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A compact Hausdorff topology that is a T_1-complement of itself

by

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Abstract. Topologies τ_1 and τ_2 on a set X are called T_1-*complementary* if $\tau_1 \cap \tau_2 = \{X \setminus F : F \subseteq X \text{ is finite}\} \cup \{\emptyset\}$ and $\tau_1 \cup \tau_2$ is a subbase for the discrete topology on X. Topological spaces (X, τ_X) and (Y, τ_Y) are called T_1-*complementary* provided that there exists a bijection $f : X \to Y$ such that τ_X and $\{f^{-1}(U) : U \in \tau_Y\}$ are T_1-complementary topologies on X. We provide an example of a compact Hausdorff space of size $2^{\mathfrak{c}}$ which is T_1-complementary to itself ($\mathfrak{c}$ denotes the cardinality of the continuum). We prove that the existence of a compact Hausdorff space of size $\mathfrak{c}$ that is T_1-complementary to itself is both consistent with and independent of ZFC. On the other hand, we construct in ZFC a countably compact Tikhonov space of size $\mathfrak{c}$ which is T_1-complementary to itself and a compact Hausdorff space of size $\mathfrak{c}$ which is T_1-complementary to a countably compact Tikhonov space. The last two examples have the smallest possible size: It is consistent with ZFC that $\mathfrak{c}$ is the smallest cardinality of an infinite set admitting two Hausdorff T_1-complementary topologies [8]. Our results provide complete solutions to Problems 160 and 161 (both posed by S. Watson [14]) from *Open Problems in Topology* (North-Holland, 1990).

1. Introduction. Recall that a topology τ on a set X is called a T_1 *topology*, and the pair (X, τ) is called a T_1-*space*, provided that all singletons $\{x\}$ of X are τ-closed. The *cofinite topology* $\{X \setminus F : F \subseteq X \text{ is finite}\} \cup \{\emptyset\}$ on X is the smallest (with respect to set inclusion) T_1 topology on X.

Two topologies τ_1 and τ_2 on an infinite set X are called:

(i) T_1-*independent* if their set-theoretic intersection $\tau_1 \cap \tau_2$ coincides with the cofinite topology on X;

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(ii) *transversal* if their set-theoretic union $\tau_1 \cup \tau_2$ is a subbase for the discrete topology $\{U : U \subseteq X\}$ on X;

(iii) T_1-*complementary* if τ_1 and τ_2 are both T_1-independent and transversal.

It is clear that T_1-independent (in particular, T_1-complementary) topologies must be T_1, so from now on we will only consider T_1 topologies. T_1-spaces (X, τ_X) and (Y, τ_Y) will be called T_1-*complementary* provided that there exists a bijection $f : X \to Y$ such that τ_X and $\{f^{-1}(U) : U \in \tau_Y\}$ are T_1-complementary topologies on X. Of special interest are those T_1-spaces that are T_1-complementary to themselves.

The set $L(X)$ of all T_1 topologies on a given set X with the order $\tau \le \sigma$ defined by set inclusion $\tau \subseteq \sigma$ forms a lattice rich enough to represent all lattices: Every lattice can be embedded into the lattice $(L(X), \le)$ of T_1-topologies on a suitable set X (see [7]). The topological notion of T_1-complementarity of topologies on X corresponds to the algebraic notion of complementarity in the lattice $(L(X), \le)$. We refer the reader to [10], [9], [1], [5] and [14, Sec. 12] for details and relevant discussions.

The notion of T_1-independent topologies was first introduced in [12] (in the context of topological groups) and studied in [8], while the notion of transversal topologies was introduced in [11] and studied thoroughly in [8]. The classical notion of T_1-complementarity has been studied for a long time (see [10], [9], [1], [5] and [11]).

Since the properties of T_1-independence and transversality look opposite to each other and appear (at least intuitively) contradictory, it comes as no surprise that T_1-complementary Tikhonov spaces are notoriously difficult to construct. In fact, the only known "real" (= using no additional axioms beyond ZFC) example of such spaces is due to S. Watson [15] who applied ingeniously sophisticated graph-theoretic techniques to produce a zero-dimensional Tikhonov space of size $2^{\mathfrak{c}}$ that is T_1-complementary to itself. (We use $\mathfrak{c}$ to denote the cardinality of the continuum.) This construction seems completely unsuitable for producing T_1-complementary Tikhonov spaces of size smaller than $2^{\mathfrak{c}}$ which naturally led S. Watson to ask the following

Question 1.1 ([14, Problem 92]). Is it consistent with ZFC that any Hausdorff topology which is its own T_1-complement must lie on a set of cardinality at least $\mathfrak{c}^+$?

