FUNDAMENTA
MATHEMATICAE
218 (2012)

Maximal equicontinuous factors and cohomology for tiling spaces

by

Marcy Barge (Bozeman, MT), **Johannes Kellendonk** (Lyon) and **Scott Schmieding** (College Park, MD)

Abstract. We study the homomorphism induced on cohomology by the maximal equicontinuous factor map of a tiling space. We will see that in degree one this map is injective and has torsion free cokernel. We show by example, however, that, in degree one, the cohomology of the maximal equicontinuous factor may not be a direct summand of the tiling cohomology.

1. Introduction. An effective procedure for studying the properties of a tiling, or point-pattern, T of $\mathbb{R}^n$ is to consider the space Ω (called the *hull* of T) of all tilings that, up to translation, are locally indistinguishable from T. Dynamical properties of the action of $\mathbb{R}^n$ on Ω, by translation, correspond to combinatorial properties of T. Regularity assumptions on T guarantee that the dynamical system $(\Omega, \mathbb{R}^n)$ is compact and minimal. There is then a *maximal equicontinuous factor* $(\Omega_{\max}, \mathbb{R}^n)$, with semiconjugacy $\pi : \Omega \to \Omega_{\max}$; $\Omega_{\max}$ is a compact abelian group on which $\mathbb{R}^n$ acts by translation and every equicontinuous factor of $(\Omega, \mathbb{R}^n)$ is a factor of $(\Omega_{\max}, \mathbb{R}^n)$.

The relationship between the hull of a tiling and its maximal equicontinuous factor is of fundamental importance in certain aspects of tiling theory. For example, if T is a (sufficiently well-behaved) distribution of "atoms" in $\mathbb{R}^n$, the diffraction spectrum of T is pure point (that is, T is a perfect quasicrystal) if and only if the dynamical spectrum of $(\Omega, \mathbb{R}^n)$ is pure discrete ([LMS], [D]), if and only if the factor map π is a.e. one-to-one (with respect to Haar measure, [BaKe]).

In this article we study the properties of the homomorphism π^* induced by the factor map π in cohomology. This is directly motivated by a recent formulation of the *Pisot Substitution Conjecture* ([BG]) in terms of

2010 *Mathematics Subject Classification*: Primary 37B50, 37B05; Secondary 54H20, 54H11, 55N05.
Key words and phrases: tiling space, maximal equicontinuous factor.

DOI: 10.4064/fm218-3-3

the homological properties of π^*. More generally, cohomology has long been a primary tool for understanding the structure of Ω ([AP], [S1], [FHK]) and, at least for tilings with a non-trivial discrete component of dynamical spectrum, the pull-back of the cohomology of the maximal equicontinuous factor represents a sort of skeleton supporting the rest of the cohomology of Ω.

The maximal equicontinuous factor of a tiling dynamical system is always a torus or solenoid so its cohomology (as a ring) is determined by its degree one cohomology. Consequently, our focus will be on π^* in degree one (this is also the important degree for deformation theory ([CS], [Ke2]) and the Pisot Substitution Conjecture), though we will have something to say in higher degrees for projection patterns, in which cohomology is tied to complexity. The main result is that π^* is injective in degree one with torsion-free cokernel. We will show by example, however, that the first cohomology of the maximal equicontinuous factor is not necessarily a direct summand of the first cohomology of Ω.

Let us say a few words about our methods. Given a continuous map $f : \Omega \to \mathbb{T}$ of the hull to the unit circle, and a vector $v \in \mathbb{R}^n$, there is a *Schwartzman winding number* $\tau(f)(v)$ of f with respect to the $\mathbb{R}$-action $T' \mapsto T' - tv$ on Ω in direction v ([Sch]). This defines a functional, $v \mapsto \tau(f)(v)$, which depends only on the homotopy class of f. As the group of homotopy classes of maps of Ω to $\mathbb{T}$ is naturally isomorphic with the first integer cohomology $H^1(\Omega)$ of Ω, τ provides a homomorphism from $H^1(\Omega)$ to $\mathbb{R}^{n*}$. We will see that the degree one cohomology of the maximal equicontinuous factor can be identified with the group $\mathcal{E}$ of continuous eigenvalues of the $\mathbb{R}^n$-action on Ω. Each eigenvalue, in turn, determines a functional on $\mathbb{R}^n$. With these identifications, $\tau \circ \pi^*$ is the identity, establishing that π^* is injective in degree one.

The homomorphism τ described above is the degree one part of the Ruelle–Sullivan map ([KP]). In the top degree n, τ has an interpretation as the homomorphism that assigns to each finite patch of a tiling T its frequency of occurrence in T. The range of τ is then the *frequency module* $\mathrm{freq}(\Omega)$ of Ω and its kernel is the group $\mathrm{Inf}(\Omega)$ of *infinitesimals* with respect to a natural order on the top degree cohomology. In the special case of one-dimensional tilings, we have two related short exact sequences with $H^1(\Omega)$ in the middle:

$$0 \to \mathcal{E} \xrightarrow{\pi^*} H^1(\Omega) \to \operatorname{coker} \pi^* \to 0$$

and

$$0 \to \mathrm{Inf}(\Omega) \to H^1(\Omega) \xrightarrow{\tau} \mathrm{freq}(\Omega) \to 0.$$

This situation is considered, in the context of symbolic substitutions, in [AR]. We consider tilings that arise from substitutions, as well as tilings that arise from projection methods. For almost canonical projection tilings, both of these sequences split. We will give conditions under which this is true for substitution tilings, as well as examples in which one, or both, do not split.

In the next section, we briefly review the basics of substitution tilings, projection methods, tiling cohomology, and the construction of the maximal equicontinuous factor. In Section 3 we consider the map induced in cohomology by the maximal equicontinuous factor map, and in Sections 4 and 5 we restrict consideration to almost canonical projection patterns and substitution tilings, respectively.

2. Preliminaries

2.1. Tilings and their properties. We will use the formulations and terminology of [BaKe] and just recall here what is necessary to set up the notation.

An n-dimensional *tiling* is an infinite collection of tiles which cover $\mathbb{R}^n$ and have pairwise disjoint interiors. Here a *tile* is a compact subset of $\mathbb{R}^n$ which is the closure of its interior. A tile may carry a mark in case a distinction between geometrically congruent tiles is necessary. A (finite) *patch* is a finite collection of tiles with pairwise disjoint interiors. Its *diameter* is the diameter of the set covered by its tiles.

The translation group $\mathbb{R}^n$ acts on tiles, patches and tilings as on all geometric objects of $\mathbb{R}^n$ and we denote this action by $t \cdot O$ or $O - t$ with $t \in \mathbb{R}^n$ and O the geometric object. A collection Ω of tilings of $\mathbb{R}^n$ has (translationally) *finite local complexity* (FLC) if for each R there are only finitely many translational equivalence classes of patches $P \subset T \in \Omega$ with diameter smaller than R. A single tiling T has FLC if $\{T\}$ has FLC. Finite local complexity of tilings will be a standing assumption in this article and we will not repeat it.

We say that a collection Ω of tilings of $\mathbb{R}^n$ constitutes an n-dimensional *tiling space* if Ω has FLC, is closed under translation, and is compact in the tiling metric d. In this metric two tilings are close if a small translate of one agrees with the other in a large neighborhood of the origin. The main example of a tiling space is the hull of an FLC tiling T:

$$\Omega_T = \{T' : T' \text{is a tiling of } \mathbb{R}^n \text{ and}$$
$$\text{every patch of } T' \text{ is a translate of a patch of } T\}.$$

If the translation action on Ω is free (i.e., $T - v = T \Rightarrow v = 0$), then Ω is said to be *non-periodic*, and T is called *non-periodic* if its hull is non-periodic.

Of particular interest for us are *repetitive* tilings which have the property that for each finite patch P of T the set of occurrences of translates of P in T is relatively dense. If T is repetitive, then the action of $\mathbb{R}^n$ on Ω_T by translation is minimal.

Another property which we will require occasionally is the existence of frequencies of patches in a tiling. The *frequency* of a patch P (up to translation) in T is the density of the set of occurrences of translates of P in T, and being able to define this properly, independent of the limiting procedure, is equivalent to the unique ergodicity of the dynamical system $(\Omega, \mathbb{R}^n)$. We denote, then, the unique ergodic measure by μ.

Let p be a *puncture map*; that is, p assigns to each tile τ a point $p(\tau) \in \tau$ so that $p(\tau + v) = p(\tau) + v$. If a tiling T has FLC then the set of its punctures $p(T) = \{p(\tau) : \tau \in T\}$ is a *Delone set*, i.e., a subset of $\mathbb{R}^n$ which is uniformly discrete and relatively dense. The puncture map p defines a discrete hull $\Xi = \{T' \in \Omega_T : 0 \in p(T')\}$; Ξ is also refered to as the *canonical transversal* as it is transversal in Ω_T to the $\mathbb{R}^n$-action reducing it to the so-called *tiling groupoid* $\mathcal{G} = \{(\omega, t) \in \Xi \times \mathbb{R}^n : \omega - t \in \Xi\}$ with multiplication $(\omega, t)(\omega', t') = (\omega, t + t')$ provided $\omega' = \omega - t$.

The definitions we have made for tilings all have analogs for Delone sets and whether we deal with tilings or Delone sets is mainly a matter of convenience. One could, for instance, represent a tiling T by the Delone set of its punctures, or a Delone set by its Voronoi tiling, and the topological dynamical systems $(\Omega, \mathbb{R}^n)$ are unchanged. Whereas substitutions are usually and more intuitively presented by tilings, the projection method produces Delone sets which are often referred to as *projection patterns* (or, under more general circumstances, *model sets*).

2.2. Substitution tilings. Suppose that $\mathcal{A} = \{\rho_1, \dots, \rho_k\}$ is a set of translationally inequivalent tiles (called *prototiles*) in $\mathbb{R}^n$ and Λ is an *expanding* linear isomorphism of $\mathbb{R}^n$, that is, all eigenvalues of Λ have modulus strictly greater than 1. A *substitution on $\mathcal{A}$ with expansion Λ* is a function $\Phi : \mathcal{A} \to \{P : P \text{ is a patch in } \mathbb{R}^n\}$ with the properties that, for each $i \in \{1, \dots, k\}$, every tile in $\Phi(\rho_i)$ is a translate of an element of $\mathcal{A}$, and $\Phi(\rho_i)$ covers the same set as $\Lambda(\rho_i)$. Such a substitution naturally extends to patches whose elements are translates of prototiles by $\Phi(\{\rho_{i(j)} + v_j : j \in J\}) := \bigcup_{j \in J}(\Phi(\rho_{i(j)}) + \Lambda v_j)$. A patch P is *allowed for Φ* if there is an $m \geq 1$, an $i \in \{1, \dots, k\}$, and a $v \in \mathbb{R}^n$ with $P \subset \Phi^m(\rho_i) - v$. The *substitution tiling space* associated with Φ is the collection $\Omega_\Phi := \{T : T \text{ is a tiling of } \mathbb{R}^n \text{ and every finite patch in } T \text{ is allowed for } \Phi\}$. Clearly, translation preserves allowed patches, so $\mathbb{R}^n$ acts on Ω_Φ by translation.

