FUNDAMENTA
MATHEMATICAE
218 (2012)

# Transference of weak type bounds of multiparameter ergodic and geometric maximal operators

by

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**Abstract.** Let $U_1, \ldots, U_d$ be a non-periodic collection of commuting measure preserving transformations on a probability space $(\Omega, \Sigma, \mu)$. Also let $\Gamma$ be a nonempty subset of $\mathbb{Z}_+^d$ and $\mathcal{B}$ the associated collection of rectangular parallelepipeds in $\mathbb{R}^d$ with sides parallel to the axes and dimensions of the form $n_1 \times \cdots \times n_d$ with $(n_1, \ldots, n_d) \in \Gamma$. The associated multiparameter geometric and ergodic maximal operators $M_{\mathcal{B}}$ and $M_\Gamma$ are defined respectively on $L^1(\mathbb{R}^d)$ and $L^1(\Omega)$ by

$$M_{\mathcal{B}}g(x) = \sup_{x \in R \in \mathcal{B}} \frac{1}{|R|} \int_R |g(y)|\, dy$$

and

$$M_\Gamma f(\omega) = \sup_{(n_1, \ldots, n_d) \in \Gamma} \frac{1}{n_1 \cdots n_d} \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_d=0}^{n_d-1} |f(U_1^{j_1} \cdots U_d^{j_d} \omega)|.$$

Given a Young function $\Phi$, it is shown that $M_{\mathcal{B}}$ satisfies the weak type estimate

$$|\{x \in \mathbb{R}^d : M_{\mathcal{B}}g(x) > \alpha\}| \le C_{\mathcal{B}} \int_{\mathbb{R}^d} \Phi(c_{\mathcal{B}}|g|/\alpha)$$

for a pair of positive constants $C_{\mathcal{B}}$, $c_{\mathcal{B}}$ if and only if $M_\Gamma$ satisfies a corresponding weak type estimate

$$\mu\{\omega \in \Omega : M_\Gamma f(\omega) > \alpha\} \le C_\Gamma \int_\Omega \Phi(c_\Gamma |f|/\alpha).$$

for a pair of positive constants $C_\Gamma$, $c_\Gamma$. Applications of this transference principle regarding the a.e. convergence of multiparameter ergodic averages associated to rare bases are given.

**1. Introduction.** Mathematicians have long been well aware of a close connection between ergodic theorems and problems in harmonic analysis related to the differentiation of integrals. This connection is associated to the fact that a.e. convergence results in both ergodic theory and differ-

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2010 *Mathematics Subject Classification*: Primary 37A45, 42B25.
*Key words and phrases*: ergodic theory, maximal functions, transference.

DOI: 10.4064/fm218-3-4  

entiation of integrals may be proven using *weak type bounds* of maximal operators. For example, N. Wiener observed in [18] that weak type $(1,1)$ inequalites may be used to prove both the Birkhoff Ergodic Theorem as well as the Lebesgue Differentiation Theorem. Moreover, in [2] A. P. Calderón showed that strong and weak type bounds of maximal operators arising in harmonic analysis can frequently be "transferred" to the ergodic setting. Especially in the context of one-parameter operators, transference has become quite well-understood, the papers by Bellow [1] and Coifman and Weiss [3] providing a nice overview of the subject. However, relatively little attention has been given to issues involving transference in scenarios involving sharp weak type bounds of *multiparameter* geometric and ergodic maximal operators.

The purpose of this paper is to provide a very general transference principle that will enable us to transfer weak type bounds of a wide class of multiparameter geometric maximal operators to corresponding weak type bounds of ergodic maximal operators and vice versa. It is the vice versa that is of primary interest here; we shall see that weak type bounds of multiparameter ergodic maximal functions associated to commuting transformations $U_1, \ldots, U_d$ on a probability space $\Omega$ that form a *non-periodic* collection as defined by Katznelson and Weiss in [13] can be transferred to weak type bounds of associated geometric maximal operators. To the best of our knowledge, this is the first general transference result enabling the transfer of weak type bounds from the ergodic to the geometric setting in the multiparameter case. For applications of this result, we shall see that in the scenario of commuting ergodic non-periodic families the classical Jessen–Marcinkiewicz–Zygmund theorem regarding strong differentiation of integrals [12] is equivalent to a result of Dunford [6] and Zygmund [19] regarding the a.e. convergence of multiparameter ergodic averages, that Stokolos' recent theorem [17] regarding sharp weak type bounds on maximal functions associated to rare bases can be transferred to the ergodic setting, and that $L \log L(\Omega)$ is the largest Orlicz class of functions on $\Omega$ whose two-parameter ergodic Córdoba averages converge a.e.

