FUNDAMENTA
MATHEMATICAE
175 (2002)

Unstable homotopy invariance for finite fields

by

Kevin P. Knudson (Mississippi State, MS)

Abstract. We prove that if k is a finite field with p^d elements, then the natural map $H_i(\mathrm{GL}_n(k),\mathbb{Z}) \to H_i(\mathrm{GL}_n(k[t]),\mathbb{Z})$ is an isomorphism for $0 \le i < d(p-1)$ and for all n.

Introduction. In [2] we proved that if k is an infinite field, then for all n the natural map

$$j_* : H_\bullet(\mathrm{SL}_n(k),\mathbb{Z}) \to H_\bullet(\mathrm{SL}_n(k[t]),\mathbb{Z})$$

induced by the inclusion is an isomorphism. The same result is then true for GL_n. The method of proof was to examine the action of $\mathrm{SL}_n(k[t])$ on a suitable building $\mathcal{X}$ and then use the resulting spectral sequence. The hypothesis that k be infinite was crucial for the computation of the homology of the various simplex stabilizers.

If k is finite, then one may use the same approach, but the calculation of the homology of the stabilizers is more difficult. In this note, we carry this out and prove the following result.

Theorem 3.1. *Let k be a finite field with p^d elements. Then for $0 \le i < d(p-1)$ and for all n, the restriction map*

$$j^* : H^i(\mathrm{GL}_n(k[t]),\mathbb{Z}) \to H^i(\mathrm{GL}_n(k),\mathbb{Z})$$

is an isomorphism.

The same result is therefore true for homology. The bound on i is optimal; see Remark 2.2 below.

Note that this result is in keeping with one's intuition that if the field k is large, then the map j^* should be an isomorphism "most of the time". Also, it is independent of n and therefore complements what van der Kallen's stability theorem [1] gives:

$$j^* : H^i(\mathrm{GL}_n(k[t]),\mathbb{Z}) \to H^i(\mathrm{GL}_n(k),\mathbb{Z})$$

2000 *Mathematics Subject Classification*: Primary 20G10.

Partially supported by NSF grant no. DMS-0070119.

is an isomorphism for $i \geq 2n+1$. One may be able to say more about the map j^* in specific cases, but Theorem 3.1 seems to be the best possible general result.

The calculation of the cohomology of the stabilizers is similar to Quillen's computation of the mod p cohomology of $\mathrm{GL}_n(k)$. Indeed, one need only compute the invariants of the action of the diagonal subgroup $T \subset \mathrm{GL}_n(k)$ on the cohomology of various Sylow p-subgroups. It was Quillen's paper [3] that motivated the proof of Theorem 3.1.

This paper is organized as follows. In Section 1 we describe the building $\mathcal{X}$ and the stabilizers of the $\mathrm{GL}_n(k[t])$ action on it. In Section 2 we calculate the cohomology of the stabilizers. Finally in Section 3 we prove Theorem 3.1.

1. The spectral sequence. Denote by $\mathcal{X}$ the Bruhat–Tits building associated to the vector space $k(t)^n$. Recall that the vertices of $\mathcal{X}$ are equivalence classes of $\mathcal{O}$-lattices in $k(t)^n$ (here, $\mathcal{O}$ consists of the set of a/b with $\deg b \geq \deg a$), where two lattices L and L' are equivalent if there is an $x \in k^\times$ with $L' = xL$. A collection of vertices $\Lambda_0, \Lambda_1, \ldots, \Lambda_m$ forms an m-simplex if there are representatives L_i of the Λ_i with

$$t^{-1}L_0 \subset L_m \subset L_{m-1} \subset \ldots \subset L_0.$$

It is possible to put a metric on $\mathcal{X}$ so that each edge in $\mathcal{X}$ has length one. When we speak of the distance between vertices we implicitly use this metric. For a more complete description of $\mathcal{X}$, see, for example, [2].

The group $\mathrm{GL}_n(k[t])$ acts on $\mathcal{X}$ with fundamental domain an infinite contractible wedge $\mathcal{T}$, which is the subcomplex of $\mathcal{X}$ spanned by the vertices

$$[e_1 t^{r_1}, \ldots, e_{n-1}t^{r_{n-1}}, e_n], \qquad r_1 \geq \ldots \geq r_{n-1} \geq 0,$$

where $e_1, \ldots, e_n$ denotes the standard basis of $k(t)^n$ (see [4]).

