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The Suslinian number and other cardinal invariants of continua

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Abstract. By the *Suslinian number* $\mathrm{Sln}(X)$ of a continuum X we understand the smallest cardinal number κ such that X contains no disjoint family $\mathcal{C}$ of non-degenerate subcontinua of size $|\mathcal{C}| > \kappa$. For a compact space X, $\mathrm{Sln}(X)$ is the smallest Suslinian number of a continuum which contains a homeomorphic copy of X. Our principal result asserts that each compact space X has weight $\le \mathrm{Sln}(X)^+$ and is the limit of an inverse well-ordered spectrum of length $\le \mathrm{Sln}(X)^+$, consisting of compacta with weight $\le \mathrm{Sln}(X)$ and monotone bonding maps. Moreover, $w(X) \le \mathrm{Sln}(X)$ if no $\mathrm{Sln}(X)^+$-Suslin tree exists. This implies that under the Suslin Hypothesis all Suslinian continua are metrizable, which answers a question of Daniel et al. [Canad. Math. Bull. 48 (2005)]. On the other hand, the negation of the Suslin Hypothesis is equivalent to the existence of a hereditarily separable non-metrizable Suslinian continuum. If X is a continuum with $\mathrm{Sln}(X) < 2^{\aleph_0}$, then X is 1-dimensional, has rim-weight $\le \mathrm{Sln}(X)$ and weight $w(X) \ge \mathrm{Sln}(X)$. Our main tool is the inequality $w(X) \le \mathrm{Sln}(X) \cdot w(f(X))$ holding for any light map $f : X \to Y$.

In this paper we introduce a new cardinal invariant related to the Suslinian property of continua. By a *continuum* we understand any compact connected Hausdorff space. Following Lelek [7], we define a continuum X to be *Suslinian* if it contains no uncountable family of pairwise disjoint non-degenerate subcontinua. The simplest example of a Suslinian continuum is the usual interval $\mathbb{I} = [0,1]$. On the other hand, the existence of non-metrizable Suslinian continua is a subtle problem. The properties of such continua were considered in [1]. It was shown in [1] that each Suslinian continuum X is perfectly normal, rim-metrizable, and 1-dimensional. Moreover, a locally connected Suslinian continuum has weight $\le \omega_1$.

The simplest examples of non-metrizable Suslinian continua are Suslin lines. However this class of examples has a consistency flavor since no Suslin

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line exists in some models of ZFC (for example, in models satisfying (MA + ¬CH)). It turns out that any example of a non-metrizable locally connected Suslinian continuum necessarily has consistency nature: the existence of such a continuum is equivalent to the existence of a Suslin line (see [1]). This implies that under the Suslin Hypothesis (asserting that no Suslin line exists) each locally connected Suslinian continuum is metrizable.

It is clear that each Suslinian continuum X has countable Suslin number $c(X)$. At this point we recall the definition of some known topological cardinal invariants. Given a topological space X let

- $c(X) = \sup\{|\mathcal{U}| : \mathcal{U}$ is a disjoint family of non-empty open subsets of $X\}$ be the *Suslin number* of X;
- $l(X) = \min\{\kappa :$ each open cover of X contains a subcover of size $\le \kappa\}$ be the *Lindelöf number* of X;
- $d(X) = \min\{|D| : D$ is a dense set in $X\}$ be the *density* of X;
- $hl(X) = \sup\{l(Y) : Y \subset X\}$ be the *hereditary Lindelöf number* of X;
- $hd(X) = \sup\{d(Y) : Y \subset X\}$ be the *hereditary density* of X;
- $w(X) = \min\{|\mathcal{B}| : \mathcal{B}$ is a base of the topology of $X\}$ be the *weight* of X;
- rim-$w(X) = \min\{\sup_{U\in\mathcal{B}} w(\partial U) : \mathcal{B}$ is a base of the topology of $X\}$ be the *rim-weight* of X.

In the context of Suslinian continua, by analogy with the Suslin number $c(X)$ it is natural to introduce a new cardinal invariant

$$\mathrm{Sln}(X) = \sup\{|\mathcal{C}| : \mathcal{C} \text{ is a disjoint family of non-degenerate subcontinua of } X\}$$

defined for any continuum X and called the *Suslinian number* of X. Thus a continuum X is Suslinian if and only if $\mathrm{Sln}(X) \le \aleph_0$.

It is clear that $\mathrm{Sln}(X) \le \mathrm{Sln}(Y)$ for any pair $X \subset Y$ of continua. It will be convenient to extend the definition of $\mathrm{Sln}(X)$ to all Tikhonov spaces letting

$$\mathrm{Sln}(X) = \min\{\mathrm{Sln}(Y) : Y \text{ is a continuum containing } X\}$$

for a Tikhonov space X.

Like many other cardinal invariants the Suslinian number is monotone.

Proposition 1. *If X is a Tikhonov space and Y is a subspace of X, then* $\mathrm{Sln}(Y) \le \mathrm{Sln}(X)$.

The cardinal invariant $\mathrm{Sln}(X)$ is not trivial since it can attain any infinite value.

Proposition 2. $\mathrm{Sln}(X) = c(X) = w(X) = \kappa$ *for the hedgehog* $X = \{(x_\alpha)_{\alpha<\kappa} : |\{\alpha < \kappa : x_\alpha \ne 0\}| \le 1\} \subset [0,1]^\kappa$ *with κ needles.*

Note that each hedgehog is *rim-finite* in the sense that it has a base of the topology consisting of sets with finite boundaries. Let us remark that a rim-finite continuum X with uncountable Suslinian number must be non-metrizable (because rim-countable metrizable continua are Suslinian, see [7]).

The Suslinian number cannot increase under monotone maps. We recall that a map $f : X \to Y$ is *monotone* if $f^{-1}(y)$ is connected for any $y \in Y$.

PROPOSITION 3. *If X and Y are compact spaces and $f : X \to Y$ is a surjective monotone map, then* $\mathrm{Sln}(Y) \le \mathrm{Sln}(X)$.

Proof. Embed X in a continuum Z with $\mathrm{Sln}(Z) = \mathrm{Sln}(X)$. Consider the following equivalence relation on Z: $x \sim y$ if either $x = y$ or $x, y \in X$ and $f(x) = f(y)$. Let $T = Z/\!\sim$ be the quotient space and $q : Z \to T$ be the quotient map. Since all the equivalence classes are connected, the quotient map q is monotone. Since the preimage of a connected set under a monotone map is connected, $\mathrm{Sln}(T) \le \mathrm{Sln}(Z)$. It remains to observe that Y can be identified with a subspace of T, which yields $\mathrm{Sln}(Y) \le \mathrm{Sln}(T) \le \mathrm{Sln}(Z) = \mathrm{Sln}(X)$. For a similar argument, we also refer the readers to Theorem 2.4.13 in [3]. ∎

PROPOSITION 4. *If X is a Tikhonov space and K is a compact subset of X, then* $\mathrm{Sln}(X/K) \le \mathrm{Sln}(X)$.

