 12 this is a nontrivial normal subgroup of G_0. As G is locally finite, X is a nontrivial normal subgroup of G, which is invariant over the parameters used to define B, and thus *-closed by Fact 4. Since it intersects B trivially, it cannot be open, contradicting the conclusion of the first paragraph of the proof of the claim. This proves the assertion. ■

By Fact 10 we may assume that G is nilpotent.

Claim. *We may assume that $G' \le Z(G)$.*

Proof of Claim. By Fact 4 the subgroups $Z_n(G)$ in the upper central series are $\emptyset$-closed for all $n \geq 1$, and there is some minimal n such that $Z_n(G)$ is infinite. Replacing G by an open subgroup intersecting $Z_{n-1}(G)$ trivially, we may assume $n = 1$. But now $\mathcal{M}(G/Z(G)) < \mathcal{M}(G)$; by inductive hypothesis $G/Z(G)$ has an open abelian subgroup $H/Z(G)$ whose preimage H in G satisfies $H' \leq Z(H)$. ∎

For $g \in G$ put $H_g := \{(hZ(G), [h,g]) : hZ(G) \in G/Z(G)\}$, a subgroup of $G/Z(G) \times Z(G)$. Since $G/Z(G) \times Z(G)$ is abelian, H_g is commensurable with an e-closed group for some $e \in \operatorname{acl}(\emptyset)$ by Theorem 15. If π_1 denotes the projection onto the first coordinate, then $[h,g] = [h,g']$ for any $hZ(G) \in \pi_1(H_g \cap H_{g'})$, whence $[h, g'g^{-1}] = 1$. However, we may choose g and g' to be two independent generic elements such that H_g and $H_{g'}$ are commensurable. Then $\pi_1(H_g \cap H_{g'})$ is a subgroup of finite index in $G/Z(G)$, and $g'g^{-1}$ is a generic element of G with $|G : C_G(g'g^{-1})|$ finite.

The set of all $g \in G$ such that $C_G(g)$ has finite index in G is a subgroup of G, which is $\emptyset$-invariant and closed; since it contains a generic element, it has finite index in G. Replacing G by an open subgroup, we finish by Lemma 16. ∎

DEFINITION 8. A *Morley sequence* in an orbit o over A is a sequence $(a_i : i < \omega)$ of elements in the orbit such that $a_i \mathop{\smile\!\!\!\!\!\!\mid}^{m}_{A} (a_j : j < i)$ and $a_k \in o(a_i/A, a_j : j < i)$ for all $i \leq k < \omega$.

Note that if o is over A, then in an m-stable theory there must be a finite $k < \omega$ such that $\mathcal{M}(A/a_i : i < k)$ is minimal possible. Then $A \mathop{\smile\!\!\!\!\!\!\mid}^{m}_{(a_i : i<k)} a_k$; as $a_k \mathop{\smile\!\!\!\!\!\!\mid}^{m}_{A} (a_i : i < k)$, the orbit $o(a_k/a_i : i < k)$ is *parallel* to o (meaning that they have a common non-m-forking extension).

THEOREM 18. *The $\mathcal{M}$-gap conjecture holds for small profinite groups: There is no orbit o in a small profinite group with $\omega \leq \mathcal{M}(o) < \infty$. In particular, a small profinite m-stable group has finite $\mathcal{M}$-rank.*

Proof. Let G be a small profinite group containing a 1-orbit o of infinite $\mathcal{M}$-rank $\alpha < \infty$. Taking m-forking extensions if necessary, we may assume $\alpha = \omega$; adding parameters, we suppose that o is over $\emptyset$. The subgroup of elements of finite $\mathcal{M}$-rank is $\emptyset$-invariant and hence closed by Fact 4; it follows that there is a bound $n < \omega$ on the $\mathcal{M}$-rank of a 1-orbit over $\emptyset$ of finite $\mathcal{M}$-rank. Let o' be an m-forking extension of o of $\mathcal{M}$-rank $> n$, and $(a_i : i < \omega)$ a Morley sequence in o'. Then there is $k < \omega$ such that $o(a_k/a_i : i < k)$ is parallel to o' and hence has $\mathcal{M}$-rank $> n$; it follows that there is a minimal $k < \omega$ such that over $(a_i : i \leq k)$ there is a 1-orbit of $\mathcal{M}$-rank $> n$. We add $(a_i : i < k)$ to the language. Then n is the maximal $\mathcal{M}$-rank of a 1-orbit of finite $\mathcal{M}$-rank over $\emptyset$, and there is $m > n$ which is the maximal $\mathcal{M}$-rank of a 1-orbit of finite $\mathcal{M}$-rank over a single realization of $o(a_k)$ (which again we call o).

We repeat: Let o' be an m-forking extension of o of $\mathcal{M}$-rank $> m$, and $(a_i : i < \omega)$ a Morley sequence in o'. Let $\overline{a} = (a_i : i < k)$ be a maximal initial segment of $(a_i : i < \omega)$ which is m-independent over $\emptyset$. The groups $H(a_i)$ of elements of finite rank over a_i are closed for all $i \geq k$, and conjugate under $\mathrm{Aut}^*(G/\overline{a})$. Let H be the closed group of elements of finite rank over $\overline{a}, a_k$. Then $H(a_i) \leq H$ for $i \geq k$, so there are only finitely many commensurability classes for $H(a_i)$ with $i \geq k$ by Theorem 15. Hence there are $i > j \geq k$ such that $H(a_i)$ and $H(a_j)$ are commensurable. But $\mathcal{M}(a_i/a_j) \geq \mathcal{M}(o') > m$, so $a_i \overset{m}{\not\perp} a_j$ by the choice of m; by Lemma 11 and Fact 13 there is an imaginary $e \in \mathrm{acl}(a_i) \cap \mathrm{acl}(a_j) = \mathrm{acl}(\emptyset)$ and an e-closed subgroup N commensurable with $H(a_i)$. But then for a generic element $g \in N$ we get $\mathcal{M}(g) = \mathcal{M}(g/e) = \mathcal{M}(N) = \mathcal{M}(H(a_i)) = m > n$, a contradiction. ∎

This concludes the proof of Theorem 1.

Corollary 19. *A small profinite m-stable group has finite exponent.*

Proof. By Theorem 1, we may replace G by an open subgroup and assume it is abelian. Let o be an open orbit in G; by local finiteness its elements have finite order n, say. Then the group generated by o is open in G and has exponent n. ∎

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