The substitution Φ is *primitive* if for each pair ρ_i, ρ_j of prototiles there is a $k \in \mathbb{N}$ so that a translate of ρ_i occurs in $\Phi^k(\rho_j)$. If Φ is primitive then Ω_Φ is repetitive.

If Φ is primitive and Ω_Φ is FLC and non-periodic then Ω_Φ is compact in the tiling metric, $\Phi : \Omega_\Phi \to \Omega_\Phi$ is a homeomorphism, and the translation action on Ω_Φ is minimal and uniquely ergodic [AP, So]. In particular, $\Omega_\Phi = \Omega_T$ for any $T \in \Omega_\Phi$. All substitutions will be assumed to be primitive, aperiodic and FLC.

2.3. Almost canonical projection patterns. We describe here almost canonical projection patterns without going into details which the reader may find in [FHK].

Consider a regular lattice $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^{n^\perp}$ such that $\mathbb{R}^n$ is in irrational position with respect to Γ, and a window K which is a compact polyhedron. Let $\pi^\| : \mathbb{R}^n \times \mathbb{R}^{n^\perp} \to \mathbb{R}^n$ be the projection onto the first factor and $\pi^\perp : \mathbb{R}^n \times \mathbb{R}^{n^\perp} \to \mathbb{R}^{n^\perp}$ be the projection onto the second factor. Define the set S of *singular points* in $\mathbb{R}^{n^\perp}$ by

$$S := \bigcup_{\gamma \in \Gamma} (\partial K - \pi^\perp(\gamma))$$

where ∂K denotes the boundary of K. We assume that

- the restrictions of $\pi^\|$ and $\pi^\perp$ to Γ are one-to-one,
- the restrictions of $\pi^\|$ and $\pi^\perp$ to Γ have dense image,
- there exists a finite set of affine hyperplanes $\{W_i\}_{i \in I}$ of codimension 1 in $R^{n^\perp}$ such that S may be alternatively described as

$$S = \bigcup_{i \in I} \bigcup_{\gamma \in \Gamma} (W_i - \pi^\perp(\gamma)).$$

We call the hyperplanes $W_i - \pi^\perp(\gamma)$, $i \in I, \gamma \in \Gamma$, *cut planes*. By the second assumption S is a dense subset of $\mathbb{R}^{n^\perp}$ but of zero Lebesgue measure. The last assumption means that, given a face f of K, the union of all $\pi^\perp(\Gamma)$-translates of f contains the affine hyperplane spanned by f; in particular the faces of K have rational orientation with respect to $\pi^\perp(\Gamma)$ and the stabilizer $\{\gamma \in \Gamma : W_i - \pi^\perp(\gamma) = W_i\}$ of an affine hyperplane W_i must have rank at least $n^\perp - 1$.

We also assume (for simplicity) that 0 is not a singular point. Then the set

$$P_K := \{\pi^\|(\gamma) : \gamma \in \Gamma, \, \pi^\perp(\gamma) \in K\}$$

is a repetitive Delone set, called the *projection pattern with window K*.

With the above rather restrictive assumptions made on the window K the projection pattern is called *almost canonical*. There are standard ways to turn P_K into a tiling which is mutually locally derivable from P_K. For instance the dual of the Voronoi tiling defined by P_K will do it.

2.4. Tiling cohomology and the order structure on the top degree. We are interested in the cohomology of a tiling (or pattern) T. This cohomology can be defined in various equivalent ways, for instance as the Čech cohomology $H(\Omega)$ of the hull Ω of T or as (continuous cocycle) cohomology $H(\mathcal{G})$ of the tiling groupoid $\mathcal{G}$ (after [Re]). The equivalence between the two formulations of tiling cohomology can be seen by either realizing that Ω is a classifying space for the groupoid, or by a further reduction: From the work of Sadun–Williams [SW] we know that we can deform the tiling into a tiling by decorated cubes without changing the topological structure of the hull (the hull of the tiling by cubes is homeomorphic to the original one). It then follows that the tiling groupoid of the tiling by cubes is a transformation groupoid $\Xi'\times\mathbb{Z}^n$ which is continuously similar to $\mathcal{G}$ [Re, FHK]. Here Ξ' is the canonical transversal of the tiling by cubes. Like Ξ it is a compact totally disconnected space. It then follows quickly that $H(\mathcal{G})\cong H(\Xi'\times\mathbb{Z}^n)$ and, by definition of the groupoid cohomology, $H(\Xi'\times\mathbb{Z}^n)$ is the dynamical cohomology $H(\mathbb{Z}^n, C(\Xi',\mathbb{Z}))$ which is the cohomology of the group $\mathbb{Z}^n$ with coefficients in the integer-valued continuous (and hence locally constant) functions.

Now what the construction of [SW] actually does on the level of spaces is to realize Ω as a fiber bundle over an n-torus whose typical fiber is Ξ' such that the above $\mathbb{Z}^n$ action corresponds to the holonomy action induced by the fundamental group of the torus. In other words, Ω is the mapping torus of that $\mathbb{Z}^n$ action. It follows (as is seen for instance from the Serre spectral sequence) that $H(\mathbb{Z}^n, C(\Xi',\mathbb{Z}))$ is isomorphic to $H(\Omega)$.

In the highest non-vanishing degree, namely in degree n, $H^n(\mathcal{G})$ is the group of co-invariants,

$$H^n(\mathcal{G})\cong C(\Xi,\mathbb{Z})/B,$$

where B is the subgroup generated by differences of indicator functions of the form $1_U - 1_{U-t}$, $U\subset\Xi$ a clopen subset and $t\in\mathbb{R}^n$ such that $U-t\subset\Xi$.

The group of co-invariants carries a natural order: an element is positive whenever it is represented by a positive function in $C(\Xi,\mathbb{Z})$. Moreover, the order structure is preserved under groupoid isomorphism, and hence the ordered group of co-invariants is a topological invariant for the tiling system.

Let us assume that the tiling system is strictly ergodic. Hence the $\mathbb{R}^n$ action on Ω as well as the groupoid action on Ξ are minimal and uniquely ergodic. Let ν be the unique ergodic measure on Ξ. Then $1_U\mapsto\nu(U)$ factors through $C(\Xi,\mathbb{Z})/B$ and hence, combined with the isomorphism $H^n(\Omega)\cong C(\Xi,\mathbb{Z})/B$, defines a group homomorphism

$$\tau : H^n(\Omega)\to\mathbb{R}.$$

Now the order can be described by saying that $x \in H^n(\Omega)$ is positive whenever $\tau(x) \geq 0$. We say that an element x is *infinitesimal* if it is neither strictly positive nor strictly negative, which is hence the case if and only if $\tau(x) = 0$. We denote the set of infinitesimal elements by $\mathrm{Inf}(\Omega)$.

It is well known that a basis of the topology of Ξ is given by the *acceptance domains* on patches, that is, by subsets containing all tilings which have a given patch at the origin. It follows from this (and the unique ergodicity) that $\nu(U_P)$ is the frequency of occurrence of the patch P in T where U_P is the acceptance domain of P. Let us denote by $\mathrm{freq}(\Omega)$ the subgroup of $\mathbb{R}$ generated by the frequencies of finite patches in T. We thus have an exact sequence

$$0 \to \mathrm{Inf}(\Omega) \to H^n(\Omega) \xrightarrow{\tau} \mathrm{freq}(\Omega) \to 0, \tag{1}$$

which splits if $\mathrm{freq}(\Omega)$ is finitely generated.

2.5. The maximal equicontinuous factor and eigenvalues. Let (X, G) be a minimal dynamical system with compact Hausdorff space X and abelian group G action. There is a maximal equicontinuous factor $(X_{\max}, G)$ of that system—unique up to conjugacy—and this factor can be obtained from the continuous eigenvalues of the action. In fact, a *continuous eigenfunction* of a dynamical system (X, G) is a non-zero function $f \in C(X)$ for which there exists a (continuous) character $\chi \in \hat{G}$ such that

$$f(t \cdot x) = \chi(t) f(x).$$

We call χ the *eigenvalue* of f. To stress that this eigenvalue is an eigenvalue to a continuous function (as opposed to an L^2-function) one also calls it a *continuous* eigenvalue. But we will here consider only eigenvalues to continuous eigenfunctions and so drop that adjective.

The set $\mathcal{E}$ of all eigenvalues forms a subgroup of the Pontryagin dual $\hat{G}$ of G. We consider $\mathcal{E}$ with discrete topology. Then the Pontryagin dual $\hat{\mathcal{E}}$ of $\mathcal{E}$ is a compact abelian group and the maximal equicontinuous factor can be identified with it, $X_{\max} \cong \hat{\mathcal{E}}$. The factor map $\pi : X \to \hat{\mathcal{E}}$ is then given by $x \mapsto j_x$, where $j_x : \mathcal{E} \to \mathbb{T}^1$ is defined by $j_x(\chi) = f_\chi(x)$, and the G-action on $\varphi \in \hat{\mathcal{E}}$ is given by $(t \cdot \varphi)(\chi) = \chi(t)\varphi(\chi)$. Here f_χ is the eigenfunction to eigenvalue χ normalized in such a way that $f_\chi(x_0) = 1$ where $x_0 \in X$ is some chosen point used to normalize all eigenfunctions.

3. The factor map and cohomology. We are interested in the map in cohomology induced by the factor map π:

$$\pi^* : H^k(\hat{\mathcal{E}}) \to H^k(X).$$

(If nothing else is said this means integer-valued Čech cohomology.) In particular, we consider the kernel and cokernel of π^*. The situation is extremely simple in degree 0: X and $X_{\max}$ are connected and so their cohomology in degree 0 is $\mathbb{Z}$, and π^* is an isomorphism in that degree. The situation is very complicated in degrees larger than one, and we will only be able to say something for almost canonical projection patterns. This will be done in the next section. In this section we will concentrate on degree 1, which is important for deformation theory [SW, CS, Ke2, Bo] and for the homological version of the Pisot conjecture [BBJS, BG].