Proof. Let Z be a continuum that contains X and has $\mathrm{Sln}(Z) = \mathrm{Sln}(X)$. Since K is a compact subspace of X, the quotient space X/K naturally embeds into the quotient space Z/K. We claim that $\mathrm{Sln}(Z/K) \le \mathrm{Sln}(Z)$. In the opposite case we would find a disjoint family $\mathcal{C}$ of subcontinua in Z/K having the cardinality $|\mathcal{C}| > \mathrm{Sln}(Z)$. At most one subcontinuum $C \in \mathcal{C}$ can contain the point $K \in Z/K = \{K\} \cup (Z \setminus K)$. Deleting this subcontinuum from the family $\mathcal{C}$, if necessary, we can assume that $\bigcup \mathcal{C} \subset Z \setminus K$. Then $\mathcal{C}$ can be thought of as a disjoint family of subcontinua of Z having size $|\mathcal{C}| > \mathrm{Sln}(Z)$, which contradicts the definition of $\mathrm{Sln}(Z)$. This contradiction witnesses that $\mathrm{Sln}(Z/K) \le \mathrm{Sln}(Z)$ and then

$$\mathrm{Sln}(X/K) \le \mathrm{Sln}(Z/K) \le \mathrm{Sln}(Z) = \mathrm{Sln}(X). \quad ∎$$

Recall that a map $f : X \to Y$ between compact Hausdorff spaces is called *light* if $f^{-1}(y)$ is zero-dimensional for each $y \in Y$.

THEOREM 1. *If X and Y are compact spaces and $f : X \to Y$ is a light map, then* $w(X) \le w(Y) \cdot \mathrm{Sln}(X)$.

For the proof of this theorem we shall need two lemmas.

LEMMA 1. *For any point z of a continuum Z there is a family $\mathcal{U}$ of closed neighborhoods of z in Z such that $|\mathcal{U}| \le \mathrm{Sln}(Z)$ and $\bigcap \mathcal{U}$ is zero-dimensional.*

Proof. We shall construct a transfinite sequence $(U_\alpha)_{\alpha<\alpha_0}$ of closed neighborhoods of z and a transfinite sequence $(K_\alpha)_{\alpha<\alpha_0}$ of pairwise disjoint, non-degenerate subcontinua of Z such that $K_\alpha \subset \bigcap_{\beta<\alpha} U_\beta$ and $U_\alpha \cap K_\alpha = \emptyset$ for each $\alpha < \alpha_0$.

To start the construction we choose any subcontinuum $K_0 \subset Z \setminus \{z\}$ and take any closed neighborhood $U_0 \subset Z$ of z missing the set K_0. Then U_0 is not zero-dimensional, and since Z is a continuum, we can find a subcontinuum $K_1 \subset U_0$ not containing the point z.

Suppose that for some ordinal α the closed neighborhoods U_β, $\beta < \alpha$, of z are already selected so that $\bigcap_{\beta<\alpha} U_\beta$ is not zero-dimensional. Choose any non-degenerate continuum $K_\alpha \subset \bigcap_{\beta<\alpha} U_\beta \setminus \{z\}$. Then choose a closed neighborhood U_α of z which is disjoint from K_α. Observe that when $\beta < \alpha$, then $K_\beta \cap U_\beta = \emptyset$ and $K_\alpha \subset U_\beta$, whence $K_\beta \cap K_\alpha = \emptyset$.

The construction should stop at some ordinal α_0 of size $|\alpha_0| \le \mathrm{Sln}(Z)$. For this ordinal the intersection $\bigcap_{\alpha<\alpha_0} U_\alpha$ is zero-dimensional. Then $\mathcal{U} = \{U_\alpha : \alpha < \alpha_0\}$ is the required family of closed neighborhoods of the point z in Z. ∎

Lemma 2. *For any closed subset K of a continuum Z there is a family $\mathcal{U}$ of closed neighborhoods of K such that $|\mathcal{U}| \le \mathrm{Sln}(Z)$ and $\bigcap\mathcal{U} \setminus K$ is zero-dimensional.*

Proof. Consider the quotient space $Z/K = \{K\} \cup (Z \setminus K)$ of Z by K and let $q : Z \to Z/K$ be the quotient map. By Lemma 1, the continuum Z/K contains a family $\mathcal{V}$ of closed neighborhoods of the point $K \in Z/K$ such that $|\mathcal{V}| \le \mathrm{Sln}(Z/K)$ and the intersection $\bigcap\mathcal{U}$ is zero-dimensional. It is easy to see that the family $\mathcal{U} = \{q^{-1}(V) : V \in \mathcal{V}\}$ of closed neighborhoods of K has the desired property: it has cardinality $|\mathcal{U}| \le |\mathcal{V}| \le \mathrm{Sln}(Z/K) \le \mathrm{Sln}(Z)$ and $\bigcap\mathcal{U} \setminus K$ is zero-dimensional (being homeomorphic to $\bigcap\mathcal{V} \setminus \{K\}$). ∎

Proof of Theorem 1. Let $f : X \to Y$ be a light map between compact Hausdorff spaces. We need to prove that the weight of X satisfies $w(X) \le \kappa$ where $\kappa = w(Y) \cdot \mathrm{Sln}(X)$. Let Z be a continuum such that $Z \supset X$ and $\mathrm{Sln}(X) = \mathrm{Sln}(Z)$. Of course, $\mathrm{Sln}(Z) \le \kappa$.

By Lemma 2, the continuum Z contains a family $\mathcal{U}$ of closed neighborhoods of the subset $X \subset Z$ such that $|\mathcal{U}| \le \mathrm{Sln}(Z) \le \kappa$ and $\bigcap\mathcal{U} \setminus X$ is zero-dimensional. The family $\mathcal{U}$ can be used to construct a map $g : Z \to \mathbb{I}^\kappa$ such that $X \subset g^{-1}(\mathbf{0}) \subset \bigcap\mathcal{U}$, where $\mathbf{0} = \{0\}^\kappa \in [0,1]^\kappa = \mathbb{I}^\kappa$. It follows that $g^{-1}(\mathbf{0}) \setminus X \subset \bigcap\mathcal{U} \setminus X$ is zero-dimensional.

