MATHEMATICAE
174 (2002)

On the non-extendibility of strongness and supercompactness through strong compactness

by

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Abstract. If κ is either supercompact or strong and $\delta < \kappa$ is α strong or α supercompact for every $\alpha < \kappa$, then it is known δ must be (fully) strong or supercompact. We show this is not necessarily the case if κ is strongly compact.

1. Introduction and preliminaries. A well-known result of Magidor [16] states that if κ is supercompact and $\delta < \kappa$ is α supercompact for all $\alpha < \kappa$, then δ is supercompact. Indeed, the following is true.

Lemma 1.1 (Folklore). *If κ is a strong cardinal and $\delta < \kappa$ is either α strong, α strongly compact, or α supercompact for every $\alpha < \kappa$, then δ must be* (*fully*) *strong, strongly compact, or supercompact.*

Proof. Let $\lambda > \kappa$ be a cardinal so that $\lambda = \beth_\lambda$, and let $\gamma = \beth_\omega(\lambda)$. Take $j : V \to M$ as an elementary embedding witnessing the γ strongness of κ. Since $V \vDash$ "δ is either α strong, α strongly compact, or α supercompact for every $\alpha < \kappa$" and $\delta < \kappa$, $M \vDash$ "$j(\delta) = \delta$ is either α strong, α strongly compact, or α supercompact for every $\alpha < j(\kappa)$". In particular, because $j(\kappa) > \gamma > \lambda$, $M \vDash$ "δ is either λ strong, λ strongly compact, or λ supercompact". As $V_\gamma \subseteq M$, $V \vDash$ "δ is either λ strong, λ strongly compact, or λ supercompact" as well. Since λ may be chosen arbitrarily large, this proves Lemma 1.1. ■

We observe that Lemma 1.1 has a local version. Specifically, if κ is measurable and $\delta < \kappa$ is either α strong, α strongly compact, or α supercompact

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2000 *Mathematics Subject Classification*: 03E35, 03E55.

Key words and phrases: supercompact cardinal, strongly compact cardinal, strong cardinal, non-reflecting stationary set of ordinals.

The author wishes to thank the referee for helpful suggestions that have been incorporated into this version of the paper.

for $\alpha < \kappa$, then δ is either $\kappa + 1$ strong, κ strongly compact, or κ supercompact. The proof is essentially the same as the one given above, with j replaced by an elementary embedding witnessing κ's measurability, and the observation that the κ closure of M with respect to V is enough to ensure that δ is either $\kappa + 1$ strong, κ strongly compact, or κ supercompact in V.

Key to the proof of Lemma 1.1 is the fact that the inner model M contains a large chunk of the universe V, something which will be true if κ is either supercompact or, more weakly, strong. It is not necessarily the case, however, that if κ is only strongly compact, then there is an elementary embedding witnessing any degree of strong compactness into an inner model M containing any more of V than $V_{\kappa+1}$. Thus, we can ask the following question: If κ is a non-supercompact strongly compact cardinal and $\delta < \kappa$ is either α supercompact or α strong for every $\alpha < \kappa$, then must δ be either (fully) supercompact or strong? Note that by a theorem of Di Prisco [7], the answer to the analogue of this question if δ is α strongly compact for every $\alpha < \kappa$ is yes.

The purpose of this paper is to show that the answer to the above question is no. Specifically, we prove the following two theorems.

THEOREM 1. *Suppose $V \vDash$ "ZFC + $\kappa_1 < \kappa_2$ are supercompact". There is then a partial ordering $\mathbb{P} \in V$ so that $V^{\mathbb{P}} \vDash$ "ZFC + κ_2 is strongly compact but not supercompact + κ_1 is α supercompact for every $\alpha < \kappa_2$ + κ_1 is not supercompact".*

THEOREM 2. *Suppose $V \vDash$ "ZFC + κ is supercompact". There is then a partial ordering $\mathbb{P} \in V$ and a strong cardinal $\delta < \kappa$ so that $V^{\mathbb{P}} \vDash$ "ZFC + κ is strongly compact but not supercompact + δ is α strong for every $\alpha < \kappa$ + δ is not strong".*

Before giving the proofs of Theorems 1 and 2, we briefly mention some preliminary information. Essentially, our notation and terminology are standard, and when this is not the case, this will be clearly noted. For $\alpha < \beta$ ordinals, $[\alpha, \beta]$, $[\alpha, \beta)$, $(\alpha, \beta]$, and (α, β) are as in standard interval notation.