3.1. The cohomology of the maximal equicontinuous factor. Note that the group $\mathcal{E}$ of eigenvalues is at most countable. This follows from the fact that $L^2(X,\mu)$ is separable (for any ergodic invariant probability measure μ), and eigenfunctions to distinct eigenvalues are orthogonal in that Hilbert space.

We suppose that $\mathcal{E}$ is torsion free, which is certainly the case if $\hat{G}$ is torsion free, in particular thus if $G=\mathbb{R}^n$. As an abelian group $\mathcal{E}$ is a $\mathbb{Z}$-module and we may consider the exterior algebra $\Lambda\mathcal{E}$, which is a graded ring.

As is well-known, $H^1(S^1)$ is a free abelian group of rank one. We pick a generator $\gamma\in H^1(S^1)$ (which amounts to choosing an orientation). Given an element of $\chi\in\mathcal{E}$, which we may view as a character on $\hat{\mathcal{E}}$, $\chi:\hat{\mathcal{E}}\to S^1$, $\chi^*(\gamma)$ defines an element in $H^1(\hat{\mathcal{E}})$ and thus a group homomorphism $\jmath:\mathcal{E}\to H^1(\hat{\mathcal{E}})$, $\jmath(\chi)=\chi^*(\gamma)$.

Theorem 1. *$\Lambda\jmath:\Lambda\mathcal{E}\to H(\hat{\mathcal{E}})$ is a graded ring isomorphism.*

Proof. As $\mathcal{E}$ is countable and torsion free we can write it as $\mathcal{E}=\varinjlim(\mathcal{E}_n,i_n^{n+1})$ where $\mathcal{E}_n$ is free abelian of finite rank and $i_n^{n+1}:\mathcal{E}_n\to\mathcal{E}_{n+1}$ is an injective group homomorphism (1). We denote by $i_n:\mathcal{E}_n\to\mathcal{E}$ the corresponding group inclusion. It follows that $\hat{\mathcal{E}}=\varprojlim(\hat{\mathcal{E}}_n,\hat{i}_n^{n+1})$. Now $\hat{\mathcal{E}}_n$ is a torus whose dimension equals the rank of $\mathcal{E}_n$ and so its cohomology is generated as a ring by its degree 1 elements, which, in turn, are the elements of the form $\jmath(\chi)$, $\chi\in\mathcal{E}_n$. This shows that $H(\hat{\mathcal{E}}_n)\cong\Lambda\mathcal{E}_n$ with ring isomorphism given by $\Lambda\jmath:\Lambda\mathcal{E}_n\to H(\hat{\mathcal{E}}_n)$. Hence $H(\hat{\mathcal{E}})=\varinjlim H(\mathcal{E}_n,i_n^{n+1*})\cong\varinjlim(\Lambda\mathcal{E}_n,\Lambda i_n^{n+1})=\Lambda\mathcal{E}$. ∎

3.2. Injectivity of π^* in degree one. Let $[X,S^1]$ denote the set of homotopy classes of maps from X to the circle $S^1:=\{z\in\mathbb{C}:|z|=1\}$. This is an abelian group (known as the *Bruschlinsky group* of X) under the operation $[f]+[g]:=[fg]$, and the map $[f]\mapsto f^*(\gamma)$ is a natural isomorphism between $[X,S^1]$ and $H^1(X)$ (see, for example, [PT]). If we take $X=\hat{\mathcal{E}}$, the maximal equicontinuous factor, then $H^1(\hat{\mathcal{E}})=\mathcal{E}$ and the isomorphism $\mathcal{E}\cong[\hat{\mathcal{E}},S^1]$ is given by $\chi\mapsto[\chi]$, where we view χ as a character on $\hat{\mathcal{E}}$ via

(1) $\mathcal{E}=\{g_n,n\in\mathbb{N}\}$ and we may take $\mathcal{E}_n$ to be the group generated by $\{g_1,\dots,g_n\}$.

Pontryagin duality. The naturality of the isomorphism implies that

$$\begin{array}{ccc} H^1(\hat{\mathcal{E}}) & \xrightarrow{\pi^*} & H^1(X) \\ \| & & \| \\ [\hat{\mathcal{E}}, S^1] & \xrightarrow{\pi^*} & [X, S^1] \end{array}$$

commutes.

We now suppose that $G = \mathbb{R}^n$ and $X = \Omega$ is a tiling space. Then $\hat{\mathbb{R}^n}$ is isomorphic to $\mathbb{R}^{n*}$, the dual of $\mathbb{R}^n$ as a vector space, the map $\mathbb{R}^{n*} \ni \beta \mapsto e^{2\pi\imath\beta} \in \hat{\mathbb{R}^n}$ providing a group isomorphism. We define $E = \{\beta : e^{2\pi\imath\beta} \in \mathcal{E}\}$, calling it also the *group of eigenvalues*, let $\imath : E \hookrightarrow \mathbb{R}^{n*}$ be the inclusion, and $\theta : E \to [\Omega, S^1]$ be the composition $\theta(\beta) = \pi^*([e^{2\pi\imath\beta}])$. If f_β is an eigenfunction to eigenvalue β, normalized so that its modulus is everywhere 1, then $\theta(\beta) = [f_\beta]$. Indeed, by minimality any two eigenfunctions differ by a multiplicative constant and hence are homotopic.

The Lie algebra of $G = \mathbb{R}^n$ is $\mathbb{R}^n$. Let $H^k(\mathbb{R}^n, C^\infty(\Omega, \mathbb{R}))$ be the Lie algebra cohomology of $\mathbb{R}^n = \mathrm{Lie}(G)$ with values in $C^\infty(\Omega, \mathbb{R})$ which are continuous functions that are smooth with respect to the derivative d defined by the Lie algebra action. Since any continuous function on Ω can be approximated in the sup norm by a smooth function, we can define a group homomorphism

$$\psi : [\Omega, S^1] \to H^1(\mathbb{R}^n, C^\infty(\Omega, \mathbb{R})), \quad \psi([f]) = \frac{1}{2\pi\imath} f^{-1} df,$$

using a smooth representative. Given an ergodic invariant probability measure μ on Ω we can define the homomorphism

$$C_\mu : H^1(\mathbb{R}^n, C^\infty(\Omega, \mathbb{R})) \to \mathbb{R}^{n*}, \qquad C_\mu(\alpha) = \int_\Omega \alpha(\omega)\, d\mu(\omega).$$

The composition $\tau = \mathcal{C}_\mu \circ \psi$ is the degree 1 part of the Ruelle–Sullivan map of [KP]. We mention that $\tau([f])(v)$ is also known as the *Schwartzman winding number* of f with respect to the $\mathbb{R}$-action $T \mapsto T - tv$ on Ω.

LEMMA 2. *$\tau \circ \theta : E \to \mathbb{R}^{n*}$ is given by $\tau \circ \theta = \imath$.*

Proof. Since $f_\beta^{-1} df_\beta = 2\pi\imath\beta$, a constant function, we have $C_\mu(\psi(\theta(\beta))) = \beta$. ∎

COROLLARY 3. *π^* is injective in degree* 1.

Proof. $\tau \circ \theta$ is injective and factors through the degree 1 part of π^*. ∎

REMARK. Lemma 2 is actually the degree 1 part of a more general result which can be obtained with the help of the full Ruelle–Sullivan map $\tau : H(\Omega) \to \Lambda\mathbb{R}^{n*}$. The composition $\tau \circ \pi^* \circ \Lambda\jmath : \Lambda E \to \Lambda\mathbb{R}^{n*}$ is $\Lambda\imath$. In particular $\operatorname{im} \tau \circ \pi^* = \Lambda\imath(E)$. The proof of these statements is a slight generalization of the one given in [KP, Thm. 13]. In degree n we may identify $\Lambda^n\mathbb{R}^{n*} \cong \mathbb{R}$ and obtain the same map as above (justifying this way the double use of τ in the notation).

3.3. The cokernel of π^* in degree one. We start with the remark that $H^1(\Omega)$ is torsion free, as follows from the universal coefficient theorem. In higher degrees, $H^k(\Omega)$ may contain torsion.

LEMMA 4 (Krasinkiewicz [Kr]). *Suppose that $f : \Omega \to S^1$ is continuous and suppose that there are $0 \neq k \in \mathbb{Z}$ and a continuous $g : \Omega \to S^1$ such that $[f] = k[g] \in [\Omega, S^1]$. Then there is a continuous $\tilde{f} : \Omega \to S^1$ so that $f = \tilde{f}^k$.*

Proof. Let $p_k : S^1 \to S^1$ be the k-fold covering map $p_k(z) := z^k$. By assumption we have $[f] = [p_k \circ g]$. Let $H : \Omega \times I \to S^1$ be a homotopy from $h := p_k \circ g$ to f. Then $\tilde{h} := g$ is a lift of h. Being a covering map, p_k has the homotopy lifting property, so there is a homotopy $\tilde{H} : \Omega \times I \to S^1$ from $\tilde{h}$ to some function $\tilde{f}$ such that $\tilde{H}(\cdot, 0) = \tilde{h}$ and $p_k \circ \tilde{H} = H$. It follows that $p_k \circ \tilde{f} = f$. ■

THEOREM 5. *The cokernel of the homomorphism $\pi^* : H^1(\hat{\mathcal{E}}) \to H^1(\Omega)$ is torsion free.*

Proof. The statement of the theorem is equivalent to saying that the cokernel of $\theta : E \to H^1(\Omega)$ is torsion free. Let $[f] \in [\Omega, S^1] \cong H^1(\Omega)$, $k \in \mathbb{N}$, and $\beta \in E$ be such that $k[f] = \theta(\beta) = [f_\beta]$. By Lemma 4 there is an $\tilde{f}_\beta : \Omega \to S^1$ so that $f_\beta = p_k \circ \tilde{f}_\beta$. Then $k([\tilde{f}_\beta] - [f]) = 0$. Since $H^1(\Omega)$ is torsion free, $[\tilde{f}_\beta] = [f]$. We claim that $\tilde{f}_\beta$ is an eigenfunction with eigenvalue β/k.