**2. A general transference principle relating weak type bounds of multiparameter geometric and ergodic maximal operators.** Let $\Gamma$ be a nonempty subset of $\mathbb{Z}_+^d$ and let $\mathcal{B}$ be the associated collection of rectangular parallelepipeds in $\mathbb{R}^d$ with sides parallel to the axes and dimensions of the form $n_1 \times \cdots \times n_d$ with $(n_1, \ldots, n_d) \in \Gamma$. Define the geometric maximal operator $M_{\mathcal{B}}$ associated to $\mathcal{B}$ by

$$M_{\mathcal{B}} f(x) = \sup_{x \in R \in \mathcal{B}} \frac{1}{|R|} \int_R |f(y)|\, dy.$$

Let $U_1, \ldots, U_d$ be measure preserving transformations acting on the probability space $\Omega$. Define the ergodic maximal operator $M_\Gamma$ on $L^1(\Omega)$ by

$$M_\Gamma f(\omega) = \sup_{(n_1,\ldots,n_d)\in\Gamma} \frac{1}{n_1\cdots n_d} \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_d=0}^{n_d-1} |f(U_1^{j_1}\cdots U_d^{j_d}\omega)|.$$

Following the terminology of Katznelson and Weiss [13], we say that a collection $\{U_1, \ldots, U_d\}$ of commuting measure preserving transformations on a measure space $(\Omega, \Sigma, \mu)$ is *non-periodic* if for any $(m_1, \ldots, m_d)$ in $\mathbb{Z}^d \setminus \{(0,\ldots,0)\}$ we have $\mu\{\omega \in \Omega : U_1^{m_1}\cdots U_d^{m_d}\omega = \omega\} = 0$.

The desired general transference principle is the following.

THEOREM 1. *Let $U_1, \ldots, U_d$ be a collection of commuting measure preserving transformations on the probability space $(\Omega, \Sigma, \mu)$, and let $\Phi$ be a Young function.*

(i) *If for every $g \in L^1(\mathbb{R}^d)$ and $\alpha > 0$ we have*

$$|\{x \in \mathbb{R}^d : M_{\mathcal{B}}g(x) > \alpha\}| \le \int_{\mathbb{R}^d} \Phi(|g|/\alpha),$$

*then for all $f \in L^1(\Omega)$ and $\alpha > 0$ we have*

$$\mu\{\omega \in \Omega : M_\Gamma f(\omega) > \alpha\} \le \int_\Omega \Phi(|f|/\alpha).$$

(ii) *Suppose in addition that $U_1, \ldots, U_d$ form a non-periodic collection. If for every $f \in L^1(\Omega)$ and $\alpha > 0$ we have*

$$\mu\{\omega \in \Omega : M_\Gamma f(\omega) > \alpha\} \le \int_\Omega \Phi(|f|/\alpha),$$

*then for every $g \in L^1(\mathbb{R}^d)$ and $\alpha > 0$ we have*

$$|\{x \in \mathbb{R}^d : M_{\mathcal{B}}g(x) > \alpha\}| \le 2^d \int_{\mathbb{R}^d} \Phi(6^d|g|/\alpha).$$

*Proof.* (i) Define the discrete maximal operator $\tilde{M}_\Gamma$ on the set of real-valued functions on $\mathbb{N}^d$ by

$$\tilde{M}_\Gamma f(k_1,\ldots,k_d) = \sup_{(n_1,\ldots,n_d)\in\Gamma} \frac{1}{n_1\cdots n_d} \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_d=0}^{n_d-1} |f(k_1+j_1,\ldots,k_d+j_d)|.$$

Let $h$ be a real-valued function on $\mathbb{N}^d$ and $\alpha > 0$. Define the function $g_h$ on $\mathbb{R}^d$ by

$$g_h(x_1,\ldots,x_d) = h(\lfloor x_1\rfloor,\ldots,\lfloor x_d\rfloor)$$

where $g_h(x_1,\ldots,x_d) = 0$ if any of $x_1,\ldots,x_d$ is less than 0.

Note that if $(n_1,\dots,n_d)\in\Gamma$ and $(k_1,\dots,k_d)\in\mathbb{N}^d$, then

$$\begin{aligned}\frac{1}{n_1\cdots n_d}\sum_{j_1=0}^{n_1-1}\cdots\sum_{j_d=0}^{n_d-1}|h(k_1+j_1,\dots,k_d+j_d)|\\ \le \frac{1}{n_1\cdots n_d}\int\limits_{u_1=k_1}^{k_1+n_1}\cdots\int\limits_{u_d=k_d}^{k_d+n_d}|g_h(u_1,\dots,u_d)|\,du_1\cdots du_d\\ \le M_{\mathcal{B}}g_h(w_1,\dots,w_d)\quad \text{for } k_1\le w_1\le k_1+1,\dots,k_d\le w_d\le k_d+1.\end{aligned}$$

So if $\tilde M_\Gamma h(k_1,\dots,k_d)>\alpha$, then

$$M_{\mathcal{B}}g_h>\alpha\quad \text{on } [k_1,k_1+1]\times\cdots\times[k_d,k_d+1].$$

Hence

$$\begin{aligned}\#\{(k_1,\dots,k_d)\in\mathbb{N}^d:\tilde M_\Gamma h(k_1,\dots,k_d)>\alpha\}\le|\{x\in\mathbb{R}^d:M_{\mathcal{B}}g_h(x)>\alpha\}|\\ \le\int\limits_{\mathbb{R}^d}\Phi(|g_h|/\alpha)=\sum_{(k_1,\dots,k_d)\in\mathbb{N}^d}\Phi(|h(k_1,\dots,k_d)|/\alpha).\end{aligned}$$

In particular, $\tilde M_\Gamma$ satisfies the weak type estimate

$$\begin{aligned}(1)\qquad \#\{(k_1,\dots,k_d)\in\mathbb{N}^d:\tilde M_\Gamma h(k_1,\dots,k_d)>\alpha\}\\ \le\sum_{(k_1,\dots,k_d)\in\mathbb{N}^d}\Phi(|h(k_1,\dots,k_d)|/\alpha).\end{aligned}$$

We now utilize transference principles developed by A. P. Calderón in [2] modified to this multiparameter setting.