Since $w(Y) \le \kappa$, the space Y can be identified with a subset of the Tikhonov cube $\mathbb{I}^\kappa$. It follows from the Tietze–Urysohn Theorem that the map f can be extended to a map $\bar{f} : Z \to \mathbb{I}^\kappa$. Now consider the map

$$h = (\bar{f}, g) : Z \to \mathbb{I}^\kappa \times \mathbb{I}^\kappa, \quad z \mapsto (\bar{f}(z), g(z)),$$

and observe that

$$X \subset h^{-1}(\mathbb{I}^\kappa \times \mathbf{0}) = g^{-1}(\mathbf{0}) \subset \bigcap \mathcal{U}.$$

It follows that for every $y \in \mathbb{I}^\kappa \times \mathbf{0}$ the preimage $h^{-1}(y)$ lies in the union $f^{-1}(y) \cup (\bigcap \mathcal{U} \setminus X)$ of two zero-dimensional spaces and hence is zero-dimensional.

Since Z is a continuum, each component of a non-empty open set U contains a non-trivial subcontinuum. Consequently, U has at most $\mathrm{Sln}(X)$ components. Denote by $\mathcal{C}_U$ the family of closures of components of U.

Let $\mathcal{B}$ be a base for the topology of $h(Z)$ with $|\mathcal{B}| \le \kappa$. Finally consider the family $\mathcal{C} = \bigcup_{B \in \mathcal{B}} \mathcal{C}_{h^{-1}(B)}$ of closed subsets of Z, which has size at most κ. Because of the compactness of X, the inequality $w(X) \le \kappa$ will follow as soon as we prove that the family $\mathcal{C}$ separates the points of X in the sense that any two distinct points $x, y \in X$ lie in disjoint elements C_x, C_y of the family $\mathcal{C}$.

If $h(x) \ne h(y)$, then we can find two basic subsets $B_x, B_y \in \mathcal{B}$ with disjoint closures such that $h(x) \in B_x$ and $h(y) \in B_y$. Let D_x be the component of $h^{-1}(B_x)$, containing the point x and D_y be the component of $h^{-1}(B_y)$, containing the point y. Then $\overline{D}_x, \overline{D}_y$ are disjoint elements of $\mathcal{C}$ separating the points x, y.

Next, suppose that $h(x) = h(y) = z$ and observe that $z \in h(X) \subset \mathbb{I}^\kappa \times \mathbf{0}$. It follows from the zero-dimensionality of $\bigcap \mathcal{U} \setminus X$ and the inclusion $h^{-1}(z) \subset f^{-1}(z) \cup (\bigcap \mathcal{U} \setminus X)$ that the set $h^{-1}(z)$ is zero-dimensional. Consequently, we can find two open subsets $O_x, O_y \subset Z$ with disjoint closures such that $x \in O_x$, $y \in O_y$ and $h^{-1}(z) \subset O_x \cup O_y$. Since the map h is closed, there is a basic neighborhood $B_z \in \mathcal{B}$ of z such that $h^{-1}(B_z) \subset O_x \cup O_y$. Then x and y lie in the closures of distinct components of $h^{-1}(B_z)$. This completes the proof that the collection $\mathcal{C}$ separates the points of X. ∎

The previous theorem allows us to generalize the classical monotone-light Factorization Theorem [12, 13.3] asserting that any map $f : X \to Y$ between compact Hausdorff spaces can be represented as the (unique) composition $\lambda \circ \mu$ of a monotone map $\mu : X \to Z$ and a light map $\lambda : Z \to Y$. Applying the preceding theorem and two propositions to the calculation of the weight of the space Z, we conclude that $w(Z) \le w(Y) \cdot \mathrm{Sln}(Z) \le w(Y) \cdot \mathrm{Sln}(X)$. In this way we obtain the following corollary.

COROLLARY 1. *Let $f : X \to Y$ be a map between compact spaces and $f = \lambda \circ \mu$ be the monotone-light decomposition of f into a monotone surjective map $\mu : X \to Z$ and a light map $\lambda : Z \to Y$. Then $w(Z) \le w(Y) \cdot \mathrm{Sln}(X)$ and the non-degeneracy set $N_\mu = \{z \in Z : |\mu^{-1}(z)| > 1\}$ of μ has size $|N_\mu| \le \mathrm{Sln}(X)$.*

In this corollary, if we assume that f is the constant map, then the space Z is the decomposition of X into its components and it is zero-dimensional. Thus we obtain the following corollary.

COROLLARY 2. *Each compact Hausdorff space X admits a monotone map $f : X \to Z$ onto a zero-dimensional space Z of weight $w(Z) \leq \mathrm{Sln}(X)$. In particular, each zero-dimensional compact space Z has weight $w(Z) \leq \mathrm{Sln}(Z)$.*

As another application of Theorem 1 we prove that each Suslinian continuum X is hereditarily decomposable, that is, X contains no indecomposable subcontinuum (a continuum X is *indecomposable* if X cannot be written as the union of two proper non-degenerate subcontinua of X).

PROPOSITION 5. *If X is a Tikhonov space with $\mathrm{Sln}(X) \leq \aleph_0$, then all compact zero-dimensional subspaces of X are metrizable and all subcontinua of X are decomposable.*

Proof. If Z is a zero-dimensional compact subset of X, then $w(Z) \leq \mathrm{Sln}(Z) \leq \mathrm{Sln}(X) \leq \aleph_0$ by the preceding corollary.

Now take any subcontinuum C of X. Then $\mathrm{Sln}(C) \leq \mathrm{Sln}(X) \leq \aleph_0$, which means that the continuum C is Suslinian. Let $f : C \to [0,1]$ be any non-constant map. By Theorem 1, the map f can be written as the composition $f = \lambda \circ \mu$ of a monotone map $\mu : C \to Z$ and a light map $\lambda : Z \to [0,1]$ of some continuum Z with $w(Z) \leq \mathrm{Sln}(C) \leq \aleph_0$. Thus, Z is a metrizable Suslinian continuum. Such a continuum is decomposable. Otherwise, since each indecomposable continuum has uncountably many composants (see [6, Theorem 7′, p. 213]), we would have $\mathrm{Sln}(Z) > \aleph_0$. Consequently, we can write $Z = A \cup B$ as the sum of two properly smaller subcontinua $A, B \subset Z$. Their preimages $\mu^{-1}(A)$ and $\mu^{-1}(B)$ under the monotone map μ are proper subcontinua of C whose union equals C. This means that the continuum C is decomposable. ■

Next we prove that the hereditary Lindelöf number of any space X is bounded from above by the Suslinian number of X. For Suslinian continua this result was proved in Theorem 1 of [1].