Denote by v_0 the vertex $[e_1, \ldots, e_n]$ and by v_i the vertex

$$[e_1 t, \ldots, e_i t, e_{i+1}, \ldots, e_n], \qquad i = 1, \ldots, n-1.$$

For a q-element subset $I = \{i_1, \ldots, i_q\}$ of $\{1, \ldots, n-1\}$, define $E_I^{(q)}$ to be the subcomplex of $\mathcal{T}$ which is the union of all rays with origin v_0 passing through the $(q-1)$-simplex $\langle v_{i_1}, \ldots, v_{i_q}\rangle$. There are $\binom{n-1}{q}$ such $E_I^{(q)}$. Observe that if $I = \{1, \ldots, n-1\}$, then $E_I^{(n-1)} = \mathcal{T}$. When we write $E_J^{(l)}$, the superscript l denotes the cardinality of the set J.

The structure of the various simplex stabilizers was described in [2]. If $(x_{ij}(t)) \in \mathrm{GL}_n(k[t])$ stabilizes the vertex $[e_1 t^{r_1}, \ldots, e_{n-1}t^{r_{n-1}}, e_n]$, then we have

$$\deg x_{ij}(t) \leq r_i - r_j$$

(set $r_n = 0$). Note that since $r_1 \geq \ldots \geq r_{n-1}$, some of the $x_{ij}(t)$ with $i > j$ may be 0. Denote the stabilizer of σ by Γ_σ. The group Γ_σ is the intersection of the stabilizers Γ_v where v ranges over the vertices of σ. In this case $\deg x_{ij}(t) \le \min_{v\in\sigma}\{r_i^{(v)} - r_j^{(v)}\}$. Observe that in any case, the group Γ_σ has a block form where the blocks along the diagonal are matrices with entries in k, blocks below are zero, and blocks above contain polynomials of bounded degree. In the case $n = 3$, we have the following block forms:

$$
\begin{aligned}
v_0 : \quad & \Gamma_{v_0} = \mathrm{GL}_3(k) \\
\sigma \in E^{(1)}_{\{1\}} : \quad & \Gamma_\sigma = \left(\begin{array}{c|cc} * & * & * \\ \hline 0 & * & * \\ 0 & * & * \end{array}\right), \\
\sigma \in E^{(1)}_{\{2\}} : \quad & \Gamma_\sigma = \left(\begin{array}{cc|c} * & * & * \\ & * & * \\ \hline 0 & 0 & * \end{array}\right), \\
\sigma \in \mathcal{T} - (E^{(1)}_{\{1\}} \cup E^{(1)}_{\{2\}}) : \quad & \Gamma_\sigma = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.
\end{aligned}
$$

We have a short exact sequence

$$1 \to C_\sigma \to \Gamma_\sigma \xrightarrow{t=0} P_\sigma \to 1$$

where P_σ is a parabolic subgroup of $\mathrm{GL}_n(k)$. From the above description of Γ_σ, we see that the group C_σ has a block form where blocks along the diagonal are identity matrices, blocks below are zero, and blocks above contain polynomials of bounded degree which are divisible by t.

Filter the complex $\mathcal{T}$ by setting $W^{(0)} = v_0$ and

$$W^{(l)} = \bigcup_{|I|=l} E^{(l)}_I, \qquad 1 \le l \le n-1.$$

Observe that if σ and τ are simplices in the same component of $W^{(i)} - W^{(i-1)}$, then $P_\sigma = P_\tau$ since on such a component the relationships among the r_i defining the vertices do not vary from vertex to vertex (i.e., if $r_i > r_{i+1}$ for one vertex in the component, then the same holds for every vertex in the component; since these relations determine which entries below the diagonal are zero, we see that the stabilizers of these vertices all have the same block form and hence so does the stabilizer of any simplex in the component).

The action of $\mathrm{GL}_n(k[t])$ on $\mathcal{X}$ gives rise to a spectral sequence converging to $H^\bullet(\mathrm{GL}_n(k[t]), \mathbb{Z})$ with E_1-term

(1)
$$E_1^{p,q} = \prod_{\dim\sigma = p} H^q(\Gamma_\sigma, \mathbb{Z})$$

where $\mathit{\Gamma}_\sigma$ denotes the stabilizer of the simplex $\sigma \in \mathcal{T}$. Now we shall compute the terms $E_2^{p,0}$.