When forcing, $q \geq p$ will mean that q is stronger than p. If G is V-generic over $\mathbb{P}$, we will use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. If $x \in V[G]$, then $\dot{x}$ will be a term in V for x. We may, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$, especially when x is some variant of the generic set G, or x is in the ground model V.

If κ is a cardinal and $\mathbb{P}$ is a partial ordering, $\mathbb{P}$ is *κ-directed closed* if for every cardinal $\delta < \kappa$ and every directed set $\langle p_\alpha : \alpha < \delta \rangle$ of elements of $\mathbb{P}$ (where $\langle p_\alpha : \alpha < \delta \rangle$ is *directed* if any two elements p_ϱ and p_ν have a common upper bound of the form p_σ) there is an upper bound $p \in \mathbb{P}$.

$\mathbb{P}$ is *κ-strategically closed* if in the two-person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages (choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued. Note that if $\mathbb{P}$ is κ-strategically closed and $f : \kappa \to V$ is a function in $V^{\mathbb{P}}$, then $f \in V$. $\mathbb{P}$ is *$\prec\kappa$-strategically closed* if in the two-person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha < \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages (again choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued.

Suppose $\kappa < \lambda$ are regular cardinals. A partial ordering $\mathbb{P}_{\kappa,\lambda}$ that will be used in this paper is the partial ordering for adding a non-reflecting stationary set of ordinals of cofinality κ to λ. Specifically, $\mathbb{P}_{\kappa,\lambda} = \{s : s$ is a bounded subset of λ consisting of ordinals of cofinality κ so that for every $\alpha < \lambda$, $s \cap \alpha$ is non-stationary in $\alpha\}$, ordered by end-extension. Two things which can be shown (see [5] or [2]) are that $\mathbb{P}_{\kappa,\lambda}$ is δ-strategically closed for every $\delta < \lambda$, and if G is V-generic over $\mathbb{P}_{\kappa,\lambda}$, in $V[G]$, a non-reflecting stationary set $S = S[G] = \bigcup\{S_p : p \in G\} \subseteq \lambda$ of ordinals of cofinality κ has been introduced. It is also virtually immediate that $\mathbb{P}_{\kappa,\lambda}$ is κ-directed closed.

We mention that we are assuming familiarity with the large cardinal notions of measurability, strongness, strong compactness, and supercompactness. Interested readers may consult [12] for further details. Also, unlike [12], we will say that the cardinal κ is λ *strong* for $\lambda > \kappa$ if there is $j : V \to M$ an elementary embedding having critical point κ so that $j(\kappa) > |V_\lambda|$ and $V_\lambda \subseteq M$. As always, κ is *strong* if κ is λ strong for every $\lambda > \kappa$.

2. The proof of Theorem 1. Let $V \models$ "ZFC $+\ \kappa_1 < \kappa_2$ are supercompact". Without loss of generality, by first using an iteration of Laver's partial ordering of [13] (such as the one given in [1]) to force κ_i for $i = 1, 2$ to have its supercompactness indestructible under κ_i-directed closed forcing, then employing an Easton support iteration to add to every measurable cardinal $\delta > \kappa_2$ a non-reflecting stationary set of ordinals of cofinality κ_2, and then forcing with a κ_1-directed closed partial ordering to ensure GCH holds at and above κ_1, we may also assume that $V \models$ "No cardinal $\lambda > \kappa_2$ is measurable $+\ \kappa_1$'s supercompactness is indestructible under κ_1-directed closed forcing $+\ 2^\delta = \delta^+$ for every cardinal $\delta \geq \kappa_1$". The fact that no cardinal above κ_2 is measurable in V follows from the Gap Forcing Theorem of [10] and [11].