Let $f\in L^1(\Omega)$. We associate to $f$ the function $F$ on $\Omega\times\mathbb{N}^d$ given by

$$F(\omega,t_1,\dots,t_d)=f(U_1^{t_1}\cdots U_d^{t_d}\omega).$$

For each $m>0$ we define an associated function $F_m$ on $\Omega\times\mathbb{N}^d$ by

$$F_m(\omega,t_1,\dots,t_d)=F(\omega,t_1,\dots,t_d)\cdot\prod_{i=1}^d\chi_{[0,m)}(t_i).$$

Note that for fixed $0\le t_1,\dots,t_d<m$ the functions

$$F_m(\omega,t_1,\dots,t_d)=F_m(\omega,t_1,\dots,t_d)(\omega)$$

and $f(\omega)$ are equimeasurable on $\Omega$.

For $\epsilon>0$ we define

$$\Gamma_\epsilon=\{(n_1,\dots,n_d)\in\Gamma:n_i<\epsilon\text{ for } i=1,\dots,d\}.$$

Fix now $\omega\in\Omega$, $f\in L^1(\Omega)$. Observe that

$$f(U_1^{t_1}\cdots U_d^{t_d}\omega)=f(U_1^{t_1}\cdots U_d^{t_d}\omega)(t_1,\dots,t_d)$$

provides a function on $\mathbb{N}^d$ on which we may act by $\tilde{M}_\Gamma$ and evaluate at $(n_1, \ldots, n_d)$. Now, if $(k_1, \ldots, k_d) \in \Gamma$,

$$\begin{aligned}\frac{1}{k_1 \cdots k_d} &\sum_{j_1=0}^{k_1-1} \cdots \sum_{j_d=0}^{k_d-1} f(U_1^{j_1} \cdots U_d^{j_d}\omega) \\ &= \frac{1}{k_1 \cdots k_d} \sum_{j_1=0}^{k_1-1} \cdots \sum_{j_d=0}^{k_d-1} f(U_1^{n_1+j_1} \cdots U_d^{n_d+j_d} U_1^{-n_1} \cdots U_d^{-n_d}\omega) \\ &= \frac{1}{k_1 \cdots k_d} \sum_{j_1=0}^{k_1-1} \cdots \sum_{j_d=0}^{k_d-1} F(U_1^{-n_1} \cdots U_d^{-n_d}\omega, n_1 + j_1, \ldots, n_d + j_d),\end{aligned}$$

taking advantage of the fact that $U_1, \ldots, U_d$ commute. As $(k_1, \ldots, k_d)$ are arbitrary in $\Gamma$ we then have, for any given $(n_1, \ldots, n_d) \in \mathbb{N}^d$,

$$M_\Gamma f(\omega) = \tilde{M}_\Gamma F(U_1^{-n_1} \cdots U_d^{-n_d}\omega, k_1, \ldots, k_d)|_{(k_1,\ldots,k_d)=(n_1,\ldots,n_d)}.$$

For notational convenience we will express

$$\tilde{M}_\Gamma F(U_1^{-n_1} \cdots U_d^{-n_d}\omega, k_1, \ldots, k_d)|_{(k_1,\ldots,k_d)=(n_1,\ldots,n_d)}$$

as

$$\tilde{M}_\Gamma F(U^{-n_1} \cdots U^{-n_d}\omega, n_1, \ldots, n_d)$$

and similarly express

$$\tilde{M}_\Gamma F_m(U_1^{-n_1} \cdots U_d^{-n_d}\omega, k_1, \ldots, k_d)|_{(k_1,\ldots,k_d)=(n_1,\ldots,n_d)}$$

as

$$\tilde{M}_\Gamma F_m(U^{-n_1} \cdots U^{-n_d}\omega, n_1, \ldots, n_d).$$

Observe we have shown that for every $(n_1, \ldots, n_d) \in \mathbb{N}^d$ we have

$$\begin{aligned}\mu\{\omega \in \Omega : &M_\Gamma f(\omega) > \alpha\} \\ &= \mu\{\omega \in \Omega : \tilde{M}_\Gamma F(U^{-n_1} \cdots U^{-n_d}\omega, n_1, \ldots, n_d) > \alpha\}.\end{aligned}$$

A similar argument shows that if $0 \le n_i \le a$ for $i = 1, \ldots, d$, we have

$$\begin{aligned}\mu\{\omega \in \Omega : &M_{\Gamma_\epsilon} f(\omega) > \alpha\} \\ &= \mu\{\omega \in \Omega : \tilde{M}_{\Gamma_\epsilon} F_{a+\epsilon}(U^{-n_1} \cdots U^{-n_d}\omega, n_1, \ldots, n_d) > \alpha\}\end{aligned}$$