PROPOSITION 1.1. *The bottom row of the spectral sequence* (1) *satisfies*

$$E_2^{p,0} = 0, \qquad p > 0.$$

Proof. The bottom row of the spectral sequence consists of the groups

$$E_1^{p,q} = \prod_{\dim \sigma = p} H^0(\mathit{\Gamma}_\sigma, \mathbb{Z}).$$

This is simply the cochain complex $C^\bullet(\mathcal{T}, \mathbb{Z})$. Since $\mathcal{T}$ is contractible, we see that $E_2^{p,0} = 0$ for $p > 0$. ■

We now consider cohomology with $\mathbb{F}_l$-coefficients, $l \neq p$. In this case we see that since in the extensions

$$1 \to C_\sigma \to \mathit{\Gamma}_\sigma \to P_\sigma \to 1$$

the subgroups C_σ are p-groups, the map $\mathit{\Gamma}_\sigma \to P_\sigma$ induces an isomorphism

$$H^\bullet(P_\sigma, \mathbb{F}_l) \to H^\bullet(\mathit{\Gamma}_\sigma, \mathbb{F}_l). \tag{2}$$

Now, for each q, we have a coefficient system $\mathcal{H}^q$ on $\mathcal{T}$ defined by

$$\mathcal{H}^q(\sigma) = H^q(\mathit{\Gamma}_\sigma, \mathbb{F}_l).$$

In this notation, then, the qth row of the spectral sequence (1) is the cochain complex $C^\bullet(\mathcal{T}, \mathcal{H}^q)$. Using the isomorphism (2), we see that on each component of $W^{(i)} - W^{(i-1)}$ the system $\mathcal{H}^q$ is constant for each q. By Lemma 5 of [4] the inclusion $v_0 \to \mathcal{T}$ induces an isomorphism

$$H^\bullet(\mathcal{T}, \mathcal{H}^q) \xrightarrow{\cong} H^\bullet(v_0, \mathcal{H}^q)$$

for all q. It follows that

$$E_2^{pq} = \begin{cases} H^q(\mathrm{GL}_n(k), \mathbb{F}_l), & p = 0, \\ 0, & p > 0. \end{cases}$$

Hence we have the following result, due to Soulé [4].

PROPOSITION 1.2. *If $l \neq p$, then the natural map $\mathrm{GL}_n(k) \to \mathrm{GL}_n(k[t])$ induces an isomorphism*

$$H^\bullet(\mathrm{GL}_n(k[t]), \mathbb{F}_l) \to H^\bullet(\mathrm{GL}_n(k), \mathbb{F}_l)$$

for all n. ■

It remains to calculate the $\mathbb{F}_p$-cohomology of the various $\mathit{\Gamma}_\sigma$; we do this in the next section.

2. Mod p cohomology. In this section cohomology is with $\mathbb{F}_p$ coefficients. We shall write $H^\bullet(G)$ for $H^\bullet(G,\mathbb{F}_p)$.

Let Γ_σ be the stabilizer of the simplex σ and consider the extension

$$1 \to C_\sigma \to \Gamma_\sigma \to P_\sigma \to 1,$$

where P_σ is a parabolic subgroup of $\mathrm{GL}_n(k)$. Observe that the Sylow p-subgroup of P_σ is the group $U_n(k)$ consisting of upper triangular unipotent matrices over k. Indeed, if Q is a Sylow p-subgroup of P_σ, then $Q \supseteq U_n(k)$ since $U_n(k)$ is a p-subgroup of P_σ. But since Q is a p-subgroup of $\mathrm{GL}_n(k)$, we must have $Q \subseteq U_n(k)$. Thus, denoting by U_σ a Sylow p-subgroup of Γ_σ, we have an extension

$$1 \to C_\sigma \to U_\sigma \to U_n(k) \to 1.$$

The group U_σ consists of matrices whose i,j entry has degree less than l_{ij}, where

$$l_{ij} = \min_{v\in\sigma}\{r_i^{(v)} - r_j^{(v)}\}$$

(see Section 1).

Now, the restriction map

$$j^* : H^\bullet(\Gamma_\sigma) \to H^\bullet(U_\sigma)$$

is injective. If T denotes the diagonal subgroup of $\mathrm{GL}_n(k)$, then since T normalizes U_σ, the image of j^* lies in the subgroup of T-invariants.

