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A model-theoretic Baire category theorem for simple theories and its applications

by

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Abstract. We prove a model-theoretic Baire category theorem for $\tilde{\tau}^f_{low}$-sets in a countable simple theory in which the extension property is first-order and show some of its applications. We also prove a trichotomy for minimal types in countable nfcp theories: either every type that is internal in a minimal type is essentially 1-based by means of the forking topologies, or T interprets an infinite definable 1-based group of finite D-rank or T interprets a strongly minimal formula.

1. Introduction. The goal of this paper is to generalize a result from [S1] and to give some applications. In [S1] the first step for proving supersimplicity of countable unidimensional simple theories eliminating hyperimaginaries is to show the existence of an unbounded type-definable forking-open set (a set defined in terms of forking by formulas, see Definition 2.1) of bounded finite SU_{se}-rank (for definition see Section 4).

In this paper we develop a general framework for this kind of result. It is a new idea of a model-theoretic Baire category theorem, namely, one deals with certain "uniformly definable" family of generalized closed sets (in complicated "logic"); roughly speaking, given a partition of a complicated open set into countably many sets, each of which is the intersection of a "uniformly definable" family of generalized closed sets, one can find a forking-open set that is contained in some generalized closed set in one of these families. So, the main point is that we obtain a very nice set (forking-open), but we can only require that it be a subset of some generalized closed set in one of these families and not in its intersection. In particular, it is not just the usual Baire category theorem for a complicated topological space. The proof is quite similar to the proof in [S1] and has some important consequences, e.g. in a countable wnfcp theory if for every non-algebraic element a (even in some fixed non-empty $\tilde{\tau}^f_{low}$-set) there is $a' \in \mathrm{acl}(a) \setminus \mathrm{acl}(\emptyset)$ of finite SU-rank,

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then there exists a weakly minimal formula. We also prove a trichotomy for countable nfcp theories as indicated in the abstract.

We assume basic knowledge of simple theories. A good textbook on simple theories is [W]. The notations follow usual conventions. T will denote a complete first-order theory with no finite models in some language L. We will work in some large saturated model $\mathcal{C}$ of T (not necessarily with elimination of imaginaries, unless stated otherwise). Ordinals will be denoted by $\alpha, \beta, \gamma, \ldots$. Sets $A, B, C, \ldots$ will be small subsets of $\mathcal{C}$, i.e. of cardinality strictly less than the cardinality of $\mathcal{C}$. The letters $a, b, c, \ldots$ denote finite tuples from $\mathcal{C}$, and $x, y, z, \ldots$ denote finite tuples of variables, unless stated otherwise. We use $p, q, r, \ldots$ to denote types (possibly partial) over some set. For an invariant set V (over some small set) and n, we denote by V^n the set of n-tuples of realizations of V.

2. Preliminaries. The forking topology is introduced in [S0] and is a variant of Hrushovski's and Pillay's topologies from [H0] and [P0], respectively. In this section T is assumed to be simple and we work in a large saturated model $\mathcal{C}$ of T.

Definition 2.1. Let $A \subseteq \mathcal{C}$ and let x be a finite tuple of variables.

(1) An invariant set $\mathcal{U}$ over A is said to be a *basic τ^f-open set over A* if there is $\phi(x, y) \in L(A)$ such that

$$\mathcal{U} = \{a \mid \phi(a, y) \text{ forks over } A\}.$$

Note that the family of basic τ^f-open sets over A is closed under finite intersections, thus forms a basis for a unique topology on $S_x(A)$. An open set in this topology is called a *τ^f-open set over A* or a *forking-open set over A*.

(2) An invariant set $\mathcal{U}$ over A is said to be a *basic τ^f_∞-open set over A* if $\mathcal{U}$ is a type-definable τ^f-open set over A. The family of basic τ^f_∞-open sets over A is a basis for a unique topology on $S_x(A)$. An open set in this topology is called a *τ^f_∞-open set over A*.

Recall that a formula $\phi(x, y) \in L$ is *low in x* if there exists $k < \omega$ such that for every $\emptyset$-indiscernible sequence $(b_i \mid i < \omega)$, the set $\{\phi(x, b_i) \mid i < \omega\}$ is inconsistent iff every subset of it of size k is inconsistent. T is low if every $\phi(x, y)$ is low in x.

Remark 2.2. Assume $\phi(x, t) \in L$ is low in t and $\psi(y, v) \in L$ is low in v ($x \cap y$, $t \cap v$ may not be $\emptyset$). Then $\theta(xy, tv) \equiv \phi(x, t) \vee \psi(y, v)$ is low in tv.

Proof. Let $k_1 < \omega$ be a witness that $\phi(x, t)$ is low in t and let $k_2 < \omega$ be a witness that $\psi(y, v)$ is low in v. Let $k = k_1 + k_2 - 1$. By adding dummy variables we may assume $x = y$ and $t = v$ (as tuples of variables).

Let $(a_i \mid i < \omega)$ be indiscernible such that $\{\phi(a_i,t) \vee \psi(a_i,t) \mid i < \omega\}$ is inconsistent. Thus, every subset of $\{\phi(a_i,t) \mid i < \omega\}$ of size k_1 is inconsistent, and every subset of $\{\psi(a_i,t) \mid i < \omega\}$ of size k_2 is inconsistent. Thus every subset of size k of $\{\phi(a_i,t) \vee \psi(a_i,t) \mid i < \omega\}$ is inconsistent.

Here we state some basic facts about the τ^f-topology.

REMARK 2.3. (1) The τ^f-topology on $S_x(A)$ refines the Stone topology of $S_x(A)$ for all x, A.

(2) A basic τ^f-open set in a low theory is type-definable and every Stone-closed subset of $(S_x(A), \tau^f)$ is a Baire topological space (i.e. the intersection of countably many dense open sets in it is dense) [S1, Remark 7.6].

(3) Let A be a small set. Let $F(x,y)$ be a type-definable relation over A and let $f(x)$ be an A-definable function. Let $\Gamma_{F,f}(x) = \exists y\ (F(x,y) \wedge y \mathop{\smile\!\!\!\!\!\!\!\mid}_A f(x))$. Then $\Gamma_{F,f}(x)$ is τ^f-closed over A ([S0, Claim 2.5] is slightly different, but the proof is the same).