By continuity of $\tilde{f}_\beta$ it is enough to verify the equation $\tilde{f}_\beta(T_0 - x) = \exp(2\pi\imath\beta(x)/k)\tilde{f}_\beta(T_0)$ for some T_0. We have $f_\beta(T_0 - x) = \exp(2\pi\imath\beta(x))f_\beta(T_0)$ for all $x \in \mathbb{R}^n$, thus $\tilde{f}_\beta^k(T_0 - x) = \exp(2\pi\imath\beta(x))\tilde{f}_\beta^k(T_0)$. Taking the kth root we obtain $\tilde{f}_\beta(T_0 - x) = u(x)\exp(2\pi\imath\beta(x)/k)\tilde{f}_\beta(T_0)$ where $u(x)$ is a kth root of unity. Continuity of $\tilde{f}_\beta$ requires that $u(x) = 1$. Hence $\tilde{f}_\beta$ is an eigenfunction with eigenvalue β/k. Hence $\theta(\beta/k) = [\tilde{f}_\beta] = [f]$ and $\operatorname{coker}\theta$ is torsion free. ■

COROLLARY 6. *If $\operatorname{coker}\pi^*$ is finitely generated then $H^1(\Omega)$ is isomorphic to the direct sum of E with $\operatorname{coker}\pi^*$.*

Proof. Under the assumption we have $\operatorname{coker}\pi^* \cong \mathbb{Z}^l$ for some finite l, as it is torsion free. ■

If $H^1(\Omega)$ is not finitely generated then it is not always the direct sum of E with $\operatorname{coker}\pi^*$, as the example in 5.3.1 shows.

4. Almost canonical projection patterns. To obtain a Cantor fiber bundle for almost canonical projection tilings we do not actually use the approach of Sadun & Williams via a deformation of the tiling, but rather consider a variant of the "rope dynamical system" of [Ke1] (see [FHK]). This way we obtain a different Cantor fiber bundle, whose fiber we shall denote by $\mathfrak{C}$. The conclusion that tiling cohomology can be formulated as group cohomology of a $\mathbb{Z}^n$-action on $\mathfrak{C}$ remains valid and we have the benefit that the structure of $\mathfrak{C}$ allows for a calculation of the cohomology groups. $\mathfrak{C}$ can be obtained from the set of singular points S by disconnecting $\mathbb{R}^{n^\perp}$ along the cut planes $W_i - \pi^\perp(\gamma)$ and moding out the action of a subgroup of $\pi^\perp(\Gamma)$ of rank $n^\perp$. This subgroup $\mathbb{Z}^{n^\perp}$ should be a direct summand, i.e. $\pi^\perp(\Gamma) = \mathbb{Z}^{n^\perp} \oplus \mathbb{Z}^n$, and it should span $\mathbb{R}^{n^\perp}$, but it can otherwise be chosen arbitrarily. So $\mathfrak{C} = F_c/\mathbb{Z}^{n^\perp}$, where F_c is the so-called cut-up space obtained by disconnecting $\mathbb{R}^{n^\perp}$, and the other summand $\mathbb{Z}^n$ yields the action on $\mathfrak{C}$. We refer the reader to [FHK, GHK2] for the precise definition of the disconnecting procedure, mentioning here only that it can be obtained via an inverse limit: For any finite collection of cut planes, disconnecting $\mathbb{R}^{n^\perp}$ along these cut planes means taking out the cut planes so that the remaining part of $\mathbb{R}^{n^\perp}$ falls into several connected components and then completing separately these connected components to obtain a closed space. The inverse limit is just geared to make that work for infinitely many cut planes.

To do the actual computation it is more convenient to work with homology. Using Poincaré duality for group (co-)homology and the fact that $\mathbb{Z}^{n^\perp}$ acts freely on F_c one obtains

$$H^k(\mathbb{Z}^n, C(\mathfrak{C},\mathbb{Z})) \cong H_{n-k}(\mathbb{Z}^n, C(\mathfrak{C},\mathbb{Z})) \cong H_{n-k}(\Gamma, C_{n^\perp}),$$

where $C_{n^\perp}$ is the $\mathbb{Z}$-module generated by indicator functions on polyhedra whose faces belong to cut planes and $\gamma \in \Gamma$ acts on such a function by pullback of the translation with $\pi^\perp(\gamma)$. Intersections of cut planes are affine subspaces of smaller dimension and we call such an affine space a *singular* space. On each singular subspace L, say of dimension k, we have a similar structure to that on $\mathbb{R}^{n^\perp}$: The intersections of the cut planes with L are affine subspaces of codimension 1 in L. We let C_k be the module generated by indicator functions on k-dimensional polyhedra in a k-dimensional singular space whose faces belong to cut planes.

The polyhedral structure and the fact that $\mathbb{R}^{n^\perp}$ is contractible give rise to an acyclic complex $C_{n^\perp} \to C_{n^\perp-1} \to \cdots \to C_0$ of Γ-modules whose differential is reminiscent of the boundary map in polyhedral complexes. As a result the homology $H_*(\Gamma, C_{n^\perp})$ may be computed by breaking the complex into $n^\perp$ short exact sequences

$$0 \to C_k^0 \to C_k \to C_{k-1}^0 \to 0, \qquad 0 \le k < n^\perp,$$

with C_k^0 equal to the image of the boundary map $\delta : C_{k+1} \to C_k$, which is, of course, the same as the kernel of $\delta : C_k \to C_{k-1}$ (and thus $C_{n^\perp} = C^0_{n^\perp-1}$), and $C^0_{-1} = \mathbb{Z}$. Each such short exact sequence gives rise to a long exact sequence in homology and in particular to a connecting homomorphism $\gamma_k : H_p(\Gamma, C^0_{k-1}) \to H_{p-1}(\Gamma, C^0_k)$. We now recall:

Theorem 7 ([BaKe]). *The maximal equicontinuous factor $\hat{\mathcal{E}}$ of a projection pattern is naturally isomorphic to the torus $\mathbb{R}^n \times \mathbb{R}^{n^\perp}/\Gamma$.*

Hence, upon identifying $\mathbb{R}^n \times \mathbb{R}^{n^\perp}$ with its dual we have $E = \Gamma$ and so we may identify $H^p(\hat{\mathcal{E}}) = \Lambda^p\Gamma = H^p(\Gamma, \mathbb{Z}) \cong H_{n+n^\perp-p}(\Gamma, \mathbb{Z})$, the last identification by Poincaré duality.

Theorem 8 ([GHK2]). *Under the identifications $H^p(\Omega) \cong H_{n-p}(\Gamma, C_{n^\perp})$ and $H^p(\hat{\mathcal{E}}) \cong H_{n+n^\perp-p}(\Gamma, \mathbb{Z})$ the map $\pi^* : H^p(\hat{\mathcal{E}}) \to H^p(\Omega)$ gets identified with the composition of connecting maps $\gamma_{n^\perp-1} \circ \cdots \circ \gamma_0 : H_{n+n^\perp-p}(\Gamma, \mathbb{Z}) \to H_{n-p}(\Gamma, C_{n^\perp})$.*

4.1. Injectivity of π^*. We now have the tools at hand to find out in which degrees π^* is injective. In fact, the long exact sequence in homology corresponding to the above exact sequence is

$$\to H_p(\Gamma, C_k) \xrightarrow{\delta'_*} H_p(\Gamma, C^0_{k-1}) \xrightarrow{\gamma_k} H_{p-1}(\Gamma, C^0_k) \to$$

where δ' is the boundary map with target space restricted to its image. Thus γ_k is injective whenever $\delta'_* = 0$. Now the module C_k decomposes as $C_k = \bigoplus_{\theta \in I_k} C_k^\theta \otimes \mathbb{Z}[\Gamma/\Gamma^\theta]$ where I_k indexes the set of Γ-orbits of singular spaces of dimension k and $\Gamma^\theta \subset \Gamma$ is the subgroup stabilizing the singular space of orbit type θ. It follows that $H_p(\Gamma, C_k) = \bigoplus_{\theta \in I_k} H_p(\Gamma^\theta, C_k^\theta)$. Since the singular spaces which make up C_k^θ are k-dimensional, $H_p(\Gamma^\theta, C_k^\theta) = 0$ if $p > k$. But we also have $H_p(\Gamma^\theta, C_k^\theta) = 0$ if $p > \operatorname{rk}\Gamma^\theta - \dim \mathbb{R}\Gamma^\theta$, because Γ^θ contains a subgroup of rank $\dim \mathbb{R}\Gamma^\theta$ which acts freely on C_k^θ. To summarize

$$H_p(\Gamma, C_k) = 0 \quad \text{ if } p > \min\{k, r_k\}$$

where $r_k = \max_{\theta \in I_k}(\operatorname{rk}\Gamma^\theta - \dim \mathbb{R}\Gamma^\theta)$. In particular, $\gamma_{n^\perp-1} : H_{p+1}(\Gamma, C^0_{n^\perp-2}) \to H_p(\Gamma, C_{n^\perp}) \cong H^{n-p}(\Omega)$ is injective if $p \ge \min\{n^\perp - 1, r_{n^\perp-1}\}$.

We now consider first the case in which the ranks of the stabilizers are minimal. Since the stabilizer of W_i must have rank at least $n^\perp - 1$, the minimal case is $r_{n^\perp-1} = 0$, which then implies that $r_k = 0$ for all k. This is in fact the generic case and it corresponds to the pattern having maximal complexity among almost canonical projection patterns; that is, the growth exponent for the complexity function is $n^\perp n$ [Ju]. We see from the above that $H_p(\Gamma, C_k^\theta) = 0$ if $p > 0$ and therefore π^* is injective in all degrees. But $H^k(\Omega)$ is infinitely generated except if $n^\perp = 1$.

The situation is different if we require that the cohomology is finitely generated. By the results of [FHK] and [Ju] this is precisely the case if $\nu := (n+n^\perp)/n^\perp$ is an integer and the rank of the stabilizer of a singular plane is ν times its dimension, i.e. $r_k = (\nu-1)k \geq k$. We find it interesting to note that this case corresponds to the case of minimal complexity, that is, the number of patches of size R grows polynomially with exponent n [Ju]. This yields the bound that $\pi^* : H^k(\hat{\mathcal{E}}) \to H^k(\Omega)$ is injective if $k \leq n - n^\perp + 1 = (\nu-2)n+1$. Furthermore, the calculations done in [FHK] (for codimension 3 patterns) show that this is the best possible bound: π^* is never injective in degree $k > (\nu-2)n+1$. In particular, for the standard tilings like the Penrose, Ammann–Beenker, Socolar, and the icosahedral tilings, $\nu = 2$, and hence π^* is injective only in degree 0 and 1.

4.2. On the cokernel of π^*. We consider here only the case of finitely generated cohomology. It comes not as a surprise that then $\operatorname{coker}\pi^*$ is also finitely generated (see [FHK]). It can, however, have torsion in higher degrees [GHK1, GHK2]: the Tübingen Triangle Tiling is an example of a 2-dimensional tiling which has torsion in its second cohomology.