Proposition 2.1. *$H^i(U_\sigma)^T = 0$ for $0 < i < d(p-1)$.*

Proof. Let Δ^+ be the set of positive roots ($t \mapsto t_i/t_j$, $1 \le j < i \le n$, where t_i is the ith entry of the diagonal matrix t). Order Δ^+ by setting $(i',j') \le (i,j)$ if either $i' < i$, or $i' = i$ and $j' \le j$. If $a \in \Delta^+$, let U_a be the subgroup of U_σ generated by the one-parameter subgroups corresponding to roots $> a$. For each $a \in \Delta^+$, we have an extension

$$1 \to (k_a)^{l_a} \to U_\sigma/U_a \to U_\sigma/U_{a'} \to 1,$$

where k_a is the T-module k with T acting via the root a, $l_a = l_{ij}$ for $a = (i,j)$, and a' is the element of Δ^+ immediately preceding a (if $a = (1,2)$, then $U_{a'} = U_\sigma$). The group T acts on this extension. Since the extension is central, the associated Hochschild–Serre spectral sequence has the form

$$E_2 = H^\bullet(U_\sigma/U_{a'}) \otimes H^\bullet(k_a)^{\otimes l_a} \Rightarrow H^\bullet(U_\sigma/U_a). \tag{3}$$

Denote by $\mathrm{PS}(M)$ the Poincaré series of the T-module M. By Lemma 15 of [3], we have

$$\mathrm{PS}(H^\bullet(k_a)) = \prod_{b=0}^{d-1}(1 + a^{-p^b}z)/(1 - a^{-p^b}z^2).$$

The spectral sequence (3) shows that

$$\mathrm{PS}(H^\bullet(U_\sigma/U_a)) \ll \mathrm{PS}(H^\bullet(U_\sigma/U_{a'}))\mathrm{PS}(H^\bullet(k_a))^{l_a},$$

where $\ll$ means that each term in the series on the left is less than or equal to the corresponding term on the right. Combining these for all $a \in \Delta^+$, we have, letting $M = \max_{a\in\Delta^+}\{l_a\}$,

$$\begin{aligned}\mathrm{PS}(H^\bullet(U_\sigma)) \ll \prod_{a\in\Delta^+}\mathrm{PS}(H^\bullet(k_a))^{l_a} &\ll \prod_{a\in\Delta^+}\mathrm{PS}(H^\bullet(k_a))^M \\ &= \Big[\prod_{a\in\Delta^+}\prod_{b=0}^{d-1}(1+a^{-p^b}z)\sum_{j\geq 0}a^{-jp^b}z^{2j}\Big]^M \\ &= \Big[\sum_I\Big(\prod_{a\in\Delta^+}a^{-M_a(I)}\Big)z^{D(I)}\Big]^M,\end{aligned}$$

where I runs over families (m_{ab}, n_{ab}) with $0 \leq m_{ab} \leq 1$ and $n_{ab} \geq 0$, and

$$M_a(I) = \sum_{b=0}^{d-1}(m_{ab}+n_{ab})p^b, \qquad D(I) = \sum_{a\in\Delta^+}\sum_{b=0}^{d-1}(m_{ab}+2n_{ab}).$$

It follows that

$$\mathrm{PS}(H^\bullet(U_\sigma)) \ll \sum_{r\geq 0}\Big(\sum_{I_1,\ldots,I_M:\sum D(I_j)=r}\Big(\prod_{a\in\Delta^+}a^{-\sum_{j=1}^M M_a(I_j)}\Big)\Big)z^r.$$

Now, it suffices to show that if $I_1, \ldots, I_M$ satisfy

$$\prod_{a\in\Delta^+}a^{-\sum_{j=1}^M M_a(I_j)} = \varepsilon$$

where ε is the trivial character of T, then

$$\sum_{j=1}^M D(I_j) \geq d(p-1).$$

Let $a_i(t) = t_{i+1}/t_i$, $1 \leq i \leq n-1$, be the simple roots. Then if $a = t_j/t_h \in \Delta^+$, we may write

$$a = \prod_{i=1}^{n-1}(a_i)^{c_{ai}}$$

where $c_{ai} = 1$ for $h \leq i < j$ and 0 otherwise. Then our equation becomes

$$\prod_a a^{-\sum_{j=1}^M M_a(I_j)} = \prod_{i=1}^{n-1}a_i^{-c_{ai}\sum_{j=1}^M M_a(I_j)} = \prod_{i=1}^{n-1}a_i^{-e_i}$$

where

$$e_i = c_{ai} \sum_{j=1}^{M} \sum_{b=0}^{d-1} (m_{ab}^{(j)} + n_{ab}^{(j)}) p^b,$$

and $I_j = \{(m_{ab}^{(j)}, n_{ab}^{(j)})\}$.