and so by the Fubini Theorem

$$\begin{aligned}
&\mu\{\omega \in \Omega : M_{\Gamma_\epsilon} f(\omega) > \alpha\} \\
&= \frac{1}{(a+1)^d} \\
&\quad \times \sum_{n_1=0}^{a} \cdots \sum_{n_d=0}^{a} \int_{\Omega} \mu\{\omega \in \Omega : \tilde{M}_\Gamma F_{a+\epsilon}(U^{-n_1} \cdots U^{-n_d}\omega, n_1, \ldots, n_d) > \alpha\} \\
&= \frac{1}{(a+1)^d} \int_{\Omega} \#\{(n_1, \ldots, n_d) \in \mathbb{Z}^d : 0 \le n_i \le a \text{ and} \\
&\qquad\qquad \tilde{M}_\Gamma F_{a+\epsilon}(U^{-n_1} \cdots U^{-n_d}\omega, n_1, \ldots, n_d) > \alpha\}\, d\omega \\
&\le \frac{1}{(a+1)^d} \int_{\Omega} \sum_{j_1=0}^{\infty} \cdots \sum_{j_d=0}^{\infty} \Phi(|F_{a+\epsilon}(U^{-n_1} \cdots U^{-n_d}\omega, j_1, \ldots, j_d)|/\alpha)\, d\omega \\
&\le \frac{1}{(a+1)^d} (a+\epsilon)^d \int_{\Omega} \Phi(|f(\omega)|/\alpha)\, d\omega.
\end{aligned}$$

Taking $a \to \infty$ gives

$$\mu\{\omega \in \Omega : M_{\Gamma_\epsilon} f(\omega) > \alpha\} \le \int_{\Omega} \Phi(|f(\omega)|/\alpha)\, d\omega.$$

Letting now $\epsilon \to \infty$ yields

$$\mu\{\omega \in \Omega : M_\Gamma f(\omega) > \alpha\} \le \int_{\Omega} \Phi(|f|/\alpha),$$

as desired.

(ii) Recall that $\Gamma$ is a nonempty subset of $\mathbb{Z}_+^d$ and $\mathcal{B}$ the associated collection of rectangular parallelepipeds in $\mathbb{R}^d$ with sides parallel to the axes and dimensions of the form $n_1 \times \cdots \times n_d$ with $\mathbf{n} = (n_1, \ldots, n_d) \in \Gamma$. It will be convenient for us to work with the subset $\mathbf{B}$ of $\mathcal{B}$ consisting of all parallelepipeds in $\mathcal{B}$ whose corners have integer coordinates.

For each $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $(\sigma_1, \ldots, \sigma_d) \in \{-1, +1\}^d$, let

$$\begin{aligned}
\mathbf{B}_x^{\sigma_1, \ldots, \sigma_d} = \{R \in \mathbf{B} : x \in R,\ &(x_1 - \epsilon_1\sigma_1, \ldots, x_d - \epsilon_d\sigma_d) \notin R \\
&\text{for any } (0, \ldots, 0) \ne (\epsilon_1, \ldots, \epsilon_d) \in \{0,1\}^d\}.
\end{aligned}$$

For each $(\sigma_1, \ldots, \sigma_d) \in \{-1, +1\}^d$ we define the maximal operator $M_{\mathbf{B}}^{\sigma_1, \ldots, \sigma_d}$ by

$$M_{\mathbf{B}}^{\sigma_1, \ldots, \sigma_d} g(x) = \sup_{x \in R \in \mathbf{B}_x^{\sigma_1, \ldots, \sigma_d}} \frac{1}{|R|} \int_R |g(y)|\, dy.$$

As any rectangle $R$ in $\mathcal{B}$ may be covered by a rectangle $\tilde{R}$ in $\mathbf{B}$ satisfying $|\tilde{R}| \le 3^d |R|$ we have

$$M_{\mathcal{B}} g(x) \le 3^d \sum_{(\sigma_1, \ldots, \sigma_d) \in \{-1,+1\}^d} M_{\mathbf{B}}^{\sigma_1, \ldots, \sigma_d} g(x).$$

It suffices then to show that, for any fixed $(\sigma_1, \ldots, \sigma_d) \in \{-1, +1\}^d$, we have

$$|\{x \in \mathbb{R}^d : M_{\mathbf{B}}^{\sigma_1,\ldots,\sigma_d} g(x) > \alpha\}| \le \int_{\mathbb{R}^d} \Phi(|g|/\alpha),$$

as then

$$\begin{aligned}
&|\{x \in \mathbb{R}^d : M_{\mathcal{B}} g(x) > \alpha\}| \\
&\quad \le \Big|\Big\{x \in \mathbb{R}^d : \sum_{(\sigma_1,\ldots,\sigma_d)\in\{-1,+1\}^d} M_{\mathbf{B}}^{\sigma_1,\ldots,\sigma_d} g(x) > \frac{\alpha}{3^d}\Big\}\Big| \\
&\quad \le \sum_{(\sigma_1,\ldots,\sigma_d)\in\{-1,+1\}^d} \Big|\Big\{x \in \mathbb{R}^d : M_{\mathbf{B}}^{\sigma_1,\ldots,\sigma_d} g(x) > \frac{\alpha}{2^d \cdot 3^d}\Big\}\Big| \\
&\quad \le 2^d \int_{\mathbb{R}^d} \Phi(6^d |g|/\alpha).
\end{aligned}$$

Let $\mathbf{M} = M_{\mathbf{B}}^{1,\ldots,1}$. By symmetry, it suffices to show that

$$|\{x \in \mathbb{R}^d : \mathbf{M} g(x) > \alpha\}| \le \int_{\mathbb{R}^d} \Phi(|g(y)|/\alpha)\, dy. \tag{2}$$

Let now $g \in L^1(\mathbb{R}^d)$ and $\alpha > 0$. We assume without loss of generality that $g \ge 0$ and moreover that $g$ and $\{x \in \mathbb{R}^d : \mathbf{M} g(x) > \alpha\}$ are supported on the $d$-cube $[0, \gamma]^d$ where $\gamma$ is a positive integer.