Transversality is in apparent contradiction with compactness, so in the same 1990 paper S. Watson asks another fundamental

Question 1.2 ([14, Problem 93]). Is there a compact Hausdorff space that is T_1-complementary to itself?

We note that even a consistent example of a space from Question 1.2 was unknown ([1]). We answer Question 1.2 affirmatively by providing a compact Hausdorff topology on a set of size $2^{\mathfrak{c}}$ that is a T_1-complement of itself. Our example is a well-understood space, the Aleksandrov duplicate of $\beta\omega \setminus \omega$, the remainder of the Stone–Čech compactification of the integers ω, and our proof that this space works uses only standard set-theoretic arguments (outlined in the proof of Lemmas 2.2 and 2.3). This offers a pleasant contrast with a sophisticated construction of an example and highly involved graph-theoretic arguments that it works found in the original manuscript by Watson [15]. (In addition, compactness comes as a bonus.) We also present an example, in ZFC, of a countably compact Tikhonov space of cardinality $\mathfrak{c}$ which is a T_1-complement of itself. This answers Question 1.1 in a strongly negative way. We prove that the existence of a compact Hausdorff space of size $\mathfrak{c}$ that is T_1-complementary to itself is both consistent with and independent of ZFC. Demonstrating the limits of the last consistency result we show, in ZFC, that a compact Hausdorff space of size $\mathfrak{c}$ can have a countably compact Tikhonov T_1-complement. The size $\mathfrak{c}$ of the examples above is the least possible: It is consistent with ZFC that no infinite set of size less than $\mathfrak{c}$ admits a pair of T_1-independent (in particular, T_1-complementary) Hausdorff topologies [8, Theorem 3.3].

2. Technical lemmas. A space X is called *subsequential* [2] if every non-closed countable infinite set A in X contains a sequence converging to a point of $X \setminus A$. It is easy to see that sequential spaces as well as sequentially compact spaces are subsequential. We will need a slight generalization of [12, Proposition 2.4]:

LEMMA 2.1. *Let τ_1 and τ_2 be T_1-independent Hausdorff topologies on a set X. If the space (X, τ_1) is subsequential, then the space (X, τ_2) is countably compact and does not contain non-trivial convergent sequences.*

Our next lemma offers a delicate refinement of the ideas from the proof of Lemma 3.7 of [8].

LEMMA 2.2. *Assume that κ is an infinite cardinal, Y and Z are T_1-spaces, Y' is a subset of Y and Z' is a subset of Z satisfying the following conditions*:

(1) $|Y'| = |Z'|$;
(2) $|Y| = |Z| = |Y \setminus Y'| = |Z \setminus Z'| = \kappa = \kappa^\omega$;
(3) *every point $y \in Y'$ has a local base at y in Y of size $\leq \kappa$*;

([1]) It should be mentioned that the existence of two infinite T_1-complementary compact Hausdorff spaces was announced to S. Watson by Bohdan Aniszczyk in 1989, but the example has never been published.

(4) *if* U *is an open subset of* Y *with* $U \cap Y' \neq \emptyset$, *then* $|U \setminus Y'| = \kappa$;

(5) *if* F *is an infinite closed subset of* Z, *then* $|F \setminus Z'| = \kappa$.

Then there exists a bijection $f : Y \to Z$ *with the following properties*:

(a) $f(Y') = Z'$;

(b) *if* Φ *is an infinite closed subset of* Y *and* $Y' \setminus \Phi \neq \emptyset$, *then* $f(\Phi)$ *is not closed in* Z.