Recall the following definition from [S0] whose roots are in [H0].

DEFINITION 2.4. We say that *the τ^f-topologies over A are closed under projections* (or *T is PCFT over A*) if for every τ^f-open set $\mathcal{U}(x,y)$ over A the set $\exists y\, \mathcal{U}(x,y)$ is τ^f-open over A. We say that *the τ^f-topologies are closed under projections* (or *T is PCFT*) if they are such over every set A.

In [BPV, Proposition 4.5] the authors proved the following equivalence which, for convenience, we will use as a definition (their definition involves extension with respect to pairs of models of T).

DEFINITION 2.5. We say that *the extension property is first-order in T* iff for any formulas $\phi(x,y), \psi(y,z) \in L$ the relation $Q_{\phi,\psi}$ defined by

$$Q_{\phi,\psi}(a) \quad \text{iff} \quad \phi(x,b) \text{ does not fork over } a \text{ for every } b \models \psi(y,a)$$

is type-definable (here a can be an infinite tuple from $\mathcal{C}$ whose sorts are fixed). We say that T has *wnfcp* if T is low and the extension property is first-order in T.

REMARK 2.6. Recall that T has nfcp (non-finite cover property) iff for every formula $\phi(x,y) \in L$ there exists $k < \omega$ such that every set $\{\phi(x,a_i) \mid i \in I\}$ of instances of $\phi(x,y)$ is consistent iff every subset of it of size k is consistent. By a theorem of Shelah, T has nfcp iff T is stable and T^{eq} eliminates the quantifier $\exists^\infty$ [Sh, Chapter 2, Theorems 4.2, 4.4]. Moreover, if T is stable then T has nfcp iff T has wnfcp [BPV].

FACT 2.7 ([S1, Corollary 3.13]). *Suppose the extension property is first-order in T. Then T is PCFT.*

We say that an A-invariant set $\mathcal{U}$ *has finite SU-rank* if $SU(a/A) < \omega$ for all $a \in \mathcal{U}$, and *has bounded finite SU-rank* if there exists $n < \omega$ such that $SU(a/A) \leq n$ for all $a \in \mathcal{U}$. The existence of a τ^f-open set of bounded finite SU-rank implies the existence of an SU-rank 1 formula (i.e. a weakly minimal formula):

FACT 2.8 ([S0, Proposition 2.13]). *Let $\mathcal{U}$ be an unbounded τ^f-open set over some set A. Assume $\mathcal{U}$ has bounded finite SU-rank. Then there exist a set $B \supseteq A$ with $|B \setminus A| < \omega$ and $\theta(x) \in L(B)$ of SU-rank 1 such that $\theta^{\mathcal{C}} \subseteq \mathcal{U} \cup \mathrm{acl}(B)$.*

In [S1] the class of $\tilde{\tau}^f$-sets and its subclass of $\tilde{\tau}^f_{st}$-sets were introduced. The class of $\tilde{\tau}^f$-sets is much wider than the class of basic τ^f-open sets. Here we look at the intermediate class of $\tilde{\tau}^f_{low}$-sets.

DEFINITION 2.9. A relation $V(x, z_1, \ldots, z_l)$ is said to be a *pre-$\tilde{\tau}^f$-set relation over $\emptyset$* if there are $\theta(\tilde{x}, x, z_1, \ldots, z_l) \in L$ and $\phi_i(\tilde{x}, y_i) \in L$ for $0 \leq i \leq l$ such that for all $a, d_1, \ldots, d_l$ from $\mathcal{C}$ we have

$$V(a, d_1, \ldots, d_l) \text{ iff } \exists \tilde{a} \left[\theta(\tilde{a}, a, d_1, \ldots, d_l) \wedge \bigwedge_{i=0}^{l} (\phi_i(\tilde{a}, y_i) \text{ forks over } d_1 \ldots d_i)\right]$$

(for $i = 0$ the sequence $d_1 \ldots d_i$ is interpreted as $\emptyset$). If each $\phi_i(\tilde{x}, y_i)$ is assumed to be low in y_i, $V(x, z_1, \ldots, z_l)$ is said to be a *pre-$\tilde{\tau}^f_{low}$-set relation.*

DEFINITION 2.10. (1) A *$\tilde{\tau}^f$-set over $\emptyset$* is a set of the form

$$\mathcal{U} = \{a \mid \exists d_1, \ldots, d_l \; V(a, d_1, \ldots, d_l)\}$$

for some pre-$\tilde{\tau}^f$-set relation $V(x, z_1, \ldots, z_l)$.

(2) A *$\tilde{\tau}^f_{low}$-set over $\emptyset$* is a set of the form

$$\mathcal{U} = \{a \mid \exists d_1, \ldots, d_l \; V(a, d_1, \ldots, d_l)\}$$

for some pre-$\tilde{\tau}^f_{low}$-set relation $V(x, z_1, \ldots, z_l)$.

REMARK 2.11. Every $\tilde{\tau}^f_{low}$-set is type-definable.

Proof. Let $\phi(x, y) \in L$ be low in x. Let $\Gamma_\phi(y, z)$ be the invariant relation defined by $\Gamma_\phi(a, c)$ iff $\phi(x, a)$ divides over c. Then $\Gamma_\phi(y, z)$ is type-definable, so the claim follows by compactness.

3. The Theorem. In this section T is assumed to be a simple theory and we work in $\mathcal{C}$ (so, T not necessarily eliminates imaginaries).

DEFINITION 3.1. Let $\Theta = \{\theta_i(x_i, x)\}_{i \in I}$ be a set of L-formulas such that $\forall x \; \exists^{<\infty} x_i \; \theta_i(x_i, x)$ for all $i \in I$. Let s be the sort of x. For $A \subseteq \mathcal{C}^s$, let $\mathrm{acl}_\Theta(A) = \{b \mid \theta_i(b, a) \text{ for some } \theta_i \in \Theta \text{ and } a \in A\}$.