THEOREM 2. *$hl(X) \leq \mathrm{Sln}(X)$ for any Tikhonov space X.*

Proof. Let $\kappa = \mathrm{Sln}(X)$ and $Z \supset X$ be a continuum with $\mathrm{Sln}(Z) = \kappa$.

First, we prove that each singleton $\{x_0\}$, $x_0 \in X$, is the intersection of κ many neighborhoods in Z. By Lemma 1, there is a family $\mathcal{N}$ of closed neighborhoods of x_0 in Z such that $|\mathcal{N}| \leq \mathrm{Sln}(Z) = \kappa$ and the intersection $\bigcap \mathcal{N}$ is zero-dimensional. The compactum $Y = \bigcap \mathcal{N}$, being zero-dimensional, admits a light map onto the singleton. Applying Theorem 1, we get $w(Y) \leq \mathrm{Sln}(Y) \leq \mathrm{Sln}(Z) = \kappa$. Consequently, we can find a family $\mathcal{N}'$ of neighborhoods of the point x_0 in Z such that $Y \cap \bigcap \mathcal{N}' = \{x_0\}$ and $|\mathcal{N}'| \le \kappa$. Then the family $\mathcal{N} \cup \mathcal{N}'$ has size $\le \kappa$ and its intersection is $\{x_0\}$.

Now, take any subspace $A \subset X$ and let $\mathcal{U}$ be a cover of A by open subsets of Z. Then $\bigcup \mathcal{U}$ is an open subset of Z and $B = Z \setminus \bigcup \mathcal{U}$ is a closed set in Z. Consider the quotient space $Z/B = (Z \setminus B) \cup \{B\}$ and let $q : Z \to Z/B$ be the quotient map. Since $\mathrm{Sln}(Z/B) \le \mathrm{Sln}(Z) = \kappa$, we may apply the previous reasoning to find a family $\mathcal{V}$ of open neighborhoods of the singleton $\{B\} \in Z/B$ with $\{B\} = \bigcap \mathcal{V}$ and $|\mathcal{V}| \le \kappa$. Then $\mathcal{W} = \{q^{-1}(V) : V \in \mathcal{V}\}$ is a family of size $\le \kappa$ with $\bigcap \mathcal{W} = B$. The complement $Z \setminus W$ of each $W \in \mathcal{W}$ is a compact subset of Z which can be covered by a finite subcollection of $\mathcal{U}$. Therefore, the union $\bigcup_{W \in \mathcal{W}} (Z \setminus W) = \bigcup \mathcal{U}$ can be covered by $\le \kappa$ elements of the cover $\mathcal{U}$. ∎

According to [3, 3.12.10(l)], $w(X) \le 2^{hl(X)}$ for any compact Hausdorff space. Hence, $w(X) \le 2^{\mathrm{Sln}(X)}$ for any Tikhonov space. In fact, we shall prove a stronger upper bound $w(X) \le \mathrm{Sln}(X)^+$.

The *Generalized Suslin Hypothesis* asserts that for any regular cardinal κ there is no κ-Suslin tree, where a tree is called *κ-Suslin* if it has height κ but contains no chain or antichain of length κ. We recall that the classical Suslin Hypothesis asserts that there is no $\aleph_1$-Suslin tree.

Below, for a cardinal κ, we denote by $\mathrm{cf}(\kappa)$ the cofinality of κ and by κ^+ the successor cardinal of κ. We identify cardinals with initial ordinals.

THEOREM 3. *Let X be a Tikhonov space. Then $w(X) \le \mathrm{Sln}(X)^+$. Moreover, if no κ^+-Suslin tree exists for $\kappa = \mathrm{Sln}(X)$, then $w(X) \le \mathrm{Sln}(X)$.*

Proof. Let $\kappa = \mathrm{Sln}(X)$ and embed X into a continuum K with $\mathrm{Sln}(K) = \mathrm{Sln}(X)$. Assuming that $\kappa^+ < w(X) \le w(K)$, we can find a continuous map $f : K \to Z$ of K onto a continuum Z of weight $w(Z) = \kappa^{++}$. Moreover, we may assume that the map f is monotone. Indeed, if f were not monotone, then it would factorize as $f = \lambda \circ \mu$ with $\mu : K \to Z_1$ monotone and $\lambda : Z_1 \to Z$ light. Then $w(Z_1) \le w(Z) \cdot \mathrm{Sln}(K) = \kappa^{++} \cdot \kappa = \kappa^{++}$. Now, let us see that the conditions $\mathrm{Sln}(Z) \le \kappa$ and $w(Z) = \kappa^{++}$ lead to a contradiction.

Express Z as the inverse limit of a well-ordered transfinite spectrum $\{Z_\alpha : \alpha < \kappa^{++}\}$ consisting of continua Z_α with $w(Z_\alpha) \le \kappa^+$. Let $p_\alpha : Z \to Z_\alpha$, $\alpha < \kappa^{++}$, denote the (surjective) limit projections of the spectrum.

Consider the family $\mathcal{T} = \{p_\alpha^{-1}(z) : z \in Z_\alpha,\ \alpha < \kappa^{++},\ \dim p_\alpha^{-1}(z) > 0\}$ of point-preimages which are not zero-dimensional. Endowed with the inverse inclusion order, this family forms a tree. This tree has no chains of length more than κ. Otherwise we would obtain a strictly decreasing sequence of length $> \kappa$ consisting of closed subsets of Z, which is impossible as $hl(Z) \le \mathrm{Sln}(Z) = \kappa$.

The tree also contains no antichain of length $> \kappa$ since otherwise we would construct a disjoint family of size $> \kappa$ consisting of components of some elements of $\mathcal{T}$. Consequently, the tree $\mathcal{T}$ has height $\leq \kappa^+$ and all levels of the tree have size $\leq \kappa$. This implies that the tree $\mathcal{T}$ contains at most κ^+ elements. Since $\kappa^+ < \kappa^{++} = \mathrm{cf}(\kappa^{++})$, we can find an ordinal $\alpha < \kappa^{++}$ such that for any point $z \in Z_\alpha$ the preimage $p_\alpha^{-1}(z)$ is zero-dimensional. This means that the limit projection $p_\alpha : Z \to Z_\alpha$ is light. Applying Theorem 1, we get a contradiction: $w(Z) \leq w(Z_\alpha) \cdot \mathrm{Sln}(Z) \leq \kappa^+$.