For $0 \le j_1, \ldots, j_d \le \gamma - 1$, let

$$Q_{j_1,\ldots,j_d} = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : j_1 \le x_1 < j_1 + 1, \ldots, j_d \le x_d < j_d + 1\}$$

and

$$a_{j_1,\ldots,j_d} = \int_{Q_{j_1,\ldots,j_d}} g(y)\, dy.$$

Define the function $\tilde{g}$ on $\mathbb{R}^d$ by

$$\tilde{g}(y) = \sum_{j_1,\ldots,j_d=0}^{\gamma-1} a_{j_1,\ldots,j_d} \chi_{Q_{j_1,\ldots,j_d}}(y).$$

Note that

$$\{x \in \mathbb{R}^d : \mathbf{M}\tilde{g}(x) > \alpha\} = \{x \in \mathbb{R}^d : \mathbf{M} g(x) > \alpha\}.$$

It is useful at this point to utilize the following refinement of the Kakutani–Rokhlin tower theorem associated to non-periodic groups, due to Katznelson and Weiss:

LEMMA 1 ([13]). *Let $\{U_1, \ldots, U_d\}$ be a non-periodic collection of commuting measure preserving transformations on the probability space $(\Omega, \Sigma, \mu)$. Then for any $\epsilon > 0$ and positive integer $\gamma$ there exist sets $A$ and $E$ in $\Omega$*

*such that $\mu(E) < \epsilon$ and*

$$\Omega = \Big( \bigcup_{j_1,\dots,j_d=0}^{\gamma-1} U_1^{j_1} \cdots U_d^{j_d} A \Big) \cup E,$$

*where the $U_1^{j_1} \cdots U_d^{j_d} A$ are pairwise disjoint.*

Let $\epsilon > 0$. By the above lemma, there exist sets $A$ and $E$ in $\Omega$ such that $\mu(E) < \epsilon$ and

$$\Omega = \Big( \bigcup_{j_1,\dots,j_d=0}^{\gamma-1} U_1^{j_1} \cdots U_d^{j_d} A \Big) \cup E,$$

where the $U_1^{j_1} \cdots U_d^{j_d} A$ are pairwise disjoint. We define $f$ on $\Omega$ by

$$f(\omega) = \sum_{j_1,\dots,j_d=0}^{\gamma-1} a_{j_1,\dots,j_d} \chi_{U_1^{j_1} \dots U_d^{j_d} A}(\omega).$$

Observe that $f(\omega) = 0$ for $\omega \in E$.

Suppose $y \in Q_{i_1,\dots,i_d}$ for some $0 \le i_1, \dots, i_d < \gamma - 1$ such that $\mathbf{M}\tilde{g}(y) > \alpha$. Then there exists $(n_1, \dots, n_d) \in \Gamma$ such that

$$\frac{1}{n_1 \cdots n_d} \int_{t_1=i_1}^{i_1+n_1} \cdots \int_{t_d=i_d}^{i_d+n_d} \tilde{g}(t_1, \dots, t_d)\, dt_1 \cdots dt_d > \alpha.$$

Note this implies $\mathbf{M}\tilde{g} > \alpha$ on all of $Q_{i_1,\dots,i_d}$. Now if $\omega \in U_1^{i_1} \cdots U_d^{i_d} A$ we have $M_\Gamma f(\omega) > \alpha$, as

$$\begin{aligned} M_\Gamma f(\omega) &\ge \frac{1}{n_1 \cdots n_d} \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_d=0}^{n_d-1} f(U_1^{j_1} \cdots U_d^{j_d} \omega) \\ &= \frac{1}{n_1 \cdots n_d} \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_d=0}^{n_d-1} \tilde{g}(y + (j_1, \dots, j_d)) \\ &= \frac{1}{n_1 \cdots n_d} \int_{t_1=i_1}^{i_1+n_1} \cdots \int_{t_d=i_d}^{i_d+n_d} \tilde{g}(t_1, \dots, t_d)\, dt_1 \cdots dt_d > \alpha. \end{aligned}$$