Definition 3.2. An invariant set $\mathcal{U}(x, y_1, \ldots, y_r)$ is said to be a *generalized uniform family of $\tilde{\tau}^f_{low}$-sets* if there is a formula $\rho(\tilde{x}, x, y_1, \ldots, y_r, z_1, \ldots, z_k) \in L$ and there are formulas $\psi_i(\tilde{x}, v_i), \mu_j(\tilde{x}, w_j) \in L$ for $0 \leq i \leq r$ and $1 \leq j \leq k$ that are low in v_i and low in w_j, respectively, such that for all $a, d_1, \ldots, d_r$ we have $\mathcal{U}(a, d_1, \ldots, d_r)$ iff $\exists \tilde{a}\ \exists e_1, \ldots, e_k$

$$\rho(\tilde{a}, a, d_1, \ldots, d_r, e_1, \ldots, e_k) \wedge \Big[\bigwedge_{i=0}^{r} (\psi_i(\tilde{a}, v_i) \text{ forks over } d_1 \ldots d_i)\Big] \wedge \Big[\bigwedge_{j=1}^{k} (\mu_j(\tilde{a}, w_j) \text{ forks over } d_1 \ldots d_r e_1 \ldots e_j)\Big].$$

Definition 3.3. An invariant set $\mathcal{F}(x, y_1, \ldots, y_r)$ is said to be a *generalized uniform family of $\tilde{\tau}^f_{low}$-closed sets* if

$$\mathcal{F}(x, y_1, \ldots, y_r) = \bigcap_i \neg \mathcal{U}_i(x, y_1, \ldots, y_r),$$

where each $\mathcal{U}_i(x, y_1, \ldots, y_r)$ is a generalized uniform family of $\tilde{\tau}^f_{low}$-sets.

The following fact [S1, Theorem 8.7] is the key ingredient of our main theorem.

Fact 3.4. *Assume the extension property is first-order in T. Let $\mathcal{U}$ be an unbounded $\tilde{\tau}^f$-set over $\emptyset$. Then there exists an unbounded τ^f-open set $\mathcal{U}^*$ over some finite set A^* such that $\mathcal{U}^* \subseteq \mathcal{U}$. In fact, if $V(x, z_1, \ldots, z_l)$ is a pre-$\tilde{\tau}^f$-set relation such that $\mathcal{U} = \{a \mid \exists d_1, \ldots, d_l\ V(a, d_1, \ldots, d_l)\}$, and $\bar{d}^* = (d^*_1, \ldots, d^*_m)$ is any maximal sequence (with respect to extension) such that $\mathcal{U}^*_{\bar{d}^*} = \exists d_{m+1}, \ldots, d_l\ V(\mathcal{C}, d^*_1, \ldots, d^*_m, d_{m+1}, \ldots, d_l)$ is unbounded, then $\mathcal{U}^*_{\bar{d}^*}$ is a τ^f-open set over $\bar{d}^*$.*

Theorem 3.5. *Let T be a countable simple theory in which the extension property is first-order. Assume:*

(1) *$\Theta = \{\theta_i(x'_i, x)\}_{i<\omega}$ is a set of L-formulas such that $\forall x\ \exists^{<\infty} x'_i\ \theta_i(x'_i, x)$ for all $i < \omega$.*
(2) *$\mathcal{U}_0(x)$ is a non-empty $\tilde{\tau}^f_{low}$-set over $\emptyset$.*
(3) *$\{F_n(x_n)\}_{n<\omega}$ is a family of $\emptyset$-invariant sets such that $F_n(\mathcal{C}) \cap \mathrm{acl}(\emptyset) = \emptyset$ for all $n < \omega$.*
(4) *For every $n < \omega$ and any variables $\bar{y} = y_1, \ldots, y_r$, let $\mathcal{F}^{\bar{y}}_n(x_n, \bar{y})$ be a generalized uniform family of $\tilde{\tau}^f_{low}$-closed sets such that $F_n(\mathcal{C}) \subseteq \mathcal{F}^{\bar{y}}_n(\mathcal{C}, \bar{d})$ for all $\bar{d}$.*

Now, assume that for all $a \in \mathcal{U}_0$ there exist $b \in \mathrm{acl}_\Theta(a)$ and $n < \omega$ such that $b \in F_n(\mathcal{C})$. Then there is an unbounded τ^f_∞-open set $\mathcal{U}^*$ over a finite tuple

$\bar{d}^*$ and variables $\bar{y}^*$ of the sort of $\bar{d}^*$, and $n^* < \omega$ such that

$$\mathcal{U}^* \subseteq \mathcal{F}_{n^*}^{\bar{y}^*}(\mathcal{C}, \bar{d}^*) \cap \mathrm{acl}_{\Theta}(\mathcal{U}_0).$$

Proof. First, we may assume Θ is downwards closed (i.e. if $\theta \in \Theta$ and $\theta' \vdash \theta$ then $\theta' \in \Theta$; note that since L is countable the closure of Θ in this sense remains countable). Assume the conclusion of the theorem is false. To get a contradiction, it will be sufficient to show the following.

Subclaim 3.6. *For every non-empty $\tilde{\tau}^f_{low}$-set $\mathcal{U} \subseteq \mathcal{U}_0$ over $\emptyset$, every $\theta \in \Theta$, and every $n < \omega$ there exists a non-empty $\tilde{\tau}^f_{low}$-set $\mathcal{U}^* \subseteq \mathcal{U}$ over $\emptyset$ such that either $\neg\exists x'\ \theta(x', a)$ for all $a \in \mathcal{U}^*$, or for all $a \in \mathcal{U}^*$ there exists $b \models \theta(x', a)$ with $b \notin F_n(\mathcal{C})$.*