A more subtle question is whether π^* is always onto a direct summand, that is, whether the exact sequence

$$0 \to H^k(\hat{\mathcal{E}}) \xrightarrow{\pi^*} H^k(\Omega) \to \operatorname{coker}\pi^* \to 0 \tag{2}$$

splits and hence the torsion in the tiling cohomology agrees with the torsion of $\operatorname{coker}\pi^*$. While this is generally true if $n^\perp \leq 2$, it remains an open question in higher codimension.

4.3. The frequency module. In this section we will prove that the frequency module, $\operatorname{freq}(\Omega)$, of an almost canonical projection pattern is always finitely generated and hence the sequence (2) splits.

We start with some known background material. It is known that the factor map π is almost everywhere one-to-one and the measure on Ω is the push forward of the (normalized) Haar-measure on $\hat{\mathcal{E}}$. This implies that the frequency module is generated by the volumes of all polyhedron in $\mathbb{R}^{n^\perp}$ whose faces lie in S. Here the volume of a polyhedron is measured with the help of the Lebesgue measure normalized so that the window K has volume 1.

THEOREM 9. *The frequency module* $\operatorname{freq}(\Omega)$ *of an almost canonical projection pattern is finitely generated.*

Proof. We call a point $x \in \mathbb{R}^{n^\perp}$ a *cut point* if it is the unique point in the intersection of $n^\perp$ cut planes. Clearly, any polyhedron whose faces lie in S has vertices which are cut points. Given that the cut points are dense we may subdivide any such polyhedron into simplices of dimension $n^\perp$ such that all vertices of the simplices are cut points (we do not care whether the newly introduced faces lie in S). The theorem thus follows if we can show that the $\mathbb{Z}$-module of volumes of all $n^\perp$-simplices whose vertices are cut points is finitely generated.

Denote by $\mathcal{P}$ the set of cut points. We claim that $\mathcal{P}-\mathcal{P}$ is contained in a finitely generated $\mathbb{Z}$-module. Suppose first that $x, y \in \mathcal{P}$ lie in a common singular space L of dimension 1 and both on the intersection with the same class of cut plane, i.e. $\{x\} = L \cap (W_i - \pi^\perp(\gamma_x))$ and $\{y\} = L \cap (W_i - \pi^\perp(\gamma_y))$. Then $x-y$ belongs to $\pi^L_{W_i}(\varGamma)$, the projection along W_i onto L of $\pi^\perp(\varGamma)$. This group is, of course, finitely generated and since there are only finitely many W_i we see that $(\mathcal{P}\cap L)-(\mathcal{P}\cap L)$ is contained in a finitely generated $\mathbb{Z}$-module. Now we can go from any cut point x to any other cut point y along singular lines, and since there are only finitely many directions of singular lines, $x-y$ lies in a finitely generated $\mathbb{Z}$-module M, say.

The volume of an $n^\perp$-simplex with vertices $(x_0, \ldots, x_{n^\perp})$ is one half of the determinant of the $n^\perp$ vectors $x_i - x_0$, $i = 1, \ldots, n^\perp$, which all lie in M. Hence the determinant also lies in a finitely generated $\mathbb{Z}$-module. ■

Corollary 10. *For almost canonical projection patterns the sequence* (2) *splits.*

5. Substitution tilings. We recall quickly how to calculate the cohomology of a substitution tiling space $\varOmega = \varOmega_\varPhi$ referring the reader to [AP] for more details.

The *collared Anderson–Putnam complex* Y is an n-dimensional CW-complex whose n-cells are collared prototiles. Two of these cells are glued along $(n-1)$-faces if some translates of the corresponding collared prototiles meet along a translate of that face in some tiling in $\varOmega$. There is a natural map $p : \varOmega \to Y$ assigning to a tiling the point in Y which corresponds to the position of the origin 0 in the collared tile of the tiling that contains 0. Furthermore, the substitution $\varPhi$ induces a continuous surjection $F : Y \to Y$ with $p \circ \varPhi = F \circ p$. By the universality property of the inverse limit, p induces a map $\hat{p} : \varOmega \to \varprojlim(Y, F)$ where $\varprojlim(Y, F)$ denotes the inverse limit of the stationary system $\cdots \to Y \xrightarrow{F} Y \xrightarrow{F} Y$. It is shown in [AP] that $\hat{p}$ is a homeomorphism that conjugates $\varPhi$ with the shift $\hat{F}$ on $\varprojlim(Y, F)$ and hence $\hat{p} : H^k(\varprojlim(Y, F)) \to H^k(\varOmega)$ is an isomorphism. By the continuity property of Čech cohomology, $H^k(\varprojlim(Y, F))$ is naturally isomorphic with $\varinjlim(H^k(Y), F^*)$. This direct limit is computable and we are interested in the case $k = 1$.

$H^1(Y)$ is a free abelian group of finite rank, and so the stationary system $H^1(Y) \xrightarrow{F^*} H^1(Y) \xrightarrow{F^*} \cdots$ is of the form $\mathbb{Z}^N \xrightarrow{A} \mathbb{Z}^N \xrightarrow{A} \cdots$ for some N and

$N \times N$ integer matrix A. Let $\mathrm{ER}(A) = \bigcap_n A^n \mathbb{Q}^N$ be the eventual range of A. We have $\mathrm{ER}(A) = \mathbb{Q}^N$ if A has non-vanishing determinant, but always $\mathrm{ER}(A) = A^N \mathbb{Q}^N$. Then

$$\varinjlim(\mathbb{Z}^N, A) = \{v \in \mathrm{ER}(A) : \exists n \ A^n v \in \mathbb{Z}^N\} = \bigcup_n \tilde{A}^{-n} \Sigma,$$

where $\Sigma = \mathrm{ER}(A) \cap \mathbb{Z}^N$ and $\tilde{A}$ is the restriction of A to $\mathrm{ER}(A)$.

If the substitution forces its border then the above construction works already if one considers non-collared tiles [AP] for the construction of the Anderson–Putnam complex Y. In the one-dimensional context, that is, for a substitution which can be symbolically defined on an alphabet of N letters, and for a substitution which forces its border in the sense that all substituted tiles start with the same tile and all end with the same tile (that is, the symbolic substitution has a common prefix and a common suffix), one may replace the Anderson–Putnam complex Y simply by a bouquet, X, of circles, one circle for each letter. In this case, $H^1(X) \cong \mathbb{Z}^N$ and the matrix A representing $H^1(X) \xrightarrow{F^*} H^1(X)$ in the basis provided by the cohomology classes of the circles is the transpose of the incidence matrix for the substitution. It turns out that, at least for determining the cohomology, we may also work with the bouquet X as long as the symbolic substitution has either a common prefix or a common suffix [AR, BD1].

5.1. One-dimensional irreducible substitutions. For one-dimensional tilings the first cohomology group arises in both exact sequences, the degree 1 version of the sequence (2), namely

$$0 \to E \xrightarrow{\theta} H^1(\Omega) \to \operatorname{coker}\theta \to 0, \tag{3}$$

and, assuming unique ergodicity, the sequence (1):

$$0 \to \mathrm{Inf}(\Omega) \to H^1(\Omega) \xrightarrow{\tau} \mathrm{freq}(\Omega) \to 0.$$

Some of the main results of [AR] give complete information about the structure of the above sequences in the context of one-dimensional primitive, irreducible substitutions of FLC. The work in [AR] starts with symbolic substitutions which are then realized geometrically by assigning a length to each symbol so as to realize it as an interval. Our case is slightly more restrictive, and can be compared if the lengths of the symbols are obtained from the left Perron–Frobenius vector of the substitution matrix. The results of [AR] require generally that the substitution forces its border on one side in the sense that the symbolic substitution has a common prefix.

A further assumption made on the substitution is that the characteristic polynomial of its substitution matrix is irreducible or, what amounts to the same, the dilation factor λ is an algebraic integer of degree equal to the number of prototiles (letters). One simply says that the substitution is *irreducible* in that case.

Theorem 11 ([AR]). *Consider a one-dimensional substitution tiling with common prefix. Assume furthermore that the substitution is irreducible. Then* $\mathrm{Inf}(\Omega) = 0$. *Moreover, if its dilation factor is a Pisot number then* $\operatorname{coker}\theta = 0$.

Note that combined with Solomyak's result ([So]) on the existence of eigenfunctions, this yields a dichotomy: Either λ is a Pisot number and then $\operatorname{coker}\theta = 0$, or λ is not a Pisot number and then $E = 0$.

Hence we see that in one dimension, under the assumptions of irreducibility and common prefix, the two exact sequences (1) and (3) are completely degenerate. We will see below in the examples that the situation is not at all like this if we look at non-irreducible substitutions. Also, the results on projection patterns indicate that this behavior is restricted to one-dimensional tilings.

5.2. On the splitting of the exact sequence (3). For substitution tilings, there is a simple criterion guaranteeing that the sequence (3) splits. Clearly, if F^* is an isomorphism in degree 1 then $H^1(\Omega) = H^1(Y)$, and since the latter is finitely generated, we deduce from Corollary 6 that the sequence (3) splits. But we can do better.

Recall that the homeomorphism $\Phi : \Omega \to \Omega$ defined by a substitution on the substitution tiling space satisfies

$$\Phi(T - v) = \Phi(T) - \Lambda(v).$$

Let f be an eigenfunction with eigenvalue $\beta \in E$. Then

$$f(\Phi(T - v)) = e^{2\pi\imath\beta(\Lambda(v))} f(\Phi(T)),$$

showing that $f \circ \Phi$ is an eigenfunction with eigenvalue $\Lambda^T\beta$. It follows that $\Phi^*\theta(\beta) = [f \circ \Phi] = \theta(\Lambda^T\beta)$, i.e. θ intertwines the action of Λ^T on E with that of Φ^* on $H^1(\Omega)$. Thus, Φ^* induces a homomorphism $\bar{\Phi}^*$ on $\operatorname{coker}\theta$. We now work with rational coefficients, i.e. rational cohomology. Then, since $H^1(Y;\mathbb{Q})$ is a finite-dimensional vector space, also $H^1(\Omega;\mathbb{Q})$ and thus $\operatorname{coker}_{\mathbb{Q}}\theta = H^1(\Omega;\mathbb{Q})/\theta(E\otimes_{\mathbb{Z}}\mathbb{Q})$ are finite-dimensional and so we can view $\bar{\Phi}^*$ as a finite matrix with rational coefficients, which we denote by $\bar{A}$. Of course, this matrix depends on the choice of basis, but its determinant does not.