Proof. Use (1) to fix a bijection $\varphi : Y' \to Z'$. According to (2) we can select faithful enumerations $Y \setminus Y' = \{y_\alpha : \alpha < \kappa\}$ and $Z \setminus Z' = \{z_\alpha : \alpha < \kappa\}$. For every $y \in Y'$ use (3) to fix a local base $\mathcal{B}_y$ of Y at y with $|\mathcal{B}_y| \leq \kappa$, and set $\mathcal{B} = \bigcup\{\mathcal{B}_y : y \in Y'\}$. Since $Y' \subseteq Y$, condition (2) implies $|Y'| \leq \kappa$ and thus $|\mathcal{B}| \leq \kappa$. Since $|Y| = \kappa = \kappa^\omega$ by (2), we have $|[Y]^\omega \times \mathcal{B}| \leq \kappa$, where $[Y]^\omega$ denotes the set of all countable infinite subsets of Y. Let $\{(C_\alpha, U_\alpha) : \alpha < \kappa\}$ be an enumeration of the set $[Y]^\omega \times \mathcal{B}$ such that for every pair $(C, U) \in [Y]^\omega \times \mathcal{B}$, the set $\{\alpha < \kappa : (C, U) = (C_\alpha, U_\alpha)\}$ is cofinal in κ.

By recursion on $\alpha < \kappa$ we will construct $Y_\alpha \subseteq Y$, $Z_\alpha \subseteq Z$ and a map $f_\alpha : Y_\alpha \to Z_\alpha$ such that the following conditions hold:

(i$_\alpha$) $Y' \subseteq Y_\alpha$, $Z' \subseteq Z_\alpha$;

(ii$_\alpha$) $|Y_\alpha \setminus Y'| \leq \alpha \cdot \omega$ and $|Z_\alpha \setminus Z'| \leq \alpha \cdot \omega$;

(iii$_\alpha$) if $\gamma < \alpha$, then $Y_\gamma \subseteq Y_\alpha$ and $Z_\gamma \subseteq Z_\alpha$;

(iv$_\alpha$) $y_\alpha \in Y_\alpha$ and $z_\alpha \in Z_\alpha$;

(v$_\alpha$) f_α is a bijection between Y_α and Z_α extending φ;

(vi$_\alpha$) if $\gamma < \alpha$, then $f_\alpha|_{Y_\gamma} = f_\gamma$;

(vii$_\alpha$) $f_\alpha(U_\alpha \cap Y_\alpha) \cap \overline{f_\alpha(C_\alpha \cap Y_\alpha)} \neq \emptyset$ provided that $\alpha \neq 0$ and $C_\alpha \cap Y_\alpha$ is infinite.

In (vii$_\alpha$) above and later in the proof of this lemma, the symbol $\bar{A}$ denotes the closure of the set $A \subseteq Z$ in Z.

Basis of induction. Let $Y_0 = Y' \cup \{y_0\}$, $Z_0 = Y' \cup \{z_0\}$ and $f_0 = \varphi \cup \{(y_0, z_0)\}$. A trivial check shows that conditions (i$_0$)–(vii$_0$) are satisfied.

Inductive step. Let $0 < \alpha < \kappa$, and suppose that sets $Y_\beta \subseteq Y$, $Z_\beta \subseteq Z$ and a map $f_\beta : Y_\beta \to Z_\beta$ satisfying conditions (i$_\beta$)–(vii$_\beta$) have already been defined for all $\beta < \alpha$. We construct $Y_\alpha \subseteq Y$, $Z_\alpha \subseteq Z$ and a map $f_\alpha : Y_\alpha \to Z_\alpha$ satisfying conditions (i$_\alpha$)–(vii$_\alpha$).

First, define $Y'_\alpha = \bigcup\{Y_\beta : \beta < \alpha\}$, $Z'_\alpha = \bigcup\{Z_\beta : \beta < \alpha\}$ and $f'_\alpha = \bigcup\{f_\beta : \beta < \alpha\}$. Clearly, $Y' \subseteq Y'_\alpha \subseteq Y$, $Z' \subseteq Z'_\alpha \subseteq Z$, $|Y'_\alpha \setminus Y'| \leq \alpha \cdot \omega$, $|Z'_\alpha \setminus Z'| \leq \alpha \cdot \omega$ and $f'_\alpha : Y'_\alpha \to Z'_\alpha$ is a bijection.

Since $U_\alpha \in \mathcal{B} = \bigcup\{\mathcal{B}_y : y \in Y'\}$, one has $U_\alpha \in \mathcal{B}_y$ for some $y \in Y'$, and so $y \in U_\alpha \cap Y' \neq \emptyset$. From (4) it now follows that $|U_\alpha \setminus Y'| = \kappa$. Since $|Y'_\alpha \setminus Y'| \leq \alpha \cdot \omega < \kappa$, we can pick a point

$$y^*_\alpha \in (U_\alpha \setminus Y') \setminus (Y'_\alpha \setminus Y') = U_\alpha \setminus (Y' \cup Y'_\alpha) \subseteq U_\alpha \setminus Y'_\alpha.$$