If no κ^+-Suslin tree exists, then the tree $\mathcal{T}$ constructed above is not κ^+-Suslin and thus has height $\leq \kappa$. In this case we replace the condition $w(Z) = \kappa^{++}$ by $w(Z) = \kappa^+$ and see that the proof above gives that $w(Z) \leq \mathrm{Sln}(Z)$. ∎

Corollary 3. *If the Generalized Suslin Hypothesis holds, then* $w(X) \leq \mathrm{Sln}(X)$ *for any Tikhonov space* X.

Applying Theorem 3 to Suslinian continua, we obtain the answer to the second part of Problem 1 of [1].

Corollary 4. *Under the Suslin Hypothesis all Suslinian continua are metrizable.*

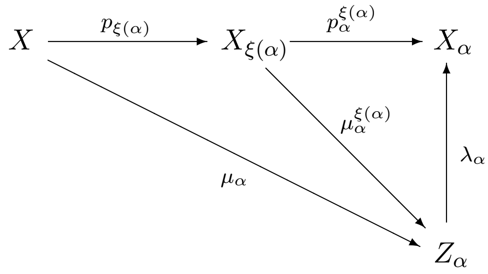
Theorem 3 allows us to describe the structure of compacta X with $w(X) > \mathrm{Sln}(X)$.

Theorem 4. *Each compact space* X *with* $w(X) > \mathrm{Sln}(X)$ *is the inverse limit of a well-ordered spectrum* $\{Z_\alpha, \pi_\alpha^\beta, \alpha \leq \beta < \mathrm{Sln}(X)^+\}$ *consisting of compacta of weight* $w(Z_\alpha) \leq \mathrm{Sln}(X)$ *and monotone bonding maps* $\pi_\alpha^\beta : Z_\beta \to Z_\alpha$.

Proof. Let $\kappa = \mathrm{Sln}(X)$. It follows from Theorem 3 that $w(X) = \kappa^+$. Therefore, we can write X as the inverse limit of a well-ordered spectrum $\mathcal{S} = \{X_\alpha, p_\alpha^\beta, \alpha \leq \beta < \kappa^+\}$ consisting of compacta of weight $\leq \kappa$ and surjective bonding maps. Since $hl(X) \leq \mathrm{Sln}(X) \leq \kappa$, this spectrum is factorizable in the sense that any continuous map $f : X \to Z$ into a compact space Z of weight $w(Z) \leq \kappa$ can be written as a composition $f = f_\alpha \circ p_\alpha$ of the limit projection $p_\alpha : X \to X_\alpha$ and a continuous map $f_\alpha : X_\alpha \to Z$ for some ordinal $\alpha < \kappa^+$ (see [5, 3.1.6]).

For each ordinal $\alpha < \kappa^+$ let $p_\alpha = \lambda_\alpha \circ \mu_\alpha$ be the (unique) monotone-light decomposition of the limit projection $p_\alpha : X \to X_\alpha$ into a monotone map $\mu_\alpha : X \to Z_\alpha$ and a light map $\lambda_\alpha : Z_\alpha \to X_\alpha$. By Proposition 3, $\mathrm{Sln}(Z_\alpha) \leq \mathrm{Sln}(X) \leq \kappa$ and by Theorem 1, $w(Z_\alpha) \leq w(X_\alpha) \cdot \mathrm{Sln}(Z_\alpha) \leq \kappa$. Then there is an ordinal $\xi(\alpha) > \alpha$ such that the monotone map $\mu_\alpha : X \to Z_\alpha$ factorizes through $X_{\xi(\alpha)}$ in the sense that $\mu_\alpha = \mu_\alpha^{\xi(\alpha)} \circ p_{\xi(\alpha)}$ for some map $\mu_\alpha^{\xi(\alpha)} : X_{\xi(\alpha)} \to Z_\alpha$.

Thus we obtain the following commutative diagram:



Let A be a cofinal subset of ordinals $< \kappa^+$ such that $\xi(\alpha) < \beta$ for any $\alpha < \beta$ in A. For any $\alpha < \beta$ in A define a bonding map $\pi_\alpha^\beta : Z_\beta \to Z_\alpha$ letting $\pi_\alpha^\beta = \mu_\alpha^{\xi(\alpha)} \circ p_{\xi(\alpha)}^\beta \circ \lambda_\beta$. We claim that the map π_α^β is monotone. This follows from the monotonicity of the map $\mu_\alpha = \mu_\alpha^{\xi(\alpha)} \circ p_{\xi(\alpha)}^\beta \circ p_\beta = \mu_\alpha^{\xi(\alpha)} \circ p_{\xi(\alpha)}^\beta \circ \lambda_\beta \circ \mu_\beta = \pi_\alpha^\beta \circ \mu_\beta$. Indeed, for any point $y \in Z_\alpha$, the preimage $(\pi_\alpha^\beta)^{-1}(y) = \mu_\beta(\mu_\alpha^{-1}(y))$ is connected, being the image of the connected set $\mu_\alpha^{-1}(y)$.

It is easy to see that $\pi_\alpha^\gamma = \pi_\alpha^\beta \circ \pi_\beta^\gamma$ for any ordinals $\alpha < \beta < \gamma$ in A, which means that $\mathcal{S}' = \{Z_\alpha, \pi_\alpha^\beta : \alpha, \beta \in A\}$ is an inverse spectrum. Let $Z = \lim \mathcal{S}'$ be the limit of this spectrum. Observe that the monotone maps $\mu_\alpha : X \to Z_\alpha$, $\alpha \in A$, induce a surjective map $\mu : X \to Z$ while the light maps $\lambda_\alpha : Z_\alpha \to X_\alpha$, $\alpha \in A$, induce a surjective map $\lambda : Z \to X$. Since $\lambda_\alpha \circ \mu_\alpha = p_\alpha$ for all $\alpha \in A$, the composition $\lambda \circ \mu : X \to X$ is the identity map of X. Consequently, both λ and μ are homeomorphisms and thus X can be identified with the limit Z of the spectrum $\mathcal{S}'$ of length κ^+ consisting of compacta of weight $\le \kappa$ and monotone bonding maps. ∎

The following particular case of Theorems 3 and 4 answers the remaining part of Problem 1 from [1].