Proposition 12. *Suppose that* $\bar{A}$ *is as above and that* $\det(\bar{A}) = \pm 1$. *Then the sequence* (3) *splits with* $H^1(\Omega) \cong E \oplus \mathbb{Z}^l$, $l = \dim\operatorname{coker}_{\mathbb{Q}}\theta$.

Proof. Recall from above that we may identify $H^1(Y) = \mathbb{Z}^N$ and $F^* = A$ so that $H^1(\Omega) \cong \{v \in \mathrm{ER}(A) : \exists n\ A^n v \in \Sigma\}$ where $\Sigma = \mathbb{Z}^N \cap \mathrm{ER}(A)$. We now let $\mathrm{ER}_{\mathbb{Z}} := \{v \in \mathrm{ER}(A) : \exists n\ A^n v \in \Sigma\}$ and denote by $V_{\mathbb{Z}} \subset \mathrm{ER}_{\mathbb{Z}}$ the subgroup corresponding to $\theta(E)$ under the above isomorphism. Then (3) can be identified with

$$0 \to V_{\mathbb{Z}} \hookrightarrow \mathrm{ER}_{\mathbb{Z}} \to \mathrm{ER}_{\mathbb{Z}}/V_{\mathbb{Z}} \to 0 \tag{4}$$

and our aim is to show that there is a splitting map $s : \mathrm{ER}_{\mathbb{Z}} \to V_{\mathbb{Z}}$.

Note that $H^1(\Omega;\mathbb{Q}) \cong \mathrm{ER}(A)$ and let V denote the subspace corresponding to $\theta(E \otimes_{\mathbb{Z}} \mathbb{Q})$ under this isomorphism; V is the rational span of $V_{\mathbb{Z}}$. Then $\bar{A}$ can be seen as the linear map induced on $W = \mathrm{ER}(A)/V$ by A. Let π_W denote the natural projection of $\mathrm{ER}(A)$ onto W and let $\Gamma := \pi_W(\Sigma)$. Then Σ and Γ are forward invariant under A and $\bar{A}$, respectively, and it follows from $\det(\bar{A}) = \pm 1$ that $\bar{A}$ restricts to an isomorphism of Γ. Working with rational vector spaces, the corresponding exact sequence $0 \to V \to \mathrm{ER}(A) \to W \to 0$ splits and there is a linear map $s' : W \to \mathrm{ER}(A)$ such that $\pi_W \circ s' = \mathrm{id}$. Let $\pi_V : \mathrm{ER}(A) \to V$ be the projection onto V with kernel $s'(W)$. We claim that $\pi_V(\mathrm{ER}_{\mathbb{Z}}) = V_{\mathbb{Z}}$, which then shows that the restriction of π_V to $\mathrm{ER}_{\mathbb{Z}}$ is a splitting map s for the sequence (4).

By Theorem 5, $\mathrm{ER}_{\mathbb{Z}}/V_{\mathbb{Z}}$ is torsion free and hence $V_{\mathbb{Z}} = V \cap \mathrm{ER}_{\mathbb{Z}}$. Indeed, if $x \in V \cap \mathrm{ER}_{\mathbb{Z}}$, there is p such that $px \in V_{\mathbb{Z}}$ and thus, if $x \notin V_{\mathbb{Z}}$, it would map to a p-torsion element in the quotient $\mathrm{ER}_{\mathbb{Z}}/V_{\mathbb{Z}}$. So we only need to show that $\pi_V(\mathrm{ER}_{\mathbb{Z}}) \subset \mathrm{ER}_{\mathbb{Z}}$.

Let $x \in \mathrm{ER}_{\mathbb{Z}}$, i.e. $x \in \mathrm{ER}(A)$ and there is n such that $A^n x \in \Sigma$. We have $A^n x = v + \gamma$ with $v = \pi_V(A^n x) \in V$ and $\gamma = s' \circ \pi_W(A^n x) \in s'(\Gamma)$. Hence $\tilde{A}^{-n} A^n x = \tilde{A}^{-n} v + \tilde{A}^{-n}\gamma \in V + s'(\Gamma)$ as $\tilde{A}$, the restriction of A to $\mathrm{ER}(A)$, is an isomorphism of $\mathrm{ER}(A)$ preserving V and inducing an isomorphism $\bar{A}$ on Γ. So we may write $x = v' + \gamma'$ with $v' \in V$ and $\gamma' \in s'(\Gamma)$. Then $v' = \pi_V(x)$ and so we get $A^n \pi_V(x) = A^n(x) - A^n\gamma' \in \Sigma$. ∎

By a result of [KS] (see also [BG]), the collection of eigenvalues of the linear transformation $A|_V$ equals the collection of all algebraic conjugates of the eigenvalues of the linear inflation Λ. Moreover, the multiplicity of any λ as an eigenvalue of Λ is no larger than the multiplicity of λ as an eigenvalue of A. A non-unit determinant of $\bar{A}$ implies the existence of a non-unit (and non-zero) eigenvalue of A that has multiplicity greater than its multiplicity as an eigenvalue of Λ (see the example in 5.3.1).

Corollary 13. *Suppose that every eigenvalue of A that is not an algebraic unit has the same multiplicity as it does as an eigenvalue of Λ. Then the sequence* (3) *splits, as in Proposition* 12.

5.3. Examples of non-irreducible substitutions. We present one-dimensional substitution tilings for which the sequences (1) and/or (3) do not split.

Note that by Lemma 2 the group $\mathrm{Inf}(\Omega)$ of infinitesimal elements necessarily has trivial intersection with the image of θ and so the sum $\mathrm{Inf}(\Omega) + \mathrm{im}\,\theta$ is direct.

5.3.1. *Non-splitting example.* We consider the substitution

$$a \mapsto abb, \qquad b \mapsto aaa,$$

whose letters have equal length when realized as tiles of a one-dimensional tiling. We fix this length to be 1; this fixes the action of $\mathbb{R}$ on the continuous hull Ω. The expansion matrix is $\Lambda = (3)$. By Host's characterization of the eigenvalues, $\beta \in E$ if and only if $\beta 3^n \bmod 1 \xrightarrow{n\to\infty} 0$, which is clearly only possible if $\beta 3^n \in \mathbb{Z}$ for some n and hence $E = \mathbb{Z}[1/3]$. It is not difficult to write down an eigenfunction f_1 for $\beta = 1$: take any $T_0 \in \Omega$ and define $f_1(T_0 - t) = \exp(2\pi\imath t)$. Given that all tiles have length 1, the value of $f_1(T_0 - t)$ depends only on the relative position of the tile on 0 (f is *strongly pattern equivariant*) and hence f extends by continuity to Ω. But the substitution is aperiodic and primitive and hence recognizable. This means that one can recognize the three-letter words in any $T \in \Omega$ which arise from a substitution of a letter and hence also $f_{3^{-n}}(T_0 - t) = \exp(2\pi\imath 3^{-n} t)$ is strongly pattern equivariant and so extends. In particular, and as it should be, $[f_{3^{-n}}] = [f_1 \circ \Phi^n]$.

Note that the substitution has a common prefix. As explained above, the first cohomology group of Ω is therefore given by the direct limit defined by the transpose of the incidence matrix of the substitution, which is

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}.$$

Since this matrix is invertible over the rationals, we obtain

$$\begin{aligned} H^1(\Omega) &\cong \bigcup_{n\in\mathbb{N}} A^{-n}\mathbb{Z}^2 = \bigcup_{n\in\mathbb{N}} \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3^{-n} & 0 \\ 0 & (-2)^{-n} \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}\mathbb{Z}^2 \\ &= \mathbb{Z}[1/3]\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathbb{Z}[1/2]\begin{pmatrix} -2 \\ 3 \end{pmatrix} + \bigcup_{k=1}^{4}\begin{pmatrix} k \\ 0 \end{pmatrix}. \end{aligned}$$

Taking into account the isomorphism between $[\Omega, S^1]$ and $H^1(\Omega)$ it is easily seen that $[f_1]$ corresponds to the element $\binom{1}{1}$ and consequently $[f_{3^{-n}}]$ is represented as $A^{-n}\binom{1}{1} = 3^{-n}\binom{1}{1}$. Thus

$$\theta(E) = \mathbb{Z}[1/3]\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We now consider the map τ, which, according to the general theory [AR], is given by the pairing $\tau(x) = \langle \nu, x\rangle$ with the left Perron–Frobenius eigenvector ν of A normalized to $\nu_1 + \nu_2 = 1$. This is $\nu = \frac{1}{5}(3,2)$. So

$$\tau(A^{-n}v) = \frac{1}{5}\cdot 3^{-n}(3v_1 + 2v_2).$$

It follows that

$$\mathrm{freq}(\Omega) = \mathrm{im}\,\tau = \frac{1}{5}\mathbb{Z}[1/3] \quad \text{and} \quad \mathrm{Inf}(\Omega) = \ker\tau = \mathbb{Z}[1/2]\begin{pmatrix}-2\\3\end{pmatrix}.$$

In particular, $\mathrm{im}\,\theta$ and $\mathrm{Inf}(\Omega)$ are subgroups of $H^1(\Omega)$ with trivial intersection but they generate only a subgroup of index 5 in $H^1(\Omega)$. Likewise, $\tau\circ\theta(E) = \mathbb{Z}[1/3]$ is a subgroup of index 5 in $\mathrm{im}\,\tau$.

Proposition 14. *Neither of the exact sequences* (1) *or* (3) *splits.*

Proof. Suppose the sequence (3) splits. There is then a $\mathbb{Z}$-module homomorphism $s : \bigcup_{n\in\mathbb{N}} A^{-n}\mathbb{Z}^2 \to E$ with $s\circ\theta = \mathrm{id}$. This s extends to a $\mathbb{Q}$-linear map $s : \mathbb{Q}^2 \cong \bigcup_{n\in\mathbb{N}} A^{-n}\mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q} \cong \mathbb{Z}[1/3]\otimes_{\mathbb{Z}}\mathbb{Q}$. There is then a $w\in\mathbb{Q}^2$ so that $s(v) = \langle w, v\rangle$ for all $v\in\mathbb{Q}^2$.