If $C_\alpha \cap Y'_\alpha$ is infinite, then $F_\alpha = \overline{f'_\alpha(C_\alpha \cap Y'_\alpha)}$ is an infinite closed subset of Z and so $|F_\alpha \setminus Z'| = \kappa$ by (5), which combined with $|Z'_\alpha \setminus Z'| \le \alpha \cdot \omega < \kappa$ allows us to pick a point

$$z^*_\alpha \in (F_\alpha \setminus Z') \setminus (Z'_\alpha \setminus Z') = F_\alpha \setminus (Z' \cup Z'_\alpha) \subseteq F_\alpha \setminus Z'_\alpha.$$

In case $C_\alpha \cap Y'_\alpha$ is finite, since $|Z \setminus Z'| = \kappa$ by (2) and $|Z'_\alpha \setminus Z'| \le \alpha \cdot \omega < \kappa$, we can pick a point

$$z^*_\alpha \in (Z \setminus Z') \setminus (Z'_\alpha \setminus Z') = Z \setminus (Z' \cup Z'_\alpha) \subseteq Z \setminus Z'_\alpha.$$

It is easy to define sets $Y_\alpha \subseteq Y$, $Z_\alpha \subseteq Z$ and a bijection $f_\alpha : Y_\alpha \to Z_\alpha$ such that $Y'_\alpha \cup \{y_\alpha, y^*_\alpha\} \subseteq Y_\alpha$, $Z'_\alpha \cup \{z_\alpha, z^*_\alpha\} \subseteq Y_\alpha$, $|Y_\alpha \setminus Y'_\alpha| \le 2$, $|Z_\alpha \setminus Z'_\alpha| \le 2$, f_α extends f'_α and $f_\alpha(x^*_\alpha) = y^*_\alpha$.

Clearly, conditions (i_α)–(vi_α) hold. Let us verify condition (vii_α). Suppose that $C_\alpha \cap Y_\alpha$ is infinite. Since $|Y_\alpha \setminus Y'_\alpha| \le 2$, the intersection $C_\alpha \cap Y'_\alpha$ must also be infinite. By our construction, $y^*_\alpha \in U_\alpha \cap Y_\alpha$ and $f_\alpha(y^*_\alpha) = z^*_\alpha \in F_\alpha = \overline{f'_\alpha(C_\alpha \cap Y'_\alpha)} \subseteq \overline{f_\alpha(C_\alpha \cap Y_\alpha)}$, which yields $z^*_\alpha \in f_\alpha(U_\alpha \cap Y_\alpha) \cap \overline{f_\alpha(C_\alpha \cap Y_\alpha)} \neq \emptyset$. The inductive step is complete.

We can now define the bijection $f : Y \to Z$. From (iv_α) for all $\alpha < \kappa$ and our choice of y_α's and z_α's it follows that $Y = \bigcup\{Y_\alpha : \alpha < \kappa\}$ and $Z = \bigcup\{Z_\alpha : \alpha < \kappa\}$. Define $f = \bigcup\{f_\alpha : \alpha < \kappa\}$. Since ($\mathrm{iii}_\alpha$), ($\mathrm{v}_\alpha$) and ($\mathrm{vi}_\alpha$) hold for all $\alpha < \kappa$, f is a bijection between Y and Z.

From (vi_α) for all $\alpha < \kappa$ it follows that $f\restriction_{Y_0} = f_0$, and from ($\mathrm{i}_0$) and ($\mathrm{v}_0$) we conclude that $f_0|_{Y'} = \varphi$, which yields $f|_{Y'} = \varphi$. Since $\varphi : Y' \to Z'$ is a bijection, we get $f(Y') = f\restriction_{Y'}(Y') = \varphi(Y') = Z'$. Thus (a) holds.