Take now $\mathbb{P}_0$ as the Easton support iteration of length κ_2 which adds, to every measurable cardinal $\delta \in (\kappa_1, \kappa_2)$, a non-reflecting stationary set of ordinals of cofinality κ_1. $\mathbb{P}_0$ can be defined so as to have cardinality κ_2. Since $V \models$ "No cardinal $\lambda > \kappa_2$ is measurable $+\ 2^\delta = \delta^+$ for every cardinal

$\delta \geq \kappa_1$", a theorem of Magidor (whose proof is given in Theorem 2 of [3]) tells us that $V^{\mathbb{P}_0} \models$ "There are no measurable cardinals in the interval (κ_1, κ_2) $+ \kappa_2$ is strongly compact". It then immediately follows that $V^{\mathbb{P}^0} \models$ "κ_2 is not $2^{\kappa_2} = \kappa_2^+$ supercompact". Further, since $\mathbb{P}_0$, by its definition, is κ_1-directed closed, $V^{\mathbb{P}_0} \models$ "κ_1 is supercompact".

Work in $V_0 = V^{\mathbb{P}^0}$. For the remainder of this paper, for α an arbitrary ordinal, let λ_α be the least measurable cardinal above α. Since $V_0 \models$ "κ_1 is supercompact $+ \kappa_2$ is the least measurable cardinal above κ_1", by reflection, $A = \{\delta < \kappa_1 : \delta \text{ is } \lambda_\delta \text{ supercompact}\}$ is unbounded in κ_1. Therefore, we may define $\mathbb{P}_1$ in V_0 as the Easton support iteration of length κ_1 which first adds a Cohen subset of ω and then adds, to every $\delta \in A$, a non-reflecting stationary set of ordinals of cofinality ω. In analogy to the definition of $\mathbb{P}_0$, $\mathbb{P}_1$ can be defined so as to have cardinality κ_1.

Lemma 2.1. *$V_1 = V_0^{\mathbb{P}^1} \models$ "κ_1 is α supercompact for every $\alpha < \kappa_2$".*

Proof. Let $\eta < \kappa_2$ be an arbitrary inaccessible cardinal in the interval (κ_1, κ_2), and let $j : V_0 \to M$ be an elementary embedding witnessing the η supercompactness of κ_1 so that $M \models$ "κ_1 is not η supercompact". Since η is below the least measurable cardinal above κ_1, $M \models$ "κ_1 is not λ_{κ_1} supercompact". This means $j(\mathbb{P}_1) = \mathbb{P}_1 * \dot{\mathbb{Q}}$, where $\dot{\mathbb{Q}}$ is a term for a partial ordering that does not add a non-reflecting stationary set of ordinals of cofinality ω to κ_1, and the least M-cardinal above κ_1 to which $\dot{\mathbb{Q}}$ is forced to add a non-reflecting stationary set of ordinals of cofinality ω must also be above η.

Let G_0 be V_0-generic over $\mathbb{P}_1$, and let H be $V_0[G_0]$-generic over $\mathbb{Q}$. Standard arguments show that $M[G_0]$ remains η closed with respect to $V_0[G_0]$. Further, $j"G_0 \subseteq G_0 * H$. This means that in $V_0[G_0][H]$, j lifts to $j : V_0[G_0] \to M[G_0][H]$. By its definition, the closure properties of $M[G_0]$, and the last sentence of the preceding paragraph, H is $V_0[G_0]$-generic over a partial ordering which is η-strategically closed in both $V_0[G_0]$ and $M[G_0]$. Therefore, $V_0[G_0] \models$ "κ_1 is α supercompact for every $\alpha < \eta$". Since η was chosen as an arbitrary inaccessible cardinal in the interval (κ_1, κ_2), this proves Lemma 2.1. ■

We remark that by the observation made immediately following the proof of Lemma 1.1, Lemma 2.1 actually shows that κ_1 is κ_2 supercompact in V_1.

Lemma 2.2. *$V_1 = V_0^{\mathbb{P}^1} \models$ "κ_1 is not $2^{\kappa_2} = \kappa_2^+$ supercompact".*

Proof. We begin by noting that $V_1 = V_0^{\mathbb{P}^1} \models$ "κ_2 is the least measurable cardinal above $\kappa_1 + \kappa_2$ is strongly compact but is not $2^{\kappa_2} = \kappa_2^+$ supercompact". This follows by the fact κ_2 is both the least measurable and least strongly compact cardinal above κ_1 in V_0, the fact that $\mathbb{P}_1$ has cardinality $\kappa_1 < \kappa_2$ in V_0, and the Lévy–Solovay results [14].