First, we show this is sufficient. Construct a decreasing sequence $(\mathcal{U}_m \mid m < \omega)$ of non-empty $\tilde{\tau}^f_{low}$-sets that begins at $\mathcal{U}_0$, and for every $m < \omega$ the set $\mathcal{U}_{m+1}$ is obtained from $\mathcal{U}_m$ by applying Subclaim 3.6 for an appropriate pair (θ, n) (that corresponds to m by a fixed bijection of $\Theta \times \omega$ with ω). By Remark 2.11 and compactness, $\bigcap \mathcal{U}_m \neq \emptyset$, so there exists $a^* \in \mathcal{U}_0$ such that for all $\theta \in \Theta$ either $\neg\exists x'\ \theta(x', a^*)$, or for every $n < \omega$ there exists $b_{n,\theta} \models \theta(x', a^*)$ such that $b_{n,\theta} \notin F_n(\mathcal{C})$. Now, by the assumption of the theorem there exist $\theta(x', x) \in \Theta$, b^* and $n^* < \omega$ such that $\theta(b^*, a^*)$ and $b^* \in F_{n^*}(\mathcal{C})$. As Θ is downwards closed, there exists $\theta^*(x', x) \in \Theta$ such that $\theta^*(x', x) \vdash \theta(x', x)$ and $\theta^*(x', a^*)$ isolates $\mathrm{tp}(b^*/a^*)$ (as it is algebraic). By the above property of a^*, there exists $b^{**} \models \theta^*(x', a^*)$ with $b^{**} \notin F_{n^*}(\mathcal{C})$, contradicting the fact that $\theta^*(x', a^*)$ isolates $\mathrm{tp}(b^*/a^*)$ and the assumption that $F_{n^*}(\mathcal{C})$ is $\emptyset$-invariant.

Proof of Subclaim 3.6. Let $\mathcal{U}$, θ and $n < \omega$ be given. Let $V(x, z_1, \ldots, z_l)$ be a pre-$\tilde{\tau}^f_{low}$-set relation such that

$$\mathcal{U} = \{a \mid \exists d_1, \ldots, d_l\ V(a, d_1, \ldots, d_l)\},$$

where V is defined by:

$$V(a, d_1, \ldots, d_l) \text{ iff } \exists \tilde{a} \left[\sigma(\tilde{a}, a, d_1, \ldots, d_l) \wedge \bigwedge_{i=0}^{l} (\phi_i(\tilde{a}, t_i) \text{ forks over } d_1 \ldots d_i)\right]$$

for some $\sigma(\tilde{x}, x, z_1, \ldots, z_l) \in L$ and $\phi_i(\tilde{x}, t_i) \in L$ which are low in t_i for $0 \leq i \leq l$. Let V_θ be defined by: for all $b, d_1, \ldots, d_l \in \mathcal{C}$,

$$V_\theta(b, d_1, \ldots, d_l) \quad \text{iff} \quad \exists a\ (\theta(b, a) \wedge V(a, d_1, \ldots, d_l)),$$

and let

$$\mathcal{U}_\theta = \{b \mid \exists d_1, \ldots, d_l\ V_\theta(b, d_1, \ldots, d_l)\}.$$

Since by the assumption $F_n(\mathcal{C}) \cap \mathrm{acl}(\emptyset) = \emptyset$, we may assume $\mathcal{U}_\theta \cap \mathrm{acl}(\emptyset) = \emptyset$ and $\mathcal{U}_\theta$ is non-empty. Now, let $\bar{d}^* = (d_1^*, \ldots, d_m^*)$ be a maximal sequence, with respect to extension ($0 \le m \le l$), such that

$$\tilde{V}_\theta(x') \equiv \exists d_{m+1},\dots,d_l\ V_\theta(x', d_1^*,\dots,d_m^*, d_{m+1},\dots,d_l)$$

is non-algebraic. We may assume $m < l$ (by choosing V appropriately). By Fact 3.4, $\tilde{V}_\theta(\mathcal{C})$ is an unbounded basic τ_∞^f-open set over $\bar{d}^*$. Since we assume the conclusion of the theorem is false, $\tilde{V}_\theta(\mathcal{C}) \not\subseteq \mathcal{F}_n^{\bar{y}^*}(\mathcal{C},\bar{d}^*)$ where $\bar{y}^* = y_1^*,\dots,y_m^*$ has the same sort as $\bar{d}^*$. Now, let each $\mathcal{U}_{s,n}(x_n,\bar{y}^*)$ for $s < \alpha$ be a generalized uniform family of $\tilde{\tau}_{low}^f$-sets such that $\mathcal{F}_n(x_n,\bar{y}^*) = \bigcap_{s<\alpha} \neg\mathcal{U}_{s,n}(x_n,\bar{y}^*)$. Let $b^* \in \tilde{V}_\theta(\mathcal{C}) \setminus \mathcal{F}_n^{\bar{y}^*}(\mathcal{C},\bar{d}^*)$. So, there exists $s^* < \alpha$ such that $b^* \in \mathcal{U}_{s^*,n}(\mathcal{C},\bar{d}^*)$. Let $\rho(\tilde{x}', x_n, y_1^*,\dots,y_m^*, z_1',\dots,z_k') \in L$ and let $\psi_i(\tilde{x}', v_i), \mu_j(\tilde{x}', w_j) \in L$ for $0 \le i \le m$ and $1 \le j \le k$ be low in v_i and low in w_j respectively, such that for all $b, d_1,\dots,d_m$ we have $\mathcal{U}_{s^*,n}(b, d_1,\dots,d_m)$ iff $\exists \tilde{b}\ \exists e_1,\dots,e_k$

$$\rho(\tilde{b}, b, d_1,\dots,d_m, e_1,\dots,e_k) \wedge \Big[\bigwedge_{i=0}^{m} (\psi_i(\tilde{b}, v_i) \text{ forks over } d_1\dots d_i)\Big] \\ \wedge \Big[\bigwedge_{j=1}^{k} (\mu_j(\tilde{b}, w_j) \text{ forks over } d_1\dots d_m e_1\dots e_j)\Big].$$

Now, let $d_{m+1}^*,\dots,d_l^*$ and $a^*, \tilde{a}^*$ and $E^* = (e_1^*,\dots,e_k^*)$ and $\tilde{b}^*$ be such that

$$(*1)\quad \theta(b^*, a^*) \wedge \sigma(\tilde{a}^*, a^*, d_1^*,\dots,d_l^*) \wedge \bigwedge_{i=0}^{l} (\phi_i(\tilde{a}^*, y_i) \text{ forks over } d_1^*\dots d_i^*),$$

$$(*2)\quad \rho(\tilde{b}^*, b^*, d_1^*,\dots,d_m^*, e_1^*,\dots,e_k^*),$$

$$(*3)\quad \Big[\bigwedge_{i=0}^{m} (\psi_i(\tilde{b}^*, v_i) \text{ forks over } d_1^*\dots d_i^*)\Big] \\ \wedge \Big[\bigwedge_{j=1}^{k} (\mu_j(\tilde{b}^*, w_j) \text{ forks over } d_1^*\dots d_m^* e_1^*\dots e_j^*)\Big].$$