Since $k^\times$ is cyclic of order $p^d - 1$, for each $1 \le i < n$ we have

$$\sum_{j=1}^{M} \sum_{b=0}^{d-1} c_{ai} (m_{ab}^{(j)} + n_{ab}^{(j)}) p^b \equiv 0 \bmod (p^d - 1),$$

and hence

$$\sum_{b=0}^{d-1} \sum_{j=1}^{M} c_{ai} (m_{ab}^{(j)} + n_{ab}^{(j)}) p^b \equiv 0 \bmod (p^d - 1).$$

By Lemma 16 of [3], we have

$$\sum_{b=0}^{d-1} \sum_{j=1}^{M} c_{ai} (m_{ab}^{(j)} + n_{ab}^{(j)}) \ge d(p-1),$$

provided at least one of the $\sum_j c_{ai}(m_{ab}^{(j)} + n_{ab}^{(j)})$ is nonzero.

Now, suppose that for some j, $D(I_j) > 0$. Then for some a, b, we have $m_{ab} + n_{ab} > 0$ and hence $\sum_j c_{ai}(m_{ab}^{(j)} + n_{ab}^{(j)}) > 0$ for some b and i. Thus,

$$\begin{aligned} \sum_{j=1}^{M} D(I_j) &= \sum_{j=1}^{M} \Big(\sum_{a,b} (m_{ab}^{(j)} + 2n_{ab}^{(j)}) \Big) \ge \sum_{j=1}^{M} \Big(\sum_{a,b} (m_{ab}^{(j)} + n_{ab}^{(j)}) \Big) \\ &\ge \sum_{j=1}^{M} \Big(\sum_{a,b} c_{ai} (m_{ab}^{(j)} + n_{ab}^{(j)}) \Big) \ge d(p-1) \end{aligned}$$

as required. ∎

REMARK 2.2. Note that this bound is optimal. Indeed, if $k = \mathbb{F}_p$, $\sigma = v_0$, then $\Gamma_\sigma = \mathrm{GL}_n(k)$, $U_\sigma = U_n(k)$, and $H^{p-1}(U_n(k))$ contains $H^1(k_a)^{\otimes(p-1)}$, which is a trivial T-module by Fermat's Little Theorem. Another example is provided by the group $\mathrm{GL}_2(\mathbb{F}_2[t])$. Using the amalgamated free product decomposition of this group, one can show that $H_1(\mathrm{GL}_2(\mathbb{F}_2[t]), \mathbb{Z})$ contains an infinite-dimensional $\mathbb{F}_2$-vector space, while $H_1(\mathrm{GL}_2(\mathbb{F}_2), \mathbb{Z})$ is finite.

3. The main theorem

THEOREM 3.1. *If* $0 \le i < d(p-1)$, *then the restriction map*

$$j^* : H^i(\mathrm{GL}_n(k[t]), \mathbb{Z}) \to H^i(\mathrm{GL}_n(k), \mathbb{Z})$$

is an isomorphism.

Proof. We know that j^* is an isomorphism for all i with $\mathbb{F}_l$-coefficients, and for $i = 0$ with $\mathbb{Z}$-coefficients. Let us consider $\mathbb{F}_p$-cohomology. By Proposition 2.1, the E_1-term of the spectral sequence (1) satisfies

$$E_1^{p,q} = 0, \qquad 0 < q < d(p-1).$$

It follows then that $E_\infty^{p,q} = 0$ for $0 < q < d(p-1)$. Since $E_\infty^{p,0} = 0$ for $p > 0$, we see that

$$H^i(\mathrm{GL}_n(k[t]), \mathbb{F}_p) = 0, \qquad 0 < i < d(p-1).$$

Since the same is true for $H^i(\mathrm{GL}_n(k), \mathbb{F}_p)$ (see [3]), we conclude that j^* is an isomorphism for $0 \leq i < d(p-1)$. ∎

References

[1] W. van der Kallen, *Homology stability for linear groups*, Invent. Math. 60 (1980), 269–295.

[2] K. Knudson, *The homology of special linear groups over polynomial rings*, Ann. Sci. École Norm. Sup. (4) 30 (1997), 385–415.

[3] D. Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math. (2) 96 (1972), 552–586.

[4] C. Soulé, *Chevalley groups over polynomial rings*, in: Homological Group Theory, C. T. C. Wall (ed.), London Math. Soc. Lecture Note Ser. 36, Cambridge Univ. Press, Cambridge, 1979, 359–367.

Department of Mathematics and Statistics
Mississippi State University
P.O. Drawer MA
Mississippi State, MS 39762, U.S.A.
E-mail: knudson@math.msstate.edu

Received 28 February 2002