It only remains to prove (b). Let Φ be an infinite closed subset of Y such that $Y' \setminus \Phi \neq \emptyset$. Pick a point $y \in Y' \setminus \Phi \subseteq Y \setminus \Phi$. Clearly, $Y \setminus \Phi$ is an open subset of Y, so $y \in U \subseteq Y \setminus \Phi$ for some $U \in \mathcal{B}_y$. Since $y \in Y'$, one has $U \in \mathcal{B}$. Note that the cofinality of the cardinal κ is uncountable since $\kappa = \kappa^\omega$. From $Y = \bigcup\{Y_\alpha : \alpha < \kappa\}$ and (iii_α) for $\alpha < \kappa$ we conclude that $\Phi \cap Y_\beta$ must be infinite for some $\beta < \kappa$. Choose an infinite countable set $C \subseteq \Phi \cap Y_\beta$ and note that $(C, U) \in [Y]^\omega \times \mathcal{B}$. Since the set $\{\alpha < \kappa : (C, U) = (C_\alpha, U_\alpha)\}$ is cofinal in κ, $(C, U) = (C_\alpha, U_\alpha)$ for some α with $\beta < \alpha < \kappa$. From $Y_\alpha \supseteq Y_\beta$ and $C_\alpha = C \subseteq Y_\beta$ we get $C_\alpha \cap Y_\alpha \supseteq C_\alpha \cap Y_\beta = C \cap Y_\beta = C$, and since the last set is infinite, so is $C_\alpha \cap Y_\alpha$. From $\alpha > \beta \ge 0$ and (vii_α) one gets $f_\alpha(U_\alpha \cap Y_\alpha) \cap \overline{f_\alpha(C_\alpha \cap Y_\alpha)} \neq \emptyset$. Since f extends f_α and $\Phi \supseteq C = C_\alpha$, it follows that

$$f(U_\alpha) \cap \overline{f(\Phi)} \supseteq f(U_\alpha) \cap \overline{f(C_\alpha)} \supseteq f_\alpha(U_\alpha \cap Y_\alpha) \cap \overline{f_\alpha(C_\alpha \cap Y_\alpha)} \neq \emptyset.$$

Therefore, there exists $y^* \in U_\alpha$ such that $f(y^*) \in \overline{f(\Phi)}$. From $U_\alpha = U \subseteq Y \setminus \Phi$ one gets $y^* \notin \Phi$. Since f is a bijection between Y and Z, this yields $f(y^*) \notin f(\Phi)$. Thus $f(y^*) \in \overline{f(\Phi)} \setminus f(\Phi)$, i.e., the set $f(\Phi)$ is not closed in Z. This finishes the proof of (b). ∎

Lemma 2.3. *Suppose that* κ, Y, Z, Y' *and* Z' *satisfy the assumptions of Lemma* 2.2. *Furthermore, assume that the following additional conditions hold*:

(6) *all points of* $Y \setminus Y'$ *are isolated in* Y;
(7) *all points of* Z' *are isolated in* Z;
(8) Z' *is dense in* Z.

Then the spaces Y *and* Z *are* T_1*-complementary.*

Proof. Let τ_Y be the topology of Y and τ_Z be the topology of Z. Consider a bijection $f : Y \to Z$ from the conclusion of Lemma 2.2. We are going to prove that the topologies τ_Y and $\sigma = \{f^{-1}(V) : V \in \tau_Z\}$ on the set Y are T_1-complementary.

T_1*-independence.* We need to check that the intersection $\tau_Y \cap \sigma$ is exactly the smallest T_1 topology η on Y. Since both τ_Y and τ_Z (and thus σ) are T_1 topologies, $\tau_Y \cap \sigma$ is also a T_1 topology on Y, and thus $\eta \subseteq \tau_Y \cap \sigma$ by minimality of η. To prove the reverse inclusion we need to show that no infinite proper subset of Y is closed in the topology $\tau_Y \cap \sigma$ or, equivalently, that no infinite proper τ_Y-closed set is σ-closed. Using the definition of σ we conclude that the last property is equivalent to the following one: If Φ is an infinite proper closed subset of Y, then $f(\Phi)$ is *not* closed in Z. To check the latter property, let Φ be an arbitrary infinite proper closed subset of Y. If $Y' \setminus \Phi \neq \emptyset$, then $f(\Phi)$ is not closed in Z by item (b) of Lemma 2.2. It only remains to consider the case $Y' \subseteq \Phi$. In this case, $Z' = f(Y') \subseteq f(\Phi)$ by item (a) of Lemma 2.2 and, therefore, $f(\Phi)$ is dense in Z by (8). If $f(\Phi)$ were closed in Z, we would have $f(\Phi) = Z$ and thus $\Phi = Y$ since $f : Y \to Z$ is a bijection. This contradicts the fact that Φ is a proper subset of Y. Hence, $f(\Phi)$ cannot be closed in Z.