By maximality of $\bar{d}^*$, we know $b^* \in \operatorname{acl}(\bar{d}^* d_{m+1}^*)$. Thus, by taking a non-forking extension of $\operatorname{tp}(\tilde{b}^* E^*/\operatorname{acl}(\bar{d}^* d_{m+1}^*))$ over $\operatorname{acl}(d_1^*\dots d_l^* a^* \tilde{a}^*)$ we may assume E^* is independent of $d_1^*\dots d_l^* a^* \tilde{a}^*$ over $\bar{d}^* d_{m+1}^*$ and $(*1)$–$(*3)$ still hold. We conclude that

$$\bigwedge_{i=m+1}^{l} (\phi_i(\tilde{a}^*, t_i) \text{ forks over } d_1^*\dots d_i^* E^*).$$

Now, we define the $\tilde{\tau}_{low}^f$-set $\mathcal{U}^*$. First, define a relation V^* by:

$$V^*(a, d_1,\dots,d_m, e_1,\dots,e_k, d_{m+1},\dots,d_l) \quad \text{iff} \quad \exists \tilde{a}, b, \tilde{b}\ (\theta^* \wedge V_0^* \wedge V_1^* \wedge V_2^*),$$

where θ^* is defined by: $\theta^*(\tilde{a}, b, \tilde{b}, a, d_1, \ldots, d_m, e_1, \ldots, e_k, d_{m+1}, \ldots, d_l)$ iff

$$\theta(b, a) \wedge \sigma(\tilde{a}, a, d_1, \ldots, d_l) \wedge \rho(\tilde{b}, b, d_1, \ldots, d_m, e_1, \ldots, e_k),$$

V_0^* is defined by: $V_0^*(\tilde{a}, \tilde{b}, d_1, \ldots, d_m)$ iff

$$\bigwedge_{i=0}^{m} (\phi_i(\tilde{a}, t_i) \vee \psi_i(\tilde{b}, v_i) \text{ forks over } d_1 \ldots d_i),$$

V_1^* is defined by: $V_1(\tilde{b}, d_1, \ldots, d_m, e_1, \ldots, e_k)$ iff

$$\bigwedge_{j=1}^{k} (\mu_j(\tilde{b}, w_j) \text{ forks over } d_1 \ldots d_m e_1 \ldots e_j),$$

and V_2^* is defined by: $V_2(\tilde{a}, d_1, \ldots, d_m, e_1, \ldots, e_k, d_{m+1}, \ldots, d_l)$ iff

$$\bigwedge_{i=m+1}^{l} (\phi_i(\tilde{a}, t_i) \text{ forks over } d_1 \ldots d_i e_1 \ldots e_k).$$

Note that V^* is a pre-$\tilde{\tau}^f_{low}$-set. Let

$$\mathcal{U}^* = \{a \mid \exists d_1, \ldots, d_m, e_1, \ldots, e_k, d_{m+1}, \ldots, d_l \\ V^*(a, d_1, \ldots, d_m, e_1, \ldots, e_k, d_{m+1}, \ldots, d_l)\}.$$

By the definition of $\mathcal{U}^*$, we have $\mathcal{U}^* \subseteq \mathcal{U}$. Moreover $\mathcal{U}^*$ is a $\tilde{\tau}^f_{low}$-set by Remark 2.2. By construction, $\mathcal{U}^* \neq \emptyset$. Now, let $a \in \mathcal{U}^*$. By the definition of $\mathcal{U}^*$, there are $\tilde{b}, b, d_1, \ldots, d_m, e_1, \ldots, e_k$ such that $\theta(b, a)$, $\rho(\tilde{b}, b, d_1, \ldots, d_m, e_1, \ldots, e_k)$, and

$$\bigwedge_{i=0}^{m} (\psi_i(\tilde{b}, v_i) \text{ forks over } d_1 \ldots d_i),$$

$$\bigwedge_{j=1}^{k} (\mu_j(\tilde{b}, w_j) \text{ forks over } d_1 \ldots d_m e_1 \ldots e_j).$$

Thus $\mathcal{U}_{s^*,n}(b, d_1 \ldots d_m)$ and therefore $\neg \mathcal{F}_n^{\bar{y}^*}(b, d_1 \ldots d_m)$. Hence $b \notin F_n$ as required.

4. Applications. In this section we give some applications of Theorem 3.5. In fact, we will show several instances of this theorem that are apparently new even for stable theories. In this section T is assumed to be a simple theory and we work in $\mathcal{C}$.

We start by pointing out that Theorem 3.5 generalizes [S1, Theorem 9.4] that is one of the essential steps towards the proof of supersimplicity of countable simple unidimensional theories with elimination of hyperimaginaries. First recall the following definitions from [S1] of stable independence and SU_{se}-rank.

Definition 4.1. For $a \in \mathcal{C}$, $A, B \subseteq \mathcal{C}$, $a \underset{A}{\not\perp}^{s} B$ if for some stable $\phi(x,y) \in L$, there are $b \subseteq A \cup B$ and $a' \in \phi(\mathcal{C}, b) \cap \mathrm{dcl}(Aa)$ such that $\phi(x,b)$ forks over A.

Definition 4.2. The *SU_{se}-rank* of $\mathrm{tp}(a/A)$ is defined by induction on α: if $\alpha = \beta + 1$, then $SU_{se}(a/A) \geq \alpha$ if there exist $B_1 \supseteq B_0 \supseteq A$ such that $a \underset{B_0}{\not\perp}^{s} B_1$ and $SU_{se}(a/B_1) \geq \beta$. For limit α, $SU_{se}(a/A) \geq \alpha$ if $SU_{se}(a/A) \geq \beta$ for all $\beta < \alpha$.