So

$$Q_{j_1,\dots,j_d} \subset \{x \in \mathbb{R}^d : \mathbf{M}\tilde{g}(x) > \alpha\} \quad \text{implies} \quad U_1^{j_1} \cdots U_d^{j_d} A \subset \{\omega \in \Omega : M_\Gamma f(\omega) > \alpha\}$$

and

$$\begin{aligned}|\{x\in\mathbb{R}^d:\mathbf{M}\tilde{g}(x)>\alpha\}| &\le \frac{1}{\mu(A)}\mu\{\omega\in\Omega: M_\Gamma f(\omega)>\alpha\}\\ &\le \frac{1}{\mu(A)}\int\limits_{\Omega}\Phi(f/\alpha) \quad \text{(by hypothesis)}\\ &= \frac{1}{\mu(A)}\sum_{j_1,\dots,j_d=0}^{\gamma-1}\ \int\limits_{U_1^{j_1}\dots U_d^{j_d}A}\Phi(f/\alpha) = \sum_{j_1,\dots,j_d=0}^{\gamma-1}\ \int\limits_{Q_{j_1,\dots,j_d}}\Phi(\tilde{g}/\alpha)\\ &= \sum_{j_1,\dots,j_d=0}^{\gamma-1}\Phi\Big(\int\limits_{Q_{j_1,\dots,j_d}} g/\alpha\Big) \quad \text{(by the definition of } \tilde{g})\\ &\le \sum_{j_1,\dots,j_d=0}^{\gamma-1}\ \int\limits_{Q_{j_1,\dots,j_d}}\Phi(g/\alpha) \quad \text{(by Jensen's inequality)}\\ &\le \int\limits_{\mathbb{R}^d}\Phi(g/\alpha).\end{aligned}$$

Hence (2) holds and we have

$$|\{x\in\mathbb{R}^d: M_{\mathcal{B}}g(x)>\alpha\}|\le 2^d\int\limits_{\mathbb{R}^d}\Phi(6^d g/\alpha),$$

as desired. ∎

REMARK. Theorem 1 is false without the assumption that $U_1,\dots,U_d$ form a *non-periodic* collection of measure preserving transformations. For example, if $d\ge 2$ and $U_1,\dots,U_d$ were all the identity transformation on $\Omega$ and $\Gamma=\mathbb{Z}_+^d$, then $M_\Gamma$ would be bounded on $L^1(\Omega)$ although $M_{\mathcal{B}}$ would not be of weak type $(1,1)$.

## 3. Applications

**3.1. The Jessen–Marcinkiewicz–Zygmund and Dunford–Zygmund theorems.** Let $\mathcal{B}_d$ denote the set of rectangular parallelepipeds in $\mathbb{R}^d$ with sides parallel to the coordinate axes. In 1935 Jessen, Marcinkiewicz, and Zygmund proved in [12] that $\mathcal{B}_d$ differentiates $L(\log^+ L)^{d-1}(\mathbb{R}^d)$, i.e.

$$\lim_{\substack{x\in R\in\mathcal{B}_d\\ \operatorname{diam}(R)\to 0}}\frac{1}{|R|}\int\limits_R g(y)\,dy = g(x) \quad \text{a.e.} \tag{3}$$

provided $g\in L(\log^+ L)^{d-1}(\mathbb{R}^d)$.

The Jessen–Marcinkiewicz–Zygmund theorem has a counterpart in ergodic theory. In particular, Dunford [6] and Zygmund [19] independently proved that if $U_1, \ldots, U_d$ form a collection of measure preserving transformations of a measure space $\Omega$ of finite measure onto itself and $f \in L(\log^+ L)^{d-1}(\Omega)$, then

$$\lim_{m_1,\ldots,m_d\to\infty} \frac{1}{m_1\cdots m_d} \sum_{j_1=0}^{m_1-1} \cdots \sum_{j_d=0}^{m_d-1} f(U_1^{j_1}\cdots U_d^{j_d}\omega) \tag{4}$$

exists for a.e. $\omega$ in $\Omega$.

Each of these convergence results may be seen to be a consequence of a weak type estimate for a corresponding maximal operator. The *strong maximal operator* $M_S$ is defined on $L^1(\mathbb{R}^d)$ by

$$M_S f(x) = \sup_{x\in R\in\mathcal{B}_d} \frac{1}{|R|} \int_R |f(y)|\,dy$$

and the *ergodic strong maximal operator* $M_{U_1,\ldots,U_d}$ is defined on $L^1(\Omega)$ by

$$M_{U_1,\ldots,U_d} f(\omega) = \sup_{(n_1,\ldots,n_d)\in\mathbb{Z}_+^d} \frac{1}{n_1\cdots n_d} \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_d=0}^{n_d-1} |f(U_1^{j_1}\cdots U_d^{j_d}\omega)|.$$

In 1972 N. A. Fava showed the following:

Theorem 2 ([7]). *The strong maximal operator $M_S$ satisfies the weak type estimate*

$$|\{x\in\mathbb{R}^d : M_S g(x) > \alpha\}| \le C \int_{\mathbb{R}^d} \frac{|g|}{\alpha}\biggl(1+\log^+\biggl(\frac{|g|}{\alpha}\biggr)\biggr)^{d-1} \tag{5}$$

*and the ergodic strong maximal operator $M_{U_1,\ldots,U_d}$ satisfies the weak type estimate*

$$\mu\{\omega\in\Omega : M_{U_1,\ldots,U_d} f(\omega) > \alpha\} \le C \int_{\Omega} \frac{|f|}{\alpha}\biggl(1+\log^+\biggl(\frac{|f|}{\alpha}\biggr)\biggr)^{d-1}. \tag{6}$$

An immediate implication of this theorem is that (3) holds for any function $g$ in $L(\log^+ L)^{d-1}(\mathbb{R}^d)$ and (4) holds for any $f$ in $L(\log^+ L)^{d-1}(\Omega)$. Conversely, by the work of Stein [16] on limits of sequences of operators we see that the weak type inequality (5) in Fava's theorem is a consequence of the a.e. convergence result (3) of Jessen, Marcinkiewicz, and Zygmund. Moreover, by the work of Sawyer [14] we find that, provided $U_1, \ldots, U_d$ form a commuting ergodic family of measure preserving transformations (in particular, satisfying the condition that $U_1^{m_1}\cdots U_d^{m_d}A = A$ for every $(m_1,\ldots,m_d)\in\mathbb{Z}_+^d$ implies $\mu(A)=0$ or $\mu(A)=1$), the weak type estimate (6) follows from the convergence result (4) of Dunford and Zygmund.