Transversality. Let us check that the union $\tau_Y \cup \sigma$ is a subbase for the discrete topology on Y. It suffices to verify that for every $y \in Y$, there exist $U_y \in \tau_Y$ and $W_y \in \sigma$ such that $\{y\} = U_y \cap W_y$. Let $y \in Y$ be arbitrary. If $y \in Y \setminus Y'$, then from (6) it follows that $\{y\} \in \tau_Y$, so $U_y = \{y\}$ and $W_y = Y$ do the job. Assume now that $y \in Y'$. Since $f : Y \to Z$ is a bijection and $f(Y') = Z'$ by item (a) of Lemma 2.2, $f(y) \in Z'$ and so $\{f(y)\} \in \tau_Z$ by our assumption (7). Since f is a bijection, $\{y\} = \{f^{-1}(f(y))\} \in \sigma$ according to the definition of σ. Now $U_y = Y$ and $W_y = \{y\}$ do the job. ■

3. Main results. Let (X, τ) be a T_1-space and $x \mapsto x^*$ be a bijection of X onto its copy X^* disjoint from X. For $Z \subseteq X$, let $Z^* = \{x^* : x \in Z\}$. We set $A(X) = X \cup X^*$ and consider the following topology τ' on $A(X)$. Each point $x^* \in X^*$ is isolated in τ' and, for every $x \in X$, the family

$$\{(U \cup U^*) \setminus K : U \in \tau,\ x \in U,\ K \subseteq X^* \text{ is finite}\}$$

is a local base of τ' at x. The set $A(X)$ together with the topology τ' is called the *Aleksandrov duplicate* of (X,τ). We will omit τ when no confusion is possible.

We use $\chi(X)$ to denote the character of a space X.

THEOREM 3.1. *Let X be a T_1-space and κ be an infinite cardinal satisfying the following conditions*:

(i) $\chi(X) \le |X| = \kappa = \kappa^\omega$;

(ii) *every non-empty open subset of X is of cardinality κ*;

(iii) *each infinite closed subset of X is of cardinality κ.*

Then the Aleksandrov duplicate $A(X)$ of X is a T_1-complement of itself.

Proof. In view of Lemma 2.3 it suffices to check that κ, $Y = Z = A(X)$, $Y' = X$ and $Z' = X^*$ satisfy conditions (1)–(8) from Lemmas 2.2 and 2.3.

Conditions (1), (2) and (3) follow from (i).

To prove (4) assume that U is an open subset of $A(X)$ such that $U \cap X \neq \emptyset$. Pick a point $x \in U \cap X$. According to the definition of a base at x there exist an open subset V of X containing x and a finite set $K \subseteq X^*$ with $(V \cup V^*) \setminus K \subseteq U$. Then $|V^*| = |V| = \kappa$ by (ii) and thus $|V^* \setminus K| = \kappa$. From $V^* \setminus K \subseteq U \cap X^* = U \setminus X \subseteq A(X)$ and $|A(X)| = |X| = \kappa$ it now follows that $|U \setminus X| = \kappa$.

To check (5) assume that F is an infinite closed subset of $A(X)$. It suffices to prove that the intersection $F\cap X$ is infinite, since then $F\cap X$ is an infinite closed subset of X and, hence, $|F \cap X| = \kappa$ by (iv). The conclusion of (5) now follows from $F \setminus X^* = F \cap X$. Therefore, it only remains to prove that the set $F \cap X$ is infinite. If $F \cap X^*$ is finite, then $F \cap X = F \setminus (F \cap X^*)$ must be infinite, as required. So we can assume that $F \cap X^*$ is infinite and choose a countable infinite set $C \subseteq X$ with $C^* \subseteq F$. Let Φ be the closure of C in X. Then Φ is an infinite closed subset of X, so $|\Phi| = \kappa$ by (iii). Since $\kappa^\omega = \kappa$ by (i), it follows that $\kappa > \omega$ and thus $|\Phi \setminus C| = \kappa$. In particular, the set $\Phi \setminus C$ is infinite. Let $x \in \Phi \setminus C$ be arbitrary. Since X is a T_1-space and Φ is the closure of C in X, for every open set U of X with $x \in U$, the intersection $U \cap C$ must be infinite. From the definition of the topology of $A(X)$ it now follows that x belongs to the closure of the set C^* in $A(X)$. Since $C^* \subseteq F$ and F is a closed subset of $A(X)$, we get $x \in F$. We have proved that $\Phi \setminus C \subseteq F$ and, therefore, $F \cap X$ contains the infinite set $\Phi \setminus C$.