Work in V_0. For any α, write $\mathbb{P}_1 = \mathbb{Q}_0 * \dot{\mathbb{Q}}_1$, where $\mathbb{Q}_0$ adds non-reflecting stationary sets of ordinals of cofinality ω to cardinals at most α, and $\dot{\mathbb{Q}}_1$ is a term for the rest of $\mathbb{P}_1$. Since $|\mathbb{Q}_0| \le 2^\alpha < \lambda_\alpha$, the results of [14] and the fact $\Vdash_{\mathbb{Q}_0}$ "$\dot{\mathbb{Q}}_1$ is λ_α-strategically closed" together imply that $(\lambda_\alpha)^{V_0} = (\lambda_\alpha)^{V_1}$.

Write $\mathbb{P}_1 = \mathbb{P}' * \dot{\mathbb{P}}''$, where $|\mathbb{P}'| = \omega$ and $\Vdash_{\mathbb{P}'}$ "$\dot{\mathbb{P}}''$ is $\aleph_1$-strategically closed". In Hamkins' terminology of [9], [10], and [11], $\mathbb{P}_1$ "admits a gap at $\aleph_1$", so by the Gap Forcing Theorem of [10] and [11], any cardinal δ which is λ_δ supercompact in V_1 had to have been λ_δ supercompact in V_0. Since by its definition, forcing with $\mathbb{P}_1$ over V_0 destroys the weak compactness of any cardinal $\delta < \kappa_1$ that was λ_δ supercompact in V_0, the preceding sentence implies that $V_1 = V_0^{\mathbb{P}^1} \vDash$ "No cardinal $\delta < \kappa_1$ is λ_δ supercompact". This immediately implies that $V_1 \vDash$ "κ_1 is not $2^{\kappa_2} = \kappa_2^+$ supercompact", since otherwise, by choosing $k : V_1 \to N$ as an elementary embedding witnessing the 2^{κ_2} supercompactness of κ_1 and reflecting the fact that $N \vDash$ "κ_1 is κ_2 supercompact and κ_2 is the least measurable cardinal above κ_1", we would infer that $\{\delta < \kappa_1 : \delta \text{ is } \lambda_\delta \text{ supercompact}\}$ is unbounded in κ_1 in V_1. This proves Lemma 2.2. ∎

By defining $\mathbb{P} = \mathbb{P}_0 * \dot{\mathbb{P}}_1$, Lemmas 2.1 and 2.2 complete the proof of Theorem 1. ∎

We conclude Section 2 with some observations. It is possible to change the definition of $\mathbb{P}_1$ so as to ensure κ_1 will satisfy a greater degree of supercompactness in V_1. If, e.g., we modify the definition of $\mathbb{P}_1$ so that we add non-reflecting stationary sets of ordinals of cofinality ω to every cardinal $\delta < \kappa_1$ which is $\beth_\delta(\lambda_\delta)$ supercompact (and by the supercompactness of κ_1, there are unboundedly in κ_1 many such cardinals), then in V_1, κ_1 will be $\beth_{\kappa_1}(\kappa_2)$ supercompact but not $2^{[\beth_{\kappa_1}(\kappa_2)]^{<\kappa_1}} = 2^{\beth_{\kappa_1}(\kappa_2)} = (\beth_{\kappa_1}(\kappa_2))^+$ supercompact. However, due to the restrictions on the proof of Theorem 2 of [3], we need to know that $V \vDash$ "No cardinal $\lambda > \kappa_2$ is measurable". No such restrictions, however, are required in the proof of Theorem 2 of this paper, which we give below.

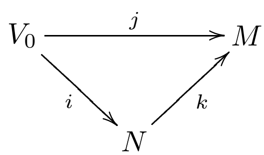
3. The proof of Theorem 2. Let $V \vDash$ "ZFC + κ is supercompact". By Lemma 2.1 of [4] and the succeeding remark, we know that $\{\delta < \kappa : \delta$ is a strong cardinal$\}$ is unbounded in κ. Without loss of generality, by first forcing GCH, then choosing a strong cardinal $\delta < \kappa$, and then forcing with Gitik and Shelah's indestructibility partial ordering of [8] (which can be defined so as to have cardinality δ), we may further assume that $V \vDash$"GCH holds for cardinals at and above δ + δ is a strong cardinal whose strongness is indestructible under forcing with an iteration of Prikry forcing as defined by Magidor in [15] which adds Prikry sequences to cardinals above δ".