Since each element of $\mathbb{Z}[1/2]\binom{-2}{3}$ is infinitely divisible by 2 in $\bigcup_{n\in\mathbb{N}} A^{-n}\mathbb{Z}^2$, also $s(v) = \langle w, v\rangle$ is infinitely divisible by 2 in $\mathbb{Z}[1/3]$ for each $v\in\mathbb{Z}[1/2]\binom{-2}{3}$. This means that w is orthogonal to $\mathbb{Z}[1/2]\binom{-2}{3}$, say $w = t\binom{3}{2}$ with $t\in\mathbb{Q}$. Then $s\big(\binom{1}{0}\big) = \big\langle t\binom{3}{2}, \binom{1}{0}\big\rangle = 3t$ must lie in $\mathbb{Z}[1/3]$; that is, $t\in\mathbb{Z}[1/3]$. Let $t = t_0/3^{n_0}$ with $t_0, n_0\in\mathbb{Z}$. Now the restriction of s to $\theta(E) = \mathbb{Z}[1/3]\binom{1}{1}$ is surjective (since $s\circ\theta = id$) so there is an $x\in\mathbb{Z}[1/3]\binom{1}{1}$, say $x = x_0/3^{m_0}\binom{1}{1}$, with $x_0, m_0\in\mathbb{Z}$, so that $s\big(x\binom{1}{1}\big) = \big\langle t\binom{3}{2}, x\binom{1}{1}\big\rangle = (t_0/3^{n_0})(x_0/3^{m_0})5 = 1$. But this implies that 5 divides 3.

The argument for the sequence (1) is completely similar, one only has to interchange the roles of the eigenvectors of A. ∎

5.3.2. *Period doubling.* We consider the substitution

$$a\mapsto ab, \qquad b\mapsto aa,$$

whose letters have equal length when realized as tiles of a one-dimensional tiling. We fix this length to be 1; this fixes the action of $\mathbb{R}$ on the continuous hull Ω. The expansion matrix is $\Lambda = (2)$. As above one sees that $E = \mathbb{Z}[1/2]$ and that $f_{2^{-n}}(T_0 - t) = \exp(2\pi\imath 2^{-n}t)$ is strongly pattern equivariant and so extends to an eigenfunction to eigenvalue 2^{-n}.

The transpose of the incidence matrix of the substitution is

$$A = \begin{pmatrix}1 & 1\\ 2 & 0\end{pmatrix}.$$

Since the substitution has a common prefix we get

$$\begin{aligned} H^1(\Omega) \cong \bigcup_{n\in\mathbb{N}} A^{-n}\mathbb{Z}^2 &= \bigcup_{n\in\mathbb{N}} \frac{1}{3}\begin{pmatrix}1 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix}2^{-n} & 0\\ 0 & (-1)^{-n}\end{pmatrix}\begin{pmatrix}2 & 1\\ -1 & 1\end{pmatrix}\mathbb{Z}^2 \\ &= \mathbb{Z}[1/2]\begin{pmatrix}1\\1\end{pmatrix} + \mathbb{Z}\begin{pmatrix}-1\\2\end{pmatrix} + \bigcup_{k=0}^{2}\begin{pmatrix}k\\0\end{pmatrix}. \end{aligned}$$

As above one sees that $[f_{2^{-n}}]$ is represented as $A^{-n}\binom{1}{1} = 2^{-n}\binom{1}{1}$. Thus

$$\theta(E) = \mathbb{Z}[1/2]\begin{pmatrix}1\\1\end{pmatrix}.$$

We now consider the map τ. The left Perron–Frobenius eigenvector of A normalized to $\nu_1 + \nu_2 = 1$ is $\nu = \frac{1}{3}(2,1)$. So

$$\tau(A^{-n}v) = \frac{1}{3}\cdot 2^{-n}(2v_1 + 1v_2).$$

It follows that

$$\mathrm{freq}(\Omega) = \frac{1}{3}\mathbb{Z}[1/2] \quad \text{and} \quad \mathrm{Inf}(\Omega) = \mathbb{Z}\begin{pmatrix}-1\\2\end{pmatrix}.$$

Hence $\operatorname{im}\theta + \mathrm{Inf}(\Omega)$ is a subgroup of index 3 in $H^1(\Omega)$.

Proposition 15. *The exact sequence* (3) *splits and hence* $H^1(\Omega) \cong \mathbb{Z}[1/2] \oplus \mathbb{Z}$. *This splitting does not respect the order as the infinitesimal elements form an index* 3 *subgroup of the second summand.*

Proof. Observe that $\operatorname{coker}\theta$ is given by the quotient $\bigcup_{n\in\mathbb{N}} A^{-n}\mathbb{Z}^2/{\sim}$ where $x \sim y$ if there exist k, n such that $x - y = 2^{-n}\cdot k\binom{1}{1}$. It follows that $\binom{3}{0} \sim \binom{-1}{2}$, $A^{-1}\binom{1}{0} = \binom{0}{1} \sim -\binom{1}{0}$, and $A^{-2}\binom{1}{0} = \frac{1}{2}\binom{1}{-1} \sim \binom{1}{0}$. Thus, $\operatorname{coker}\theta$ is generated by the equivalence class of the element $\binom{1}{0}$. In particular it is finitely generated and so the sequence splits.

Now it is clear that the infinitesimal elements form an index 3 subgroup of the second summand. The only possible orderings on $\mathbb{Z}$ are the trivial order, in which case all elements are infinitesimal, or the standard order, in which case only 0 is infinitesimal. The above splitting of $H^1(\Omega)$ is thus not an order preserving splitting into a direct sum of ordered groups. ∎

Proposition 16. *The exact sequence* (1) *does not split.*

Proof. The proof is like the above: Suppose the sequence

$$0 \to \mathbb{Z} \xrightarrow{\psi} \bigcup_{n\in\mathbb{N}} A^{-n}\mathbb{Z}^2 \to \frac{1}{3}\mathbb{Z}[1/2] \to 0$$

splits, where $\psi(1) = \binom{1}{-2}$. There is then a $\mathbb{Z}$-module homomorphism $s : \bigcup_{n\in\mathbb{N}} A^{-n}\mathbb{Z}^2 \to \mathbb{Z}$ with $s\circ\psi = \mathrm{id}$. This s extends to a $\mathbb{Q}$-linear map $s : \mathbb{Q}^2 \cong \bigcup_{n\in\mathbb{N}} A^{-n}\mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q} \cong \mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Q}$. There is then a $w \in \mathbb{Q}^2$ so that $s(v) = \langle w, v\rangle$ for all $v \in \mathbb{Q}^2$.

Since each element of $\mathbb{Z}[1/2]\binom{1}{1}$ is infinitely divisible by 2 in $\bigcup_{n\in\mathbb{N}} A^{-n}\mathbb{Z}^2$, also $s(v) = \langle w, v\rangle$ is infinitely divisible by 2 in $\mathbb{Z}$ for each $v \in \mathbb{Z}[1/2]\binom{1}{1}$. This means that w is orthogonal to $\binom{1}{1}$, say $w = t\binom{1}{-1}$ with $t \in \mathbb{Q}$. Then $s\big(\binom{1}{0}\big) = \big\langle t\binom{1}{-1}, \binom{1}{0}\big\rangle = t$ must lie in $\mathbb{Z}$; that is, $t \in \mathbb{Z}$. Now the restriction of

s to $\psi(\mathbb{Z}) = \mathbb{Z}\binom{1}{-2}$ is surjective so there is an $x \in \mathbb{Z}$ such that $s\big(x\binom{1}{-2}\big) = \big\langle t\binom{1}{-1}, x\binom{1}{-2}\big\rangle = tx3 = 1$. But this implies that $1/3$ is an integer. ■

5.3.3. *Thue–Morse.* We finally consider the Thue–Morse substitution

$$1 \mapsto 1\bar{1}, \qquad \bar{1} \mapsto \bar{1}1,$$

whose letters have equal length when realized as tiles of a one-dimensional tiling. We fix this length to be 1; this fixes the action of $\mathbb{R}$ on the continuous hull Ω. The expansion matrix is $\Lambda = (2)$.

The substitution does not force its border and so we use the technique of collared tiles (the bracketed tile is the actual tile, the other two are the collar):

$$\begin{aligned} a &:= 1(\bar{1})1, & b &:= \bar{1}(\bar{1})1, & c &:= 1(\bar{1})\bar{1},\\ \bar{a} &:= \bar{1}(1)\bar{1}, & \bar{b} &:= 1(1)\bar{1}, & \bar{c} &:= \bar{1}(1)1. \end{aligned}$$

The Anderson–Putnam complex Γ has six edges, namely the collared tiles which we orient to the right in the tiling, and four vertices $v, \bar{v}, w, \bar{w}$. Indeed w is the vertex at the end of c and the beginning of $\bar{b}$ (and $\bar{w}$ is the vertex at the end of $\bar{c}$ and the beginning of b), and v is the vertex at the end of b, $\bar{a}$ and the beginning of a, c (and $\bar{v}$ is the vertex at the end of $\bar{b}, a$ and the beginning of $\bar{a}, \bar{c}$). The cohomology of Γ is thus that of the complex

$$0 \to \mathbb{Z}^4 \xrightarrow{\delta^T} \mathbb{Z}^6 \to 0,$$

where (2)

$$\delta = \begin{pmatrix} 1 & 1 & 0 & -1 & 0 & -1\\ -1 & 0 & -1 & 1 & 1 & 0\\ 0 & -1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

In particular, $H^1(\Gamma) \cong \mathbb{Z}^3$. We now have to determine the matrix A corresponding to the endomorphism induced by the substitution on $H^1(\Gamma)$. The latter reads on collared tiles as follows:

$$a \mapsto b\bar{c}, \quad b \mapsto a\bar{c}, \quad c \mapsto b\bar{a}, \quad \bar{a} \mapsto \bar{b}c, \quad \bar{b} \mapsto \bar{a}c, \quad \bar{c} \mapsto \bar{b}a,$$

which has incidence matrix $\sigma = (\sigma_{ij})$ (with σ_{ij} equal to the number of tiles of type i in the supertile of type j) given by

$$\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1\\ 1 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 1\\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(2) We use the bases $a, b, c, \bar{a}, \bar{b}, \bar{c}$, and $v, \bar{v}, w, \bar{w}$.