Conditions (6), (7) and (8) immediately follow from our definition of Y, Z, Y' and Z'. ■

Note that condition (i) from Theorem 3.1 implies $\kappa > \omega$, which, when combined with item (iii) of the same theorem, yields countable compactness of X. In other words, a space satisfying the assumptions of Theorem 3.1

must be countably compact, thereby justifying the appearance of countable compactness in our next result.

Corollary 3.2. *Let X be a countably compact regular T_1-space satisfying $\chi(X) \leq |X| = \mathfrak{c}$ which has neither isolated points nor non-trivial convergent sequences. Then the Aleksandrov duplicate $A(X)$ of X is regular, countably compact and T_1-complementary to itself.*

Proof. The regularity and countable compactness of $A(X)$ are immediate. From [2, Th. 6.8] it follows that every countably compact regular T_1-space of cardinality less than $\mathfrak{c}$ is sequentially compact and, hence, contains non-trivial convergent sequences (if infinite). The assumptions about X now imply that all infinite closed subsets of X have cardinality $\mathfrak{c}$. Applying countable compactness, regularity and the absence of isolated points in X one can easily deduce (via the standard binary tree argument) that all non-empty open subsets of X have size $\mathfrak{c}$. It remains to apply Theorem 3.1 with $\kappa = \mathfrak{c}$. ∎

It is worth mentioning that the space from our next corollary cannot exist in ZFC since its cardinality must be at least 2^{ω_1}.

Corollary 3.3. *Suppose that X is a dense-in-itself compact Hausdorff space of cardinality $\mathfrak{c}$ without non-trivial convergent sequences. Then the Aleksandrov duplicate $A(X)$ of X is its own T_1-complement.*

Proof. The compactness of the space X implies that $\chi(X) \leq w(X) \leq |X| = \mathfrak{c}$. Since compact Hausdorff spaces are regular T_1-spaces, we can apply Corollary 3.2. ∎

Our next result answers Question 1.1 in a strongly negative way.

Corollary 3.4. *There exists a countably compact, zero-dimensional Tikhonov space of cardinality $\mathfrak{c}$ without convergent sequences which is a T_1-complement of itself.*

Proof. Let S be a dense subset of $\beta\omega \setminus \omega$ with $|S| = \mathfrak{c}$. By [2, Fact 6.5], there exists a countably compact subspace X of $\beta\omega \setminus \omega$ such that $S \subseteq X$ and $|X| = \mathfrak{c}$. Clearly, X is zero-dimensional, dense in itself and does not contain non-trivial convergent sequences. In addition, $\chi(X) \leq w(X) \leq w(\beta\omega) = \mathfrak{c}$. Therefore, the Aleksandrov duplicate $A(X)$ is regular, countably compact and T_1-complementary to itself by Corollary 3.2. One easily verifies that $A(X)$ is a zero-dimensional space of cardinality $\mathfrak{c}$ which has no convergent sequences other than trivial. In particular, $A(X)$ is Tikhonov as every regular zero-dimensional space. ∎

Under some additional set-theoretic assumptions, the space in Corollary 3.4 can even be chosen to be compact. The symbol $\mathfrak{s}$ below denotes the splitting number (see [2]).

Corollary 3.5. *Under $\mathfrak{s} = \omega_1$ & $2^{\omega_1} = \mathfrak{c}$, there exists a compact Hausdorff space of cardinality $\mathfrak{c}$ which is its own T_1-complement.*

Proof. Fedorchuk [4] showed that, under $\mathfrak{s} = \omega_1$ and $2^{\omega_1} = \mathfrak{c}$, there exists a space X satisfying the conditions of Corollary 3.3 (see also [2, pp. 132–133]), and the result follows. ∎

It is known that every compact Hausdorff space of cardinality less than 2^{ω_1} is sequentially compact [6], and thus subsequential. Combining this fact and Lemma 2.1, we conclude that an infinite compact Hausdorff space which admits a T_1-independent (in particular, T_1-complementary) compact Hausdorff topology must have size at least 2^{ω_1} [8, Corollary 3.5]. In particular, under $\mathfrak{c} < 2^{\omega_1}$, no compact Hausdorff space of cardinality $\mathfrak{c}$ admits a T_1-complementary compact Hausdorff topology (compare this with Corollary 3.9 below). Together with Corollary 3.5, this gives:

Corollary 3.6. *The existence of a compact Hausdorff space of cardinality $\mathfrak{c}$ which is its own T_1-complement is both consistent with and independent of ZFC.*

Let us show that, dropping restrictions on the cardinality of spaces, one obtains many examples (in ZFC) of compact Hausdorff spaces which are T_1-complements of themselves.