Remark 4.3. In [S1, Lemma 6.8] it is proved that in a simple theory, in which Lstp $=$ stp over sets, $\perp^{s}$ is symmetric. In fact, $\perp^{s}$ is symmetric in any simple theory. Thus for any simple theory, if s_0 and s_1 are finite tuples of sorts and $n < \omega$ then the set $\mathcal{F}_n^{s_0,s_1}$ defined by

$$\mathcal{F}_n^{s_0,s_1} = \{(a,A) \in \mathcal{C}^{s_0} \times \mathcal{C}^{s_1} \mid SU_{se}(a/A) < n\}$$

is a generalized uniform family of $\tilde{\tau}^{f}_{low}$-closed sets.

Proof. To prove that $\perp^{s}$ is symmetric, first recall [S1, Claim 6.5]:

Fact 4.4. *Let T be simple. Let $\phi(x,y) \in L$ be stable. Assume $a \underset{A}{\perp} b$ and $a' \underset{A}{\perp} b$ and* $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(a'/A)$. *Then $\phi(a,b)$ iff $\phi(a',b)$.*

By the proof of symmetry of stable independence [S3, Lemma 6.8] it will be sufficient to prove Fact 4.4 with the weaker assumption $\mathrm{stp}(a) = \mathrm{stp}(a')$ instead of the assumption $\mathrm{Lstp}(a) = \mathrm{Lstp}(a')$ (we may clearly assume $A = \emptyset$). Indeed, assume $\mathrm{stp}(a) = \mathrm{stp}(a')$. Now, for every complete type $q \in S(\emptyset)$ let E_q be the equivalence relation defined by: $E_q(a,a')$ iff "for every $b \models q$ that is independent of aa' we have $[\phi(a,b)$ iff $\phi(a',b)]$". Then E_q is Stone-open. By Fact 4.4, equality of the Lascar strong type refines E_q. Thus E_q is a $\emptyset$-definable finite equivalence relation (as a bounded Stone-open equivalence relation is definable [S3, Lemma 7]). Now, by the assumption that $\mathrm{stp}(a) = \mathrm{stp}(a')$, $E_q(a,a')$ for all complete q. Thus, by extension we infer that for every b, if each of a and a' is independent of b, then $\phi(a,b)$ iff $\phi(a',b)$.

We now explain the last phrase. We need to show that $\neg\mathcal{F}_n^{s_0,s_1}$ is a disjunction of invariant sets, each of which is a generalized uniform family of $\tilde{\tau}^{f}_{low}$-sets for all s_0, s_1 and n as above. Indeed, by symmetry of $\not\perp^{s}$, $\neg\mathcal{F}_n^{s_0,s_1}(a,A)$ iff there are $b_1, c_1, \ldots, b_n, c_n$ such that $c_i \underset{Ab_1c_1 \ldots b_{i-1}c_{i-1}b_i}{\not\perp}^{s} a$ for all $1 \leq i \leq n$. By the definition of $\not\perp^{s}$, this can be easily seen to be equivalent to a disjunction of the required form (since any stable $\phi(x,y) \in L$ is low in both x and y).

For an A-invariant set V, we set $\mathrm{acl}_1(V) = \{a' \mid a' \in \mathrm{acl}(a)$ for some $a \in V^1\}$. The following corollary generalizes [S1, Theorem 9.4].

Corollary 4.5. *Let T be a countable simple theory in which the extension property is first-order. Let $\mathcal{U}_0$ be a non-empty $\tilde{\tau}^f_{low}$-set. Assume for every $a \in \mathcal{U}_0$ there exists $a' \in \mathrm{acl}(a) \setminus \mathrm{acl}(\emptyset)$ such that $SU_{se}(a') < \omega$. Then there exists an unbounded τ^f_∞-open set $\mathcal{U} \subseteq \mathrm{acl}_1(\mathcal{U}_0)$ over a finite set such that $\mathcal{U}$ has bounded finite SU_{se}-rank.*

Proof. Let x be the variable of $\mathcal{U}_0$, so $\mathcal{U}_0 = \mathcal{U}_0(x)$. Let

$$\Theta = \{\theta(x', x) \mid \exists^{<\infty} x'\ \theta(x', x),\ x' \text{ any variable}\}.$$

Let $\mathcal{S}$ be the set of sorts. Let $I : \omega \to \mathcal{S} \times \omega$ be a bijection, and I_1, I_2 the projections of I to the first and second coordinate, respectively. Now, for each $n < \omega$ let $F_n = \{a \in \mathcal{C}^{I_1(n)} \setminus \mathrm{acl}(\emptyset) \mid SU_{se}(a) < I_2(n)\}$. Now, for every finite tuple of variables Y and $n < \omega$ let $s(Y)$ be the finite sequence of sorts of Y and let

$$\mathcal{F}^Y_n = \{(a, A) \in \mathcal{C}^{I_1(n)} \times \mathcal{C}^{s(Y)} \mid SU_{se}(a/A) < I_2(n)\}.$$

By the definition of the SU_{se}-rank, $F_n(\mathcal{C}) \subseteq \mathcal{F}^Y_n(\mathcal{C}, A)$ for every $n < \omega$ and all Y, A. By Remark 4.3, $\mathcal{F}^Y_n$ is a generalized uniform family of $\tilde{\tau}^f_{low}$-closed sets for all Y, n. By our assumptions, we see that the assumptions of Theorem 3.5 hold for $\mathcal{U}_0(x)$, Θ, $\{F_n\}_n$ and $\{\mathcal{F}^Y_n\}_{Y,n}$, and thus by its conclusion we are done.