By Theorem 1, we then see that, provided that the measure preserving transformations $U_1,\ldots,U_d$ commute and form a non-periodic family, the two weak type inequalities in Fava's theorem are immediate consequences of one another. If additionally $U_1, \dots, U_d$ form an ergodic family, the a.e. convergence results (3),(4) are also consequences of one another. More particularly, if $U_1, \dots, U_d$ commute then (3) implies (4), and if additionally $U_1, \dots, U_d$ form a non-periodic ergodic family then (4) implies (3).

We emphasize here that the non-periodicity condition on $U_1, \dots, U_d$ enables us to transfer weak type bounds on ergodic maximal operators to corresponding weak type bounds for their geometric maximal operator counterparts. The additional condition that $U_1, \dots, U_d$ form an ergodic family enables us to pass from a.e. convergence results associated to ergodic averages involving $U_1, \dots, U_d$ to associated weak type bounds of $M_{U_1,\dots,U_d}$. The question of whether (4) implies (3) only under the condition that $U_1, \dots, U_d$ form a commuting non-periodic family will be a subject of future investigation.

**3.2. Maximal operators associated to rare bases in the plane.** Let $\mathcal{B}$ be a translation invariant collection of rectangles in the plane whose sides are parallel to the axes, and define the corresponding rare maximal operator $M_{\mathcal{B}}$ by

$$M_{\mathcal{B}} f(x) = \sup_{x \in R \in \mathcal{B}} \frac{1}{|R|} \int_{|R|} |f|.$$

As $M_{\mathcal{B}} f$ is bounded by the strong maximal function $M_S f$, by Fava's theorem we automatically have

$$|\{x \in \mathbb{R}^2 : M_{\mathcal{B}} f > \alpha\}| \leq C \int_{\mathbb{R}^2} \frac{|f|}{\alpha} \left(1 + \log^+ \frac{|f(x)|}{\alpha}\right).$$

In [17] Stokolos considered the question of whether it would be possible for $M_{\mathcal{B}}$ to satisfy an estimate strictly stronger than a weak type $(x \log^+ x, x \log^+ x)$ estimate without actually being of weak type $(1,1)$. Surprisingly, the answer is no:

THEOREM 3 ([17]). *Let $\mathcal{B}$ be a translation invariant collection of rectangles in the plane whose sides are parallel to the axes, and define the corresponding rare maximal operator $M_{\mathcal{B}}$ by*

$$M_{\mathcal{B}} f(x) = \sup_{x \in R \in \mathcal{B}} \frac{1}{|R|} \int_{R} |f|.$$

*Suppose that $\Phi$ is a convex increasing function such that $\Phi(0) = 0$ and $\Phi(x) = o(x \log^+ x)$ and $M_{\mathcal{B}}$ satisfies the weak type estimate*

$$|\{x : M_{\mathcal{B}} f(x) > \alpha\}| < \int_{\mathbb{R}^n} \Phi(|f|/\alpha).$$

*Then $M_{\mathcal{B}}$ is of weak type $(1,1)$.*

By Theorem 1, we can transfer Stokolos' result to the ergodic setting, obtaining the following:

COROLLARY 1. *Let $U_1$ and $U_2$ be a non-periodic pair of commuting measure preserving transformations on a probability space $(\Omega, \Sigma, \mu)$. Let $\Gamma$ be a nonempty subset of $\mathbb{Z}^2_+$ and $M_\Gamma$ the ergodic maximal operator defined by*

$$M_\Gamma f(\omega) = \sup_{(n_1,n_2)\in\Gamma} \frac{1}{n_1 n_2} \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} |f(U_1^{j_1} U_2^{j_2} \omega)|.$$

*Suppose that $\Phi$ is a convex increasing function such that $\Phi(0) = 0$ and $\Phi(x) = o(x \log^+ x)$ and $M_\Gamma$ satisfies the weak type estimate*

$$\mu\{\omega \in \Omega : M_\Gamma f(\omega) > \alpha\} \le \int_\Omega \Phi(|f|/\alpha).$$

*Then $M_\Gamma$ must satisfy the weak type $(1,1)$ estimate*

$$\mu\{\omega \in \Omega : M_\Gamma f(\omega) > \alpha\} \le C \int_\Omega (|f|/\alpha).$$

We remark that the above corollary also follows from the paper [11] of Hagelstein and Stokolos, although it does not explicitly appear there.

It is worthwhile to emphasize here that this corollary illustrates the usefulness of being able to transfer weak type estimates for ergodic maximal operators to their geometric maximal operator counterparts. In particular by this transference we recognize that a *sharp* weak type $(x \log^+ x, x \log^+ x)$ estimate on $M_{\mathcal{B}}$ corresponds to a *sharp* weak type $(x \log^+ x, x \log^+ x)$ estimate on the associated $M_\Gamma$ in the non-periodic setting.