Take now $\mathbb{P}_0$ as Magidor's iterated Prikry forcing of [15] which adds, to every measurable cardinal $\gamma \in (\delta, \kappa)$, a Prikry sequence. By the indestructibility properties of V and Magidor's work of [15], $V^{\mathbb{P}_0} = V_0 \vDash$"GCH holds for cardinals at and above δ + δ is a strong cardinal + κ is strongly compact + There are no measurable cardinals in the interval (δ, κ)". As in the proof of Theorem 1, $V_0 \vDash$ "κ is not $2^\kappa = \kappa^+$ supercompact".

Work in V_0. Since $V_0 \vDash$ "δ is strong + κ is the least measurable cardinal above δ", by reflection, $B = \{\gamma < \delta : \gamma \text{ is } \lambda_\gamma \text{ strong}\}$ is unbounded in δ. Therefore, in analogy to the proof of Theorem 1, we may define $\mathbb{P}_1$ in V_0 as the Easton support iteration which begins by adding a Cohen subset of ω and then adds, to every $\gamma \in B$, a non-reflecting stationary set of ordinals of cofinality ω. As in the proof of Theorem 1, $\mathbb{P}_1$ can be defined so as to have cardinality δ. By the preceding paragraph, this has as an immediate consequence that in V_1, GCH holds for cardinals at and above δ.

Lemma 3.1. $V_0^{\mathbb{P}_1} = V_1 \vDash$ "*δ is α strong for every $\alpha < \kappa$*".

Proof. The proof is very similar to the proof of Lemma 2.5 of [4]. We use the notation and terminology from the introductory section of [6]. Fix $\eta > \delta$, $\eta < \kappa$ an inaccessible cardinal which is not also a Mahlo cardinal. Let $j : V_0 \to M$ be an elementary embedding witnessing the $\eta + 1$ strongness of δ generated by a $(\delta, \eta + 1)$-extender of width δ so that $M \vDash$ "δ is not $\eta + 1$ strong", and let $i : V_0 \to N$ be the elementary embedding witnessing the measurability of δ generated by the normal ultrafilter $\mathcal{U} = \{x \subseteq \delta : \delta \in j(x)\}$. We then have the commutative diagram



where $j = k \circ i$ and the critical point of k is above δ.

Since η is below the least measurable cardinal above δ and η is not a Mahlo cardinal, $M \vDash$ "There are no measurable cardinals in the interval $(\delta, \eta]$ + δ is not λ_δ strong". Define ϱ to be the least cardinal in M above δ which is λ_ϱ strong. By the next to last sentence, we can now infer that $\varrho > \eta$.

Define $f : \delta \to \delta$ as $f(\alpha) =$ The least inaccessible cardinal above λ_α. By our choice of η and the preceding paragraph, $\delta < \eta < j(f)(\delta) < \varrho$. Observe that ϱ is also the least M-cardinal above δ to which $j(\mathbb{P}_1)$ adds a non-reflecting stationary set of ordinals of cofinality ω.

Note now that $M = \{j(g)(a) : a \in [\eta^+]^{<\omega}, \operatorname{dom}(g) = [\delta]^{|a|}, g : [\delta]^{|a|} \to V_0\}$ $= \{k(i(g))(a) : a \in [\eta^+]^{<\omega}, \operatorname{dom}(g) = [\delta]^{|a|}, g : [\delta]^{|a|} \to V_0\}$. By defining $\gamma = i(f)(\delta)$, we have $k(\gamma) = k(i(f)(\delta)) = j(f)(\delta) > \eta^+$. This means $j(g)(a) =$

$k(i(g))(a) = k(i(g)\restriction[\gamma]^{|a|})(a)$, i.e., $M = \{k(h)(a) : a \in [\eta^+]^{<\omega}, h \in N, \text{dom}(h) = [\gamma]^{|a|}, h : [\gamma]^{|a|} \to N\}$. By elementariness, we must have $N \models$ "δ is not λ_δ strong and $\delta < \gamma = i(f)(\delta) < \delta_0 =$ The least cardinal ζ in N above δ which is λ_ζ strong $=$ The least cardinal to which $i(\mathbb{P}_1) - \delta$ adds a non-reflecting stationary set of ordinals of cofinality ω", since $M \models$ "$k(\delta) = \delta$ is not λ_δ strong and $k(\delta) = \delta < k(\gamma) = k(i(f)(\delta)) = j(f)(\delta) < k(\delta_0) = \varrho$". Therefore, k can be assumed to be generated by an N-extender of width $\gamma \in (\delta, \delta_0)$.