Corollary 4.6. *Let T be a countable theory with wnfcp. Let $\mathcal{U}_0$ be an unbounded $\tilde{\tau}^f$-set over $\emptyset$ of finite SU-rank. Then there exists a finite set A and an SU-rank 1 formula $\theta \in L(A)$ such that $\theta^{\mathcal{C}} \subseteq \mathcal{U}_0 \cup \mathrm{acl}(A)$.*

Proof. First, by modifying $\mathcal{U}_0$, we may assume $\mathcal{U}_0 \cap \mathrm{acl}(\emptyset) = \emptyset$. Let $\Theta = \{x' = x\}$, $\mathcal{U}_0(x) = \mathcal{U}_0$. Let $s(x)$ be the sort of x. Now, for each $n < \omega$ let

$$F_n = \{a \in \mathcal{C}^{s(x)} \setminus \mathrm{acl}(\emptyset) \mid SU(a) < n\}.$$

For every finite tuple of variables Y and $n < \omega$ let $s(Y)$ be the finite sequence of sorts of Y and let

$$\mathcal{F}^Y_n = \{(a, A) \in \mathcal{C}^{s(x)} \times \mathcal{C}^{s(Y)} \mid SU(a/A) < n\}.$$

By symmetry of forking and the assumption that T is low, each $\mathcal{F}^Y_n$ is a generalized uniform family of $\tilde{\tau}^f_{low}$-closed sets. Clearly, $F_n(\mathcal{C}) \subseteq \mathcal{F}^Y_n(\mathcal{C}, A)$ for all $n < \omega$ and Y, A. By our assumption, the assumptions of Theorem 3.5 are satisfied for $\mathcal{U}_0$, Θ, $\{F_n\}_n$ and $\{\mathcal{F}^Y_n\}_{Y,n}$ and thus by its conclusion there exists an unbounded τ^f_∞-open set $\mathcal{U}^* \subseteq \mathcal{U}_0$ over a finite set A_0 and $\mathcal{U}^*$ has bounded finite SU-rank. By Fact 2.8, there exists a finite set $A \supseteq A_0$ and there exists an SU-rank 1 formula $\theta \in L(A)$ such that $\theta^{\mathcal{C}} \subseteq \mathcal{U}^* \cup \mathrm{acl}(A)$.

COROLLARY 4.7. *Let T be a countable theory with wnfcp. Let $\mathcal{U}_0$ be a non-empty $\tilde{\tau}^f$-set over $\emptyset$. Assume that for every $a \in \mathcal{U}_0$ there exists a' in* $\mathrm{acl}(a) \setminus \mathrm{acl}(\emptyset)$ *such that $SU(a') < \omega$. Then there exists a finite set A and an SU-rank* 1 *formula $\theta \in L(A)$ such that $\theta^{\mathcal{C}} \subseteq \mathrm{acl}_1(\mathcal{U}_0) \cup \mathrm{acl}(A)$.*

Proof. Just like the proof of Corollary 4.6.

5. Dichotomies for countable theories with wnfcp. In this section we show that the dichotomy [S1, Theorem 5.5] implies a strong dichotomy between essential 1-basedness and supersimplicity in the case T is a countable wnfcp theory that eliminates hyperimaginaries. Before we state the above dichotomy for the special case of the τ^f-topologies (simplified version), let us recall the basic definitions. In this section T is assumed to be simple and we work in $\mathcal{C} = \mathcal{C}^{\mathrm{eq}}$.

First, let us fix some notations and terminology. Let V, W be invariant sets. We say that V is *generated over* W *by a small set* B if $V \subseteq \mathrm{dcl}(W \cup B)$. We say that V is *generated over* W if it is generated over W by some small set. If V is A-invariant, we say that V is (*almost*) W*-internal over* A if for every $a \in V$ there exists $B \supseteq A$, over which W is invariant, that is independent of a over A and there exists a tuple $\bar{c}$ of realizations of W such that $a \in \mathrm{dcl}(B, \bar{c})$ ($a \in \mathrm{acl}(B, \bar{c})$, respectively). If we say that V is W*-internal* (without specifying over what set) then we mean that V is W-internal over the set that V comes with (e.g. in case it is a partial type, we consider it with its specified parameters). Note that if both V and W are A-invariant then for all $B, C \supseteq A$, V is (almost) W-internal over B iff V is (respectively, almost) W-internal over C.

DEFINITION 5.1. A type $p \in S(A)$ is said to be *essentially* 1*-based by means of the τ^f-topologies* if for every finite tuple $\bar{c}$ from p and for every type-definable τ^f-open set $\mathcal{U}$ over $A\bar{c}$, the set $\{a \in \mathcal{U} \mid \mathrm{Cb}(a/A\bar{c}) \notin \mathrm{bdd}(aA)\}$ is nowhere dense in the Stone topology of $\mathcal{U}$.

We now state [S1, Theorem 5.5] for the τ^f-topologies (in fact, it is a special case of it when working over constants). Also, as indicated at the end of the proof of this fact, the finite SU-rank τ^f-open set we obtained is almost p_0-internal.

FACT 5.2. *Let T be a countable simple theory with PCFT that eliminates hyperimaginaries. Let p_0 be a partial type over $\emptyset$ of SU-rank* 1*. Then either there exists an unbounded τ^f-open set over some countable set that is almost internal to p_0 (in particular, has finite SU-rank) or every type $p \in S(A)$, with A countable, that is internal in p_0 is essentially* 1*-based by means of the τ^f-topologies.*

Theorem 5.3. *Let T be a countable theory with wnfcp that eliminates hyperimaginaries. Let p be a partial type over $\emptyset$ of SU-rank 1. Then either*

(1) *every type $q \in S(A)$, with A countable, that is internal in p is essentially 1-based by means of the τ^f-topologies, or*
(2) *there exists a weakly minimal definable set (in $L(\mathcal{C})$) that is generated over $p(\mathcal{C})$.*

Proof. Assume (1) is false. By Fact 5.2, there exists an unbounded type-definable τ^f-open set $\mathcal{U}$ over some countable set A such that $\mathrm{tp}(a/A)$ is almost p-internal for every $a \in \mathcal{U}$.