Corollary 5. *Each non-metrizable Suslinian continuum X has weight $\aleph_1$ and is the limit of an inverse spectrum of length $\aleph_1$ consisting of metrizable Suslinian continua and monotone bonding maps.*

In the subsequent proof we shall refer to properties of the hyperspace $\exp(X)$ of a given compact Hausdorff space X. The *hyperspace* $\exp(X)$ of X is the space of all non-empty closed subsets of X, endowed with the Vietoris topology. It is well known that $\exp(X)$ is a compact Hausdorff space with $w(\exp(X)) = w(X)$. We denote by $\exp_c(X)$ the subspace of $\exp(X)$ consisting of subcontinua of X. It is easy to see that $\exp_c(X)$ is a closed subspace in $\exp(X)$.

Compacta X with small Suslinian number $\mathrm{Sln}(X) < \mathfrak{c}$ share many properties of Suslinian continua.

THEOREM 5. *If* X *is a continuum with* $\mathrm{Sln}(X) < \mathfrak{c}$, *then* $\dim X \le 1$ *and*

$$\text{rim-}w(X) \le \mathrm{Sln}(X) \le hl(\exp_c(X)) \le w(X) \le \mathrm{Sln}(X)^+.$$

Proof. Let $\kappa = \mathrm{Sln}(X)$. To show that $\text{rim-}w(X) \le \mathrm{Sln}(X)$, take any point $x \in X$ and a neighborhood $U \subset X$ of x_0. Let $f : X \to [0,1]$ be any function with $f(x_0) = \{0\}$ and $f^{-1}([0,1)) \subset U$. Since $\mathrm{Sln}(X) < \mathfrak{c}$, the set $\{y \in (0,1) : \dim f^{-1}(y) > 0\}$ has size $\le \mathrm{Sln}(X) < \mathfrak{c}$. Consequently, we can find a point $y \in (0,1)$ whose preimage $f^{-1}(y) \subset Z$ is zero-dimensional. By Corollary 2, $w(f^{-1}(y)) \le \mathrm{Sln}(f^{-1}(y)) \le \mathrm{Sln}(X) = \kappa$.

Now consider the neighborhood $V = f^{-1}([0,y))$ whose boundary ∂V lies in $f^{-1}(y)$ and thus has weight $w(\partial V) \le \kappa$ and is zero-dimensional. This proves the inequality $\text{rim-}w(X) \le \kappa$, and shows that the small inductive dimension of X satisfies $\mathrm{ind}(X) \le 1$. By [3, 7.2.7], $\dim X \le 1$.

It remains to prove that $\kappa \le hl(\exp_c(X)) \le w(X) \le \mathrm{Sln}(X)^+$. The third inequality was proved in Theorem 3 while the second inequality follows from $hl(\exp_c(X)) \le w(\exp_c(X)) \le w(\exp(X)) = w(X)$. Assuming $hl(\exp_c(X)) < \kappa = \mathrm{Sln}(X)$, let $\lambda = hl(\exp_c(X))$ and find a disjoint family $\mathcal{C}$ of size $|\mathcal{C}| = \lambda^+$ consisting of non-degenerate subcontinua of X. This family $\mathcal{C}$ can be considered as a subset of the hyperspace $\exp_c(X)$ of subcontinua of X. Identify X with the set of all degenerate subcontinua in $\exp_c(X)$. Since $hl(\exp_c(X)) = \lambda$, the set $\mathcal{C}$ contains a subset $\mathcal{C}'$ of size $|\mathcal{C}'| = |\mathcal{C}| = \lambda^+$ whose closure in $\exp_c(X)$ misses X.

We claim that $\mathcal{C}'$ is not a scattered subspace of $\exp_c(X)$. Let us recall that a topological space is *scattered* if each of its subspaces has an isolated point. It is known (and can be easily shown) that the size of a scattered space is equal to its hereditary Lindelöf number. Since $|\mathcal{C}'| = \lambda^+ > \lambda = hl(\exp_c(X)) \ge hl(\mathcal{C}')$, the space $\mathcal{C}'$ is not scattered and thus contains a subspace $\mathcal{C}''$ having no isolated point.

Now we shall construct a subset $\{C_t\}_{t\in T} \subset \mathcal{C}''$ indexed by elements of the binary tree $T = \bigcup_{n\in\mathbb{N}}\{0,1\}^n$ as follows. The binary tree T consists of finite binary sequences. Given two binary sequences $t = (t_0,\dots,t_n)$, $s = (s_0,\dots,s_m)$ in T we write $t \le s$ if $n \le m$ and $t_i = s_i$ for all $i \le n$.

Take any distinct elements $C_0, C_1 \in \mathcal{C}''$ and observe that the subcontinua C_0, C_1 are disjoint (because the family $\mathcal{C}$ is disjoint). Hence, they have open neighborhoods $U_0, U_1 \subset X$ with disjoint closures.

Assuming that for some binary sequence $s = (s_0,\dots,s_n)$ the subcontinuum $C_s \in \mathcal{C}''$ and its neighborhood $U_s \subset X$ are constructed, consider the open subset $\mathcal{U}_s = \{C \in \mathcal{C}'' : C \subset U_s\}$ of the space $\mathcal{C}''$ and take any two distinct (and hence disjoint) subcontinua $C_{s\hat{}0}, C_{s\hat{}1} \in \mathcal{U}_s$. Next, choose two open neighborhoods $U_{s\hat{}0}, U_{s\hat{}1} \subset U_s$ of $C_{s\hat{}0}, C_{s\hat{}1}$ with disjoint closures. This finishes the inductive step.

Now, for any infinite binary sequence $s = (s_i)$ let C_s be a cluster point of the set $\{C_{(s|n)} : n \in \mathbb{N}\}$ in $\exp(X)$, where $s|n = (s_0, \ldots, s_{n-1})$. It is easy to see that $\{C_s : s \in \{0,1\}^\omega\}$ is a disjoint family of subcontinua of X, lying in the closure of the set $\mathcal{C}''$. Since this closure misses the set X, each continuum C_s, $s \in \{0,1\}^\omega$, is non-degenerate. Thus, $\kappa = \mathrm{Sln}(X) \geq |\{C_s : s \in \{0,1\}^\omega\}| = \mathfrak{c}$, which is a contradiction. ∎

Problem 1. *Is* $\text{rim-}w(X) \leq \mathrm{Sln}(X)$ *for any compact Hausdorff space* X*?*

Let us remark that all examples of non-metrizable Suslinian continua considered in the introduction or in [1] contain a copy of a Suslin line and hence fail to be hereditarily separable. However (consistent) examples of non-metrizable, hereditarily separable Suslinian continua can be constructed as well. For such a construction we need the following definitions and the lemma.