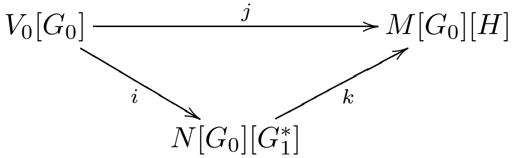
Write $i(\mathbb{P}_1) = \mathbb{P}_1 * \dot{\mathbb{Q}}_0$, where $\dot{\mathbb{Q}}_0$ is a term for the portion of $i(\mathbb{P}_1)$ adding non-reflecting stationary sets of ordinals of cofinality ω to N-cardinals in the interval $[\delta, i(\delta))$. Since $N \models$ "δ is not λ_δ strong", $\dot{\mathbb{Q}}_0$ is actually a term for a partial ordering adding non-reflecting stationary sets of ordinals of cofinality ω to N-cardinals in the interval $(\delta, i(\delta))$, or more precisely, to N-cardinals in the interval $[\delta_0, i(\delta))$.

Let G_0 be V_0-generic over $\mathbb{P}_1$. By the definition of $\mathbb{P}_1$ and the fact GCH holds in V_0 for cardinals at and above δ, $N[G_0] \models$ "$|\mathbb{Q}_0| = i(\delta) + |2^{\mathbb{Q}_0}| = i(\delta^+) = (i(\delta))^+$". As N is an ultrapower via a normal measure over δ, this means $V_0 \models$ "$|(i(\delta))^+| = \delta^+$", so we can let $\langle D_\alpha : \alpha < \delta^+\rangle \in V_0[G_0]$ be an enumeration of the dense open subsets of $\mathbb{Q}_0$ present in $N[G_0]$. For the purposes of the argument to be given below, we also assume that $\langle D_\alpha : \alpha < \delta^+\rangle$ has been defined so that for every dense open subset $D \subseteq \mathbb{Q}_0$ found in $N[G_0]$, for some odd ordinal $\beta + 1$, $D = D_{\beta+1}$. Further, since $V_0 \models$ "$|\mathbb{P}_1| = \delta$", standard arguments show that $N[G_0]$ remains δ closed with respect to $V_0[G_0]$. Therefore, as $N[G_0] \models$ "$\mathbb{Q}_0$ is $\prec\delta^+$-strategically closed", this fact is true in $V_0[G_0]$ as well.

We can now construct an $N[G_0]$-generic object, G_1^*, in $V_0[G_0]$ as follows. Players I and II play a game of length δ^+. The initial pair of moves is generated by player II choosing the trivial condition q_0 and player I responding by choosing $q_1 \in D_1$. Then, at an even stage $\alpha + 2$, player II picks $q_{\alpha+2} \geq q_{\alpha+1}$ by using some fixed strategy $\mathcal{S}$, where $q_{\alpha+1}$ was chosen by player I to be so that $q_{\alpha+1} \in D_{\alpha+1}$ and $q_{\alpha+1} \geq q_\alpha$. If α is a limit ordinal, player II uses $\mathcal{S}$ to pick q_α extending each q_β for $\beta < \alpha$. By the $\prec\delta^+$-strategic closure of $\mathbb{Q}_0$ in both $N[G_0]$ and $V[G_0]$, the sequence $\langle q_\alpha : \alpha < \delta^+\rangle$ as just described exists. By construction, $G_1^* = \{p \in \mathbb{Q}_0 : \exists\alpha < \delta^+ \, [q_\alpha \geq p]\}$ is our $N[G_0]$-generic object over $\mathbb{Q}_0$. Since $i''G_0 \subseteq G_0 * G_1^*$, i lifts to $i : V_0[G_0] \to N[G_0][G_1^*]$, and since $k''G_0 = G_0$ and $k(\delta) = \delta$, k lifts to $k : N[G_0] \to M[G_0]$. By Fact 3 of Section 1.2.2 of [6], $k : N[G_0] \to M[G_0]$ can also be assumed to be generated by an extender of width $\gamma \in (\delta, \delta_0)$.