**3.3. Córdoba bases and the Zygmund program.** Several decades ago, A. Zygmund made the far-reaching conjecture that any $k$-parameter basis of parallelepipeds consisting of members of $\mathcal{B}_d$ whose sidelengths are of the form $\phi_1(t_1, \ldots, t_k), \ldots, \phi_d(t_1, \ldots, t_k)$, where the $\phi_i$ are nonnegative functions assuming arbitrarily small values and increasing in each variable separately, necessarily differentiates $L(\log^+ L)^{k-1}(\mathbb{R}^d)$. Although this conjecture is now known not to hold in its full generality (see a counterexample due to F. Soria in [15]), many special cases of this conjecture remain a subject of considerable interest in differentiation theory and significant positive results have been obtained. An especially prominent result is the following one due to A. Córdoba:

THEOREM 4 ([5]). *Let $\mathcal{B}_{\mathcal{C}}$ consist of the rectangular parallelepipeds in $\mathbb{R}^3$ whose sides are parallel to the coordinate axes and whose dimensions are given by $s \times t \times st$. Then the associated geometric maximal operator $M_{\mathcal{C}}$*

*satisfies the weak type estimate*

$$|\{x \in \mathbb{R}^3 : M_{\mathcal{C}} f(x) > \alpha\}| \le C \int_{\mathbb{R}^3} \frac{|f|}{\alpha}\left(1 + \log^+\left(\frac{|f|}{\alpha}\right)\right)$$

*and*

$$\lim_{\substack{x \in R \in \mathcal{B}_{\mathcal{C}} \\ \operatorname{diam}(R) \to 0}} \frac{1}{|R|} \int_R f(y)\, dy = f(x)$$

*holds a.e. for any* $f \in L \log^+ L(\mathbb{R}^3)$.

An ergodic theory analogue of Córdoba's result following from Theorems 1 and 4 is the following:

COROLLARY 2. *Let* $U_1, U_2, U_3$ *be a collection of commuting measure preserving transformations of a probability space* $(\Omega, \Sigma, \mu)$ *to itself,* $\Gamma$ *be the set of points in* $\mathbb{Z}_+^3$ *of the form* $(m, n, mn)$, *and* $M_\Gamma$ *the associated ergodic maximal operator given by*

$$M_\Gamma f(\omega) = \sup_{(n_1,n_2) \in \mathbb{Z}_+^2} \frac{1}{n_1^2 n_2^2} \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} \sum_{j_3=0}^{n_1 n_2 - 1} |f(U_1^{j_1} U_2^{j_2} U_3^{j_3} \omega)|.$$

*Then*

$$\mu\{\omega \in \Omega : M_\Gamma f(\omega) > \alpha\} \le C \int_{\Omega} \frac{|f|}{\alpha}\left(1 + \log^+\left(\frac{|f|}{\alpha}\right)\right)$$

*and for every* $f \in L \log^+ L(\Omega)$ *the limit*

$$\lim_{n_1, n_2 \to \infty} \frac{1}{n_1^2 n_2^2} \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} \sum_{j_3=0}^{n_1 n_2 - 1} f(U_1^{j_1} U_2^{j_2} U_3^{j_3} \omega)$$

*exists for a.e.* $\omega \in \Omega$.

Note that the condition of non-periodicity is not necessary in the above corollary as the proof entails just the transference of a weak type bound of a geometric maximal operator to a similar weak type bound of the ergodic maximal operator counterpart. If $U_1, U_2, U_3$ do form a non-periodic collection, the above weak type estimate on $M_\Gamma$ is sharp, a fact following from Theorem 1 and the sharpness of Theorem 4 (as can be seen by acting by $M_{\mathcal{C}}$ on test functions). If additionally $U_1, U_2, U_3$ form an ergodic family the a.e. convergence result of Corollary 2 would be sharp as well, as can be seen by the aforementioned arguments of Sawyer and Stein in [14], [16].

We remark that the question of whether the collection of parallelepipeds in $\mathbb{R}^4$ with sides parallel to the axes and whose dimensions are given by $s \times t \times u \times stu$ differentiates $L(\log^+ L)^2(\mathbb{R}^4)$ is a difficult unsolved problem, although significant inroads have been made toward its solution by R. Fefferman and J. Pipher [8]. The corresponding question of whether or not, given a non-periodic collection $U_1, \ldots, U_4$ of commuting measure preserving transformations of a probability space $(\Omega, \Sigma, \mu)$ to itself and $f \in L(\log^+ L)^2(\Omega)$, the limit

$$\lim_{n_1,n_2,n_3\to\infty} \frac{1}{n_1^2 n_2^2 n_3^2} \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} \sum_{j_3=0}^{n_3-1} \sum_{j_4=0}^{n_1 n_2 n_3 - 1} f(U_1^{j_1} U_2^{j_2} U_3^{j_3} U_4^{j_4} \omega)$$

necessarily exists for a.e. $\omega \in \Omega$, is open as well.

**Acknowledgements.** This work was partially supported by a grant from the Simons Foundation (#208831 to Paul Hagelstein). The authors wish to thank Joseph Rosenblatt for his helpful comments and suggestions regarding this paper.

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*Received 30 April 2012;*
*in revised form 16 August 2012*