We recall that a surjective map $f : X \to Y$ is *irreducible* if $f(Z) \neq Y$ for any proper closed subset Z of X. This is equivalent to saying that a set $D \subset X$ is dense in X provided $f(D)$ is dense in Y.

Following [4, III.1.15] we call a monotone map $f : X \to Y$ between two continua *atomic* if for every non-degenerate subcontinuum $Z \subset Y$ the map $f|f^{-1}(Z) : f^{-1}(Z) \to Z$ is irreducible. This is equivalent to saying that $\overline{D} = f^{-1}(\overline{f(D)})$ for every subset $D \subset X$ whose image $f(D)$ is dense in some non-degenerate subcontinuum of Y. An atomic map $f : X \to Y$ will be called *I-atomic* if for every $y \in Y$ the preimage is a singleton or an arc in X.

The following lemma will be our basic tool in the subsequent inductive construction.

Lemma 3. *For any non-degenerate metrizable Suslinian continuum* Y *and any countable set* $Z \subset Y$ *there are a metrizable Suslinian continuum* X *and an I-atomic map* $f : X \to Y$ *whose non-degeneracy set* $N(f) = \{y \in Y : |f^{-1}(y)| > 1\}$ *equals* Z.

Proof. For every $z \in Z$ fix a decreasing neighborhood base $(O_n(z))_{n\in\omega}$ at z such that $\overline{O_{n+1}(z)} \subset O_n(z)$ for all $n \in \omega$. Let $\{q_n : n \in \omega\}$ be a countable dense set in $\mathbb{I} = [0,1]$. Fix a map $h_z : Y \setminus \{z\} \to \mathbb{I}$ such that $h_z(\partial O_n(z)) = \{q_n\}$ where $\partial O_n(z)$ stands for the boundary of $O_n(z)$ in Y. Such a choice of the map h_z guarantees that $h_z(C \setminus \{z\}) = \mathbb{I}$ for any non-degenerate subcontinuum $C \subset Y$ containing z.

Now consider the set $X = (Y \setminus Z) \cup (Z \times \mathbb{I})$ and the map $f : X \to Y$ which is the identity on $Y \setminus Z$ and $f(z,t) = z$ for each $(z,t) \in Z \times \mathbb{I} \subset X$. For every $z \in Z$ let $r_z : X \to \{z\} \times \mathbb{I}$ be a unique map such that

- $r_z(y) = (z, h_z(y))$ for every $y \in Y \setminus Z \subset X$;
- $r_z(y,t) = (z, h_z(y))$ for every $(y,t) \in (Z \setminus \{z\}) \times \mathbb{I} \subset X$;
- $r_z(z,t) = (z,t)$ for every $t \in \mathbb{I}$.

Endow the space X with the weakest topology making the maps $f : X \to Y$ and $r_z : X \to \{z\} \times \mathbb{I}$, $z \in Z$, continuous. According to [4, III.1.2] the resulting space X is metrizable and compact. It is easy to check that the map f is I-atomic (see also [4, III.1.15]).

Using the atomic property of f and the Suslinian property of Y it is easy to check that X is Suslinian too. ∎

Now, we are ready for the construction of our example. We note that similar constructions using atomic maps have been done before, for instance in [8], [10] and [11].

THEOREM 6. *Under the negation of the Suslin hypothesis there exists a hereditarily separable non-metrizable Suslinian continuum X. Moreover, each non-degenerate subcontinuum of X is neither metrizable nor locally connected.*

Proof. Assuming the negation of the Suslin hypothesis, fix a Suslin tree $(T, \le)$ such that each node $t \in T$ has uncountably many successors in T and infinitely many immediate successors in T. Denote by $h(t)$ the height of a node $t \in T$ and for a countable ordinal α let $T_\alpha = \{t \in T : h(t) = \alpha\}$ stand for the αth level of T. For two countable ordinals $\alpha < \beta$ let $\mathrm{pr}^\beta_\alpha : T_\beta \to T_\alpha$ denote the map assigning to a node $t \in T_\beta$ a unique node $t' \in T_\alpha$ with $t' < t$. We may additionally assume that the tree T is continuous in the sense that for any limit countable ordinal α and distinct nodes $t, t' \in T_\alpha$ there is $\beta < \alpha$ such that $\mathrm{pr}^\alpha_\beta(t) \ne \mathrm{pr}^\alpha_\beta(t')$.

We shall use transfinite induction to construct a well-ordered continuous spectrum $\{X_\alpha, \pi^\beta_\alpha : \alpha < \beta < \omega_1\}$ consisting of metrizable Suslinian continua X_α and atomic bonding maps $\pi^\beta_\alpha : X_\beta \to X_\alpha$, and a sequence $(i_\alpha : T_\alpha \to X_\alpha)_{\alpha<\omega_1}$ of injective maps such that

(1) for any countable ordinals $\alpha < \beta$ the diagram

$$\begin{array}{ccc} T_\beta & \xrightarrow{\ i_\beta\ } & X_\beta \\ {\scriptstyle \mathrm{pr}^\beta_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle \pi^\beta_\alpha} \\ T_\alpha & \xrightarrow{\ i_\alpha\ } & X_\alpha \end{array}$$

is commutative;

(2) for every $t \in T_\alpha$ the set $i_{\alpha+1}((\mathrm{pr}^{\alpha+1}_\alpha)^{-1}(t))$ is dense in $(\pi^{\alpha+1}_\alpha)^{-1}(i_\alpha(t))$;

(3) the short projections $\pi^{\alpha+1}_\alpha : X_{\alpha+1} \to X_\alpha$ are I-atomic maps with non-degeneracy set $N(\pi^{\alpha+1}_\alpha) = i_\alpha(T_\alpha)$.

We start the induction with a singleton X_0 and the injective map $i_0 : T_0 \to X_0$ assigning to the root of T the only point of X_0. Assume that for some countable ordinal α the Suslinian continua X_β, atomic bonding maps $\pi^\beta_\gamma : X^\beta \to X_\gamma$, and injective maps $i_\beta : T_\beta \to X_\beta$ have been constructed for all $\gamma \leq \beta < \alpha$.

If α is a limit ordinal, let X_α be the inverse limit of the countable spectrum $\{X_\beta, \pi^\beta_\gamma : \gamma \leq \beta < \alpha\}$ and let $\pi^\alpha_\beta : X_\alpha \to X_\beta$ stand for the limit projections of this spectrum. They are atomic as limits of atomic bonding maps. For every $t \in T_\alpha$ let $i_\alpha(t)$ be the unique point of X_α such that $\pi^\alpha_\beta(i_\alpha(t)) = i_\beta(\mathrm{pr}^\alpha_\beta(t))$ for every $\beta < \alpha$. The continuity of the tree T implies that the resulting map $i_\alpha : T_\alpha \to X_\alpha$ is injective. The Suslinian property of X_α follows from that property of the continua X_β, $\beta < \alpha$, and the atomicity of the limit projections π^α_β.