In analogy to the above, write $j(\mathbb{P}_1) = \mathbb{P}_1 * \dot{\mathbb{Q}}_1$. By the last sentence of the preceding paragraph and the fact δ_0 is the least N-cardinal to which $\dot{\mathbb{Q}}_0$ is forced to add a non-reflecting stationary set of ordinals of cofinality ω, we can use Fact 2 of Section 1.2.2 of [6] to infer that $H = \{p \in \mathbb{Q}_1 :$

$\exists q \in k''G_1^* \ [q \geq p]\}$ is $M[G_0]$-generic over $k(\mathbb{Q}_0) = \mathbb{Q}_1$. Thus, k lifts to $k : N[G_0][G_1^*] \to M[G_0][H]$, and we get the new commutative diagram



Since $\varrho > \eta$, the M-cardinals to which $\dot{\mathbb{Q}}_1$ is forced to add non-reflecting stationary sets of ordinals of cofinality ω lie in the interval $(\eta^+, j(\delta))$. Therefore, as $V_{\eta+1} \subseteq M$, $V_{\eta+1}[G_0] \subseteq M[G_0]$, and as $\mathbb{Q}_1$ adds non-reflecting stationary sets of ordinals of cofinality ω to certain inaccessible M-cardinals in the interval $(\eta^+, j(\delta))$, $V_{\eta+1}[G_0]$ is the set of all sets of rank below $\eta + 1$ in $M[G_0][H]$. Hence, j is an $\eta + 1$ strong embedding. Since η was an arbitrary non-Mahlo inaccessible cardinal below κ, this proves Lemma 3.1. ∎

We remark that by the observation made immediately following the proof of Lemma 1.1, Lemma 3.1 actually shows that δ is $\kappa + 1$ strong in V_1.

LEMMA 3.2. *$V_1 \models$ "δ is not $\kappa + 2$ strong".*

Proof. We argue in analogy to the proof of Lemma 2.2. We again begin by noting that $V_1 = V_0^{\mathbb{P}^1} \models$ "κ is the least measurable cardinal above δ + κ is strongly compact but is not $2^\kappa = \kappa^+$ supercompact". This follows by the fact κ is both the least measurable and least strongly compact cardinal above δ in V_0, the fact that $\mathbb{P}_1$ has cardinality $\delta < \kappa$ in V_0, and the results of [14].

Work in V_0. As in the proof of Lemma 2.2, for any ordinal α, $(\lambda_\alpha)^{V_0} = (\lambda_\alpha)^{V_1}$. Also, we can once more write $\mathbb{P}_1 = \mathbb{P}' * \dot{\mathbb{P}}''$, where $|\mathbb{P}'| = \omega$ and $\Vdash_{\mathbb{P}'}$ "$\dot{\mathbb{P}}''$ is $\aleph_1$-strategically closed". As before, $\mathbb{P}_1$ "admits a gap at $\aleph_1$", so by the Gap Forcing Theorem of [10] and [11], any cardinal ζ which is λ_ζ strong in V_1 had to have been λ_ζ strong in V_0. Since by its definition, forcing with $\mathbb{P}_1$ over V_0 destroys the weak compactness of any cardinal $\zeta < \delta$ that was λ_ζ strong in V_0, the preceding sentence implies that $V_1 = V_0^{\mathbb{P}^1} \models$ "No cardinal $\zeta < \delta$ is λ_ζ strong". This immediately implies that $V_1 \models$ "δ is not $\kappa + 2$ strong", since otherwise, by choosing $\ell : V_1 \to M^*$ as an elementary embedding witnessing the $\kappa + 2$ strongness of δ and reflecting the fact that $M^* \models$ "δ is κ strong and κ is the least measurable cardinal above δ", we would infer that $\{\zeta < \delta : \zeta \text{ is } \lambda_\zeta \text{ strong}\}$ is unbounded in δ in V_1. This proves Lemma 3.2. ∎