A subset Y of a space X is called *C^*-embedded* in X if every bounded real-valued continuous function $f : Y \to \mathbb{R}$ admits a continuous extension over X. An *F-space* is a Tikhonov space such that all its countable subsets are C^*-embedded in it.

Corollary 3.7. *Let X be a compact F-space without isolated points satisfying $\chi(X) \le \mathfrak{c}$. Then the Aleksandrov duplicate $A(X)$ of X is a T_1-complement of itself.*

Proof. By Arkhangel'skiĭ's theorem, $|X| \le 2^{\chi(X)} \le 2^{\mathfrak{c}}$. Further, in a compact F-space, every infinite closed subset contains a topological copy of $\beta\omega$ (see [13]). Hence, every infinite closed subset of X satisfies $2^{\mathfrak{c}} \le |F| \le |X| \le 2^{\mathfrak{c}}$. In particular, $|X| = 2^{\mathfrak{c}}$. Similarly, every non-empty open subset of X is of cardinality $2^{\mathfrak{c}}$. Therefore, we can apply Lemma 3.1 with $\kappa = 2^{\mathfrak{c}}$ and $Z = X$ to conclude that $A(X)$ is its own T_1-complement. ∎

Since $\beta\omega \setminus \omega$ is a compact F-space of weight $\mathfrak{c}$ without isolated points, we have the following result.

Corollary 3.8. *Let $X = \beta\omega \setminus \omega$. Then the Aleksandrov duplicate $A(X)$ of X is a compact Hausdorff space of size $2^{\mathfrak{c}}$ which is a T_1-complement of itself.*

The above corollary answers Question 1.2 in the affirmative.

Finally, we show in ZFC that a set of cardinality $\mathfrak{c}$ admits a pair of T_1-complementary Hausdorff topologies one of which is compact and the other is countably compact and zero-dimensional (hence, Tikhonov). The reader may want to compare this result with Corollaries 3.4 and 3.6.

Corollary 3.9. *There exists a compact Hausdorff space of size $\mathfrak{c}$ with only countably many non-isolated points which admits a countably compact, Tikhonov, zero-dimensional T_1-complementary topology.*

Proof. Let Y be the one-point compactification of the disjoint sum of countably many copies of the one-point compactification of the discrete space of size $\mathfrak{c}$. Denote by Y' the set of all non-isolated points of Y. Let Z be a countably compact subspace of the Stone–Čech compactification $\beta\omega$ of the integers ω such that $\omega \subseteq Z$ and all infinite closed subsets of Z have size $\mathfrak{c}$ (such a subspace Z can be constructed via standard transfinite induction arguments similar to the construction of the space X from [3, Example 3.10.19]). Finally, let Z' be ω, the set of isolated points of Z. A straightforward check shows that $\kappa = \mathfrak{c}$, Y, Z, Y' and Z' satisfy conditions (1)–(8) from Lemmas 2.2 and 2.3. Now Lemma 2.3 implies that Y and Z are T_1-complementary. ∎

The above corollary should be compared with the fact that no infinite Hausdorff space with only finitely many non-isolated points admits a T_1-independent (in particular, T_1-complementary) Hausdorff topology [8, Proposition 3.6].

Corollaries 3.6 and 3.8 make it natural to ask the following

Question 3.10. In ZFC only, does there exist a compact Hausdorff space of size 2^{ω_1} (or of size $\mathfrak{c}^+$) which is T_1-complementary to itself?

The space X from Corollary 3.8 has size $2^{\omega_1} = \mathfrak{c}^+$ under the Continuum Hypothesis $\mathfrak{c} = \omega_1$ combined with $2^{\mathfrak{c}} = \mathfrak{c}^+$.

Our constructions of T_1-complementary topologies depend essentially on Lemma 2.3 or on the use of the Aleksandrov duplicate of a space (see Theorem 3.1). Therefore, all pairs of T_1-complementary spaces presented in the article have many isolated points. Watson's self-complementary Tikhonov space [14] also has lots of isolated points. This gives rise to our second open problem:

Question 3.11. Does there exist a dense-in-itself (compact) Hausdorff space which is T_1-complementary to itself?

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