We find that the left eigenvectors of σ which are not in $\operatorname{im}\delta^T$ are:

$$\begin{array}{cl} (1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1) & \text{to eigenvalue } 2, \\ (1 \quad 1 \quad 1 \quad -1 \quad -1 \quad -1) & \text{to eigenvalue } 0, \\ (1 \quad -1 \quad 0 \quad 1 \quad -1 \quad 0) & \text{to eigenvalue } -1. \end{array}$$

It follows that the eventual range of the endomorphsim A induced by σ^T on $\mathbb{Z}^6/\operatorname{im}\delta^T$ has dimension 2 and that the restriction $\tilde{A}$ of A to its essential range is obtained by row-reducing the above left eigenvectors of σ to eigenvalues 2 and -1 with respect to the rows spanning $\operatorname{im}\delta^T + \ker\sigma^T$. The result is $(2 \quad 4 \quad 0 \quad 0 \quad 0 \quad 0)$, the left eigenvector of σ modulo $\langle \operatorname{im}\delta^T, \ker\sigma^T\rangle$ to eigenvalue 2, and $(2 \quad -2 \quad 0 \quad 0 \quad 0 \quad 0)$ the one to eigenvalue -1. It follows that

$$\tilde{A} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}.$$

Hence

$$H^1(\Omega) \cong \bigcup_n \tilde{A}^{-n}\mathbb{Z}^2 = \mathbb{Z}[1/2]\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \bigcup_{k=1}^{2}\begin{pmatrix} k \\ 0 \end{pmatrix}.$$

As above, one sees that $E = \mathbb{Z}[1/2]$ and that $f_{2^{-n}}(T_0 - t) = \exp(2\pi\imath 2^{-n}t)$ is strongly pattern equivariant and so extends to an eigenfunction to eigenvalue 2^{-n}. Furthermore the eigenfunction f_1 represents the class in $H^1(\Omega)$ given by the class of $(1,1,1,1,1,1)^T$ modulo $\langle \operatorname{im}\delta^T, \ker\sigma^T\rangle$ which corresponds to $(2,4)^T$ in $\bigcup_n \tilde{A}^{-n}\mathbb{Z}^2$. Hence

$$\theta(E) = \mathbb{Z}[1/2]\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

More generally, since $\tau(v) = 0$ for any left eigenvector of σ to an eigenvalue different from 2, the map $\tau : \bigcup_n \tilde{A}^{-n}\mathbb{Z}^2 \to \mathbb{R}$ is given by $\tau((2,4)^T) = 1$ and $\tau((1,-1)^T) = 0$. It follows that

$$\tau(A^{-n}v) = \frac{1}{6}\cdot 2^{-n}(v_1 + v_2),$$

$\operatorname{freq}(\Omega) = \frac{1}{3}\mathbb{Z}[1/2]$, and $\operatorname{Inf}(\Omega) = \mathbb{Z}\binom{1}{-1}$. Now the same calculation as for the period doubling sequence yields:

PROPOSITION 17. *The exact sequence* (3) *splits and hence* $H^1(\Omega) \cong \mathbb{Z}[1/2] \oplus \mathbb{Z}$. *This splitting does not respect the order, as the infinitesimal elements form an index* 3 *subgroup of the second summand. The exact sequence* (1) *does not split.*

The factors 1/5, 1/3, and 1/3 that appear in the frequency modules in the examples of 5.3.1, 5.3.2, and 5.3.3 are explained by a *reduced resultant* in [S2]. If the characteristic polynomial of the action of the substitution homeomorphism on $H^1(\Omega)$ is factored as $p(t) = q(t)r(t)$, with $q(t)$ the minimal polynomial of the dilation factor, then $q(t)$ and $r(t)$ are relatively prime. The intersection of $\mathbb{Z}$ with the ideal $(q(t), r(t))$, generated by $q(t)$ and $r(t)$ in $\mathbb{Z}[t]$, is a principal ideal in $\mathbb{Z}$, say $\mathbb{Z} \cap (q(t), r(t)) = (D)$. The integer D is the reduced resultant of $q(t)$ and $r(t)$. For example, in 5.3.1, $p(t) = (t-3)(t+2)$, and $D = 5$, whence the $1/5$. (We are indebted to the referee for pointing this out to us.)

5.4. The action of the substitution on first cohomology. As a final observation, let us consider complex-valued cohomology looking at possibly complex eigenvalues of the action of Φ^* on the first cohomology. It is shown in [BG] that all eigenvalues of Λ are also eigenvalues of Φ^*.

THEOREM 18. *The image of τ is invariant under the action of Λ^T. Moreover, suppose that $\Phi^* x = \lambda x$ for some cohomology element $x \in H^1(\Omega, \mathbb{C})$. If $\tau(x) \neq 0$ then λ must be an eigenvalue of Λ.*

Proof. Note that, by unique ergodicity,

$$C_\mu(\alpha) = \lim_{k\to\infty} \frac{1}{\mu(I_k)} \int_{I_k} \alpha(T_0 - x)\, d\nu(x)$$

where $I_k = [-k, k]^n$ is the cube of side length $2k$ centered at 0 and ν is the Lebesgue measure. This holds for any T_0. We have $\Phi^*(f^{-1}df)(T_0 - x) = \Lambda^T f^{-1}(\Phi(T_0) - \Lambda x) df(\Phi(T_0) - \Lambda x)$. Hence $C_\mu(\Phi^*(f^{-1}df)) = \Lambda^T C_\mu(f^{-1}df)$ implying that $\tau([f \circ \Phi]) = \Lambda^T \tau([f])$. So if x is an eigenvector of Φ^* to eigenvalue λ and $\tau(x) \neq 0$ then $\tau(x)$ is an eigenvector of Λ^T to λ. ∎

Thus, if x is in a generalized eigenspace of Φ^* corresponding to an eigenvalue that is not also an eigenvalue of Λ, then $x \in \ker \tau$. In the case $n = 1$, such $x \in [\Omega, S^1]$ must be infinitesimal (see Subsection 2.4 and the example in 5.3.2).

References

[AP] J. E. Anderson and I. F. Putnam, *Topological invariants for substitution tilings and their associated C^*-algebras*, Ergodic Theory Dynam. Systems 18 (1998), 509–537.

[AR] T. I. Andress and E. A. Robinson, Jr., *The Čech cohomology and the spectrum for* 1*-dimensional tiling systems*, preprint, 2010.

[B] M. Barge, *Subdynamics in tiling spaces*, preprint, 2010.

[BBJS] M. Barge, H. Bruin, L. Jones, and L. Sadun, *Homological Pisot substitutions and exact regularity*, Israel J. Math. 188 (2012), 281–300.

[BD1] M. Barge and B. Diamond, *Cohomology in one-dimensional substitution tiling spaces*, Proc. Amer. Math. Soc. 136 (2008), 2183–2191.

[BD2] M. Barge and B. Diamond, *Proximality in Pisot tiling spaces*, Fund. Math. 194 (2007), 191–238.

[BG] M. Barge and J.-M. Gambaudo, *Geometric realization for substitution tilings*, arXiv:1111.6641.

[BaKe] M. Barge and J. Kellendonk, *Proximality and pure point spectrum for tiling dynamical systems*, arXiv:1108.4065.

[BK] M. Barge and J. Kwapisz, *Geometric theory of unimodular Pisot substitutions*, Amer. J. Math. 128 (2006), 1219–1282.

[BO] M. Barge and C. Olimb, *Asymptotic structure in substitution tiling spaces*, arXiv:1101.4902.

[Bo] H. Boulmezaoud, *Cohomologie des pavages et leurs déformations*, PhD thesis, Univ. Lyon 1, 2010.

[CS] A. Clark and L. Sadun, *When shape matters: deformations of tiling spaces*, Ergodic Theory Dynam. Systems 26 (2006), 69–86.

[D] S. Dworkin, *Spectral theory and x-ray diffraction*, J. Math. Phys. 34 (1993), 2965–2967.

[FHK] A. Forrest, J. Hunton, and J. Kellendonk, *Topological invariants for projection method patterns*, Mem. Amer. Math. Soc. 159 (2002), no. 758.

[GHK1] F. Gähler, J. Hunton, and J. Kellendonk, *Integer Čech cohomology of icosahedral projection tilings*, Z. Kristallogr. 223 (2008), 801–804.

[GHK2] F. Gähler, J. Hunton, and J. Kellendonk, *Integer cohomology for rational projection patterns*, Algebr. Geom. Topol., to appear.

[GM] F. Gähler and G. R. Maloney, *Cohomology of one-dimensional mixed substitution tiling spaces*, arXiv:1112.1475v1 (2011).

[Ju] A. Julien, *Complexity and cohomology for cut-and-projection tilings*, Ergodic Theory Dynam. Systems 30 (2010), 489–523.

[Ke1] J. Kellendonk, *The local structure of tilings and their integer group of coinvariants*, Comm. Math. Phys. 187 (1997), 115–157.

[Ke2] J. Kellendonk, *Pattern equivariant functions, deformations and equivalence of tiling spaces*, Ergodic Theory Dynam. Systems 28 (2008), 1153–1176.

[KP] J. Kellendonk and I. F. Putnam, *The Ruelle–Sullivan map for actions of* $\mathbb{R}^n$, Math. Ann. 334 (2006), 693–711.

[KS] R. Kenyon and B. Solomyak, *On the characterization of expansion maps for self-affine tilings*, Discrete Comput. Geom. 43 (2010), 577–593.

[Kr] J. Krasinkiewicz, *Mappings onto circle-like continua*, Fund. Math. 91 (1976), 39–49.

[LMS] J.-Y. Lee, R. V. Moody, and B. Solomyak, *Consequences of pure point diffraction spectra for multiset substitution systems*, Discrete Comput. Geom. 29 (2003), 525–560.

[PT] W. Parry and S. Tuncel, *Classification Problems in Ergodic Theory*, London Math. Soc. Lecture Note Ser. 67, Cambridge Univ. Press, 1982.

[Re] J. Renault, *A Groupoid Approach to* C^**-algebras*, Lecture Notes in Math. 793, Springer, 1980.

[S1] L. Sadun, *Topology of Tiling Spaces*, Univ. Lecture Ser. 46, Amer. Math. Soc., 2008.

[S2] L. Sadun, *Exact regularity and the cohomology of tiling spaces*, Ergodic Theory Dynam. Systems 31 (2011), 1819–1834.

[SW] L. Sadun and R. F. Williams, *Tiling spaces are Cantor set fiber bundles*, Ergodic Theory Dynam. Systems 23 (2003), 307–316.

[Sch] S. Schwartzman, *Asymptotic cycles*, Ann. of Math. (2) 66 (1957), 270–284.

[So] B. Solomyak, *Dynamics of self-similar tilings*, Ergodic Theory Dynam. Systems 17 (1997), 695–738; Corrections, ibid. 19 (1999), 1685.

Marcy Barge
Department of Mathematics
Montana State University
Bozeman, MT 59717, U.S.A.
E-mail: barge@math.montana.edu

Scott Schmieding
Department of Mathematics
University of Maryland
College Park, MD 20742-4015, U.S.A.
E-mail: kerranti@gmail.com

Johannes Kellendonk
Université de Lyon
Université Claude Bernard Lyon 1
Institut Camille Jordan
CNRS UMR 5208
43 boulevard du 11 novembre 1918
F-69622 Villeurbanne Cedex, France
E-mail: kellendonk@math.univ-lyon1.fr

Received 3 April 2012;
in revised form 20 July 2012