By defining $\mathbb{P} = \mathbb{P}_0 * \dot{\mathbb{P}}_1$, Lemmas 3.1 and 3.2 complete the proof of Theorem 2. ∎

We conclude Section 3 and this paper with several observations. First, as the referee has essentially indicated, if $V \models$ "ZFC + GCH + $\delta < \kappa$ are so that δ is strong and κ is strongly compact", then we may force over V with the partial ordering $\mathbb{P}$ as just defined in order to obtain the conclusions of Theorem 2. In addition, as before, it is possible to change the definition of $\mathbb{P}_1$ so as to ensure δ will satisfy a greater degree of strongness in V_1. If, e.g., we change the definition of $\mathbb{P}_1$ so that we add non-reflecting stationary sets of ordinals of cofinality ω to every cardinal $\zeta < \delta$ which is $\beth_\zeta(\lambda_\zeta)$ strong (and by the strongness of δ, there are unboundedly in δ many such cardinals), then in V_1, δ will be $\beth_\delta(\kappa)$ strong but not $\beth_\delta(\kappa) + 1$ strong. Also, since Magidor's proof from [15] that iterated Prikry forcing preserves the strong compactness of κ is valid regardless of the large cardinal structure of the universe above κ, unlike Theorem 1, there is no need to do an initial forcing to ensure that $V \models$ "No cardinal $\lambda > \kappa$ is measurable".

Finally, we note that under the same hypotheses as in Theorem 1, i.e., that $V \models$ "ZFC + $\kappa_1 < \kappa_2$ are supercompact", it is possible to modify the definition of the partial ordering $\mathbb{P}$ of Theorem 1 so that $V^{\mathbb{P}} \models$ "ZFC + κ_2 is strongly compact but not supercompact + κ_1 is α supercompact for every $\alpha < \kappa_2$ + κ_1 is not supercompact + κ_1 is strong". To do this, we observe that Lemma 2.1 of [4] and the succeeding remark actually imply that if $j : V \to M$ is an elementary embedding witnessing (at least) the $2^{\lambda_{\kappa_1}}$ supercompactness of κ_1, then $M \models$ "κ_1 is a strong cardinal and κ_1 is λ_{κ_1} supercompact", meaning that $A = \{\delta < \kappa_1 : \delta$ is a strong cardinal and δ is λ_δ supercompact$\}$ is unbounded in κ_1. Therefore, if $\mathbb{P}_0$ is as in the definition given in the proof of Theorem 1, $V_0 = V^{\mathbb{P}_0}$, and $\mathbb{P}_1$ is defined in V_0 as the Easton support iteration of length κ_1 which first adds a Cohen subset of ω and then adds, to every $\delta \in A$, a non-reflecting stationary set of ordinals of cofinality ω, the exact same arguments as before show that $V^{\mathbb{P}} \models$ "ZFC + κ_2 is strongly compact but not supercompact + κ_1 is α supercompact for every $\alpha < \kappa_2$". If in the proof of Lemma 2.2, we replace the property "δ is λ_δ supercompact" with "δ is a strong cardinal and δ is λ_δ supercompact", then the same proof as given in Lemma 2.2 remains valid and shows $V^{\mathbb{P}} \models$ "κ_1 is not $2^{\kappa_2} = \kappa_2^+$ supercompact". Further, if we choose $\lambda > \kappa_2$ as any cardinal so that $\lambda = \aleph_\lambda = \beth_\lambda$ and $j : V_0 \to M$ as an elementary embedding witnessing the λ strongness of κ_1 so that $M \models$ "κ_1 is not λ strong", then either the argument given in the proof of Lemma 2.5 of [4] or the one in the proof of Lemma 3.1 shows that $V_0^{\mathbb{P}_1} = V^{\mathbb{P}} \models$ "κ_1 is λ strong". Since λ may be chosen arbitrarily large, this means that $V^{\mathbb{P}} \models$ "κ_1 is strong". And, in analogy to what was mentioned in the concluding remarks of Section 2, it is possible to change the definition of $\mathbb{P}_1$ to ensure that κ_1 witnesses a greater degree of supercompactness in $V^{\mathbb{P}}$, assuming that the cardinals to which non-reflecting stationary sets of ordinals of cofinality ω are added are also strong.

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Received 24 November 2001;
in revised form 24 February 2002