If $\alpha = \beta + 1$ is a successor ordinal, then we can apply Lemma 3 to find a metrizable Suslinian continuum $X_{\alpha+1}$ and an I-atomic map $\pi^{\alpha+1}_\alpha : X_{\alpha+1} \to X_\alpha$ whose non-degeneracy set coincides with $i_\alpha(T_\alpha)$. Thus we satisfy the condition (3) of the inductive construction. Since for every $t \in T_\alpha$ the set $(\pi^{\alpha+1}_\alpha)^{-1}(i_\alpha(t))$ is an arc in $X_{\alpha+1}$, we can define an injective map $i_{\alpha+1} : T_{\alpha+1} \to X_{\alpha+1}$ so that $\pi^{\alpha+1}_\alpha \circ i_{\alpha+1} = i_\alpha \circ \mathrm{pr}^{\alpha+1}_\alpha$ and $i_{\alpha+1}$ satisfies the condition (2) of the inductive construction.

After completing the inductive construction, consider the inverse limit X of the spectrum $\mathcal{S} = \{X_\alpha, \pi^\alpha_\beta : \beta < \alpha < \omega_1\}$. Using the atomicity of the bonding projections, one can check that the limit projections $\pi_\alpha : X \to X_\alpha$ are atomic as well.

Now, we establish the desired properties of the continuum X. First, we show that each non-degenerate subcontinuum C of X is neither metrizable nor locally connected. Let α be the smallest ordinal such that $|\pi_\alpha(C)| > 1$. The continuity of the spectrum $\mathcal{S}$ implies that $\alpha = \beta + 1$ for some ordinal β. Then $\pi_\beta(C)$ is a singleton and hence $\pi_\beta(C) \subset i_\beta(T_\beta)$ (otherwise C would be a singleton). Let $t \in T_\beta$ be a node of T with $\pi_\beta(C) = \{i_\beta(t)\}$. It follows that $\pi_\alpha(C)$ is a non-degenerate subcontinuum of the arc $A_t = (\pi^\alpha_\beta)^{-1}(i_\beta(t))$. The density of $i_\alpha(T_\alpha)$ in A_t implies the existence of a node $t' \in T_\alpha$ with $i_\alpha(t') \in \pi_\alpha(C)$. The atomicity of the projection pr_α implies that the continuum $C = \pi_\alpha^{-1}(\pi_\alpha(C))$ contains the subcontinuum $\mathrm{pr}_\alpha^{-1}(i_\alpha(t'))$ which is not metrizable (because t' has uncountably many successors in the tree T). Consequently, C is not metrizable either.

To show that C is not locally connected, assume the converse and, given any two distinct points $x, x' \in \mathrm{pr}^{-1}(\alpha)(i_\alpha(t'))$, find a closed connected neighborhood $U \subset C$ of x with $x' \notin U$. Since $\mathrm{pr}_\alpha^{-1}(i_\alpha(t'))$ is nowhere dense in C, the set U has non-degenerate projection $\mathrm{pr}_\alpha(U)$. Then the atomicity of pr_α implies that $x' \in \mathrm{pr}_\alpha^{-1}(\mathrm{pr}_\alpha(U)) = U$, which is a contradiction.

Next, we shall prove that the continuum X is Suslinian. Take any family $\mathcal{C}$ of pairwise disjoint non-degenerate subcontinua in X. Repeating the preceding argument, for every $C \in \mathcal{C}$ we can find a countable ordinal α and a node $t_C \in T_\alpha$ such that $C \supset \pi_\alpha^{-1}(i_\alpha(t_C))$. It follows that the nodes t_C, $C \in \mathcal{C}$, are pairwise incomparable in T (otherwise the family $\mathcal{C}$ would contain two intersecting continua). Since T is a Suslin tree, the antichain $\{t_C : C \in \mathcal{C}\}$ is at most countable and so is the family $\mathcal{C}$, witnessing the Suslinian property of X.

It remains to check that the continuum X is hereditarily separable. By [3, 3.12.9] it suffices to prove that each closed subspace F of X in separable. By Theorem 2, the continuum X, being Suslinian, is perfectly normal and hence $F = \pi_\alpha^{-1}(\pi_\alpha(F))$ for some countable ordinal α. Let $Z = \mathrm{pr}_\alpha(F)$. Since

$$F = \pi_\alpha^{-1}(Z \setminus i_\alpha(T_\alpha)) \cup \bigcup_{z \in Z \cap i_\alpha(T_\alpha)} \pi_\alpha^{-1}(z)$$

and $\pi_\alpha^{-1}(Z \setminus i_\alpha(T_\alpha))$ is homeomorphic to the metrizable separable space $Z \setminus i_\alpha(T_\alpha)$, it remains to check that for every $z \in i_\alpha(T_\alpha)$ the continuum $\pi_\alpha^{-1}(z)$ is separable. Consider the arc $A = \pi_\alpha^{\alpha+1}(z)$ in $X_{\alpha+1}$ and observe that $D = A \setminus i_{\alpha+1}(T_{\alpha+1})$ is a dense subspace of A. It follows from the construction that $\pi_{\alpha+1}^{-1}(D)$ is a topological copy of D, dense in $\pi_{\alpha+1}^{-1}(A) = \pi_\alpha^{-1}(z)$. Therefore, the continuum $\pi_\alpha^{-1}(z)$ is separable. ∎

We do not know if the preceding theorem can be generalized to higher cardinals.

Problem 2. *Does the existence of a κ^+-Suslin tree imply the existence of a continuum X with $hd(X) \le \mathrm{Sln}(X) = \kappa < w(X)$?*

Remark 1. The existence of a κ^+-Suslin tree is equivalent to the existence of a linearly ordered continuum X with $\kappa = \mathrm{Sln}(X) = c(X) < d(X) = w(X) = \kappa^+$.

The non-metrizable hereditarily separable Suslinian continuum constructed in Theorem 6 is very far from being locally connected. In [2], it was proved that separable homogeneous Suslinian continua are metrizable. This encourages us to recall the following question of [1].

Problem 3. *Is each locally connected (hereditarily) separable Suslinian continuum metrizable?*

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