Subclaim 5.4. *There exists an unbounded type-definable τ^f-open set $\mathcal{U}^*$ over A that is generated over $p(\mathcal{C})$.*

Proof. By [BW] or [S2, Corollary 4.9], for every $a \in \mathcal{U} \setminus \mathrm{acl}(A)$ there exists $a' \in \mathrm{dcl}(aA) \setminus \mathrm{acl}(A)$ such that $\mathrm{tp}(a'/A)$ has a fundamental system of solutions over $p(\mathcal{C})$ (i.e. $\mathrm{tp}(a'/A)$ is generated over $p(\mathcal{C})$ by a set of realizations of $\mathrm{tp}(a'/A)$ together with A). In particular, there exists a (finite) set A' of realizations of $\mathrm{tp}(a'/A)$ that is independent of a' over A and a tuple $\bar{c}$ of realizations of p such that $a' \in \mathrm{dcl}(A'A\bar{c})$. For any A-definable functions f, g let

$$F_{f,g} = \{a \in \mathcal{U} \mid f(a) = g(\bar{b}, \bar{c}) \notin \mathrm{acl}(A) \text{ for some } \bar{b}, \bar{c} \text{ with } f(a) \mathop{\smile\!\!\!\!\!\!\,|}_{A} \bar{b}, \text{ where } \bar{c} \text{ is a tuple of realizations of } p, \text{ and } \bar{b} \text{ is a tuple of realizations of } \mathrm{tp}(f(a)/A)\}.$$

By Remark 2.3(3), each $F_{f,g}$ is τ^f-closed over A. Thus, by the Baire category theorem for the τ^f-topology (by Remark 2.3(2), $\mathcal{U} \setminus acl(A), \tau^f)$ is a Baire space) there are A-definable functions f^*, g^* such that F_{f^*,g^*} has non-empty interior in the τ^f-topology over A. By Fact 2.7 there exists an unbounded type-definable τ^f-open set $\mathcal{U}^*$ over A such that for every $a \in \mathcal{U}^*$ there exists a tuple $\bar{b}$ of realizations of $\mathrm{tp}(a/A)$ that is independent of a over A such that $a = g^*(\bar{b}, \bar{c})$ for some tuple $\bar{c}$ of realizations of p. The subclaim follows now directly from [S2, Theorem 3.7]:

Fact 5.5. *Let $p \in S(\emptyset)$ and let $\mathcal{R}$ be $\emptyset$-invariant. Suppose the internality of p in $\mathcal{R}$ is witnessed by a generic parameter whose type q is almost-$\mathcal{R}$-internal. Then p is generated over $\mathcal{R}$ by a set of realizations of q.*

Now, as $\mathcal{U}^*$ has bounded finite SU-rank (the bound is determined by g^*), by Fact 2.8, there exists an SU-rank 1 formula $\theta(x, b)$ such that $\theta(\mathcal{C}, b) \subseteq \mathcal{U}^* \cup \mathrm{acl}(Ab)$. Thus (2) follows.

5.1. A trichotomy for countable theories with nfcp. Here we prove a trichotomy for countable theories with nfcp. In this subsection we work in a large saturated model $\mathcal{C} = \mathcal{C}^{\mathrm{eq}}$ of a simple theory T with elimination of hyperimaginaries unless stated otherwise.

We begin with some standard terminology and remarks. For a definable set D over A we denote by D^* the induced structure on D over A, namely, D^* is the set D equipped with all A-definable relations in $\mathcal{C}$ that are subsets of D^n for some n. Then clearly D^* has elimination of quantifiers and therefore saturated.

DEFINITION 5.6. Let D be a type-definable set over a set A. We say that D is 1-*based* if for every finite tuple $\bar{a}$ of realizations of D and set $B \supseteq A$, we have $\mathrm{Cb}(\bar{a}/B) \in \mathrm{acl}(\bar{a}A)$. A type-definable group G over A is said to be 1-based if its underlying set is.

REMARK 5.7. (1) A type-definable set D over A is 1-based iff $\bar{a}$ is independent of $\bar{a}'$ over $\mathrm{acl}(A\bar{a}) \cap \mathrm{acl}(A\bar{a}')$ for any finite tuples $\bar{a}$ and $\bar{a}'$ from D.

(2) Let D be a definable set over A. Then

- (i) if T is stable (simple), so is $\mathrm{Th}(D^*)$,
- (ii) if D^* is 1-based then D is 1-based (as a type-definable set),
- (iii) if D is stably embedded (e.g. T is stable), and p is a partial type of D^*, then $\mathrm{RM}_{D^*}(p) = \mathrm{RM}(p_D)$ (where p_D is just the conjunction of p with appropriate power of D, RM is the usual Morley rank in $\mathcal{C}$, and RM_{D^*} is the Morley rank in D^*).

LEMMA 5.8. *Assume L is countable and $\theta(\mathcal{C}) \subseteq \mathrm{acl}(p(\mathcal{C}))$, where p is any partial type over $\emptyset$ and $\theta(x) \in L$ is non-algebraic. Then*

(1) *there exists a $\emptyset$-definable $\theta^*(x) \vdash \theta(x)$ and $\emptyset$-definable functions f, g and $n < \omega$ such that $f[\theta^*(\mathcal{C}) \setminus \mathrm{acl}(\emptyset)] \subseteq g[p^n(\mathcal{C})]$ and $f[\theta^*(\mathcal{C})]$ is non-algebraic, and*

(2) *if p is minimal then $f[\theta^*(\mathcal{C})]$ has ordinal Morley rank and thus contains a strongly minimal formula.*

Proof. For every $a \in \theta(\mathcal{C}) \setminus \mathrm{acl}(\emptyset)$ there exist $n < \omega$ and $\bar{c} \in p^n(\mathcal{C})$ such that $a \in \mathrm{acl}(\bar{c})$. Let $e = \mathrm{Cb}(\bar{c}/a)$. Now, by elimination of hyperimaginaries there exists $e^* \in \mathrm{acl}(a) \cap \mathrm{dcl}(p(\mathcal{C})) \setminus \mathrm{acl}(\emptyset)$. Let $e^{**} = \{e' \mid \mathrm{tp}(e'/a) = \mathrm{tp}(e^*/a)\}$ (e^{**} is an imaginary element). Then clearly e^{**} is in $\mathrm{dcl}(a) \cap \mathrm{dcl}(p(\mathcal{C})) \setminus \mathrm{acl}(\emptyset)$. For any appropriate $\emptyset$-definable functions f, g let

$$F_{f,g} = \{a \in \theta(\mathcal{C}) \mid \exists \bar{c} \subseteq p(\mathcal{C})\ [f(a) = g(\bar{c}) \notin \mathrm{acl}(\emptyset)]\}.$$

Consequently, $\{F_{f,g}\}_{f,g}$ is a countable family of Stone-closed sets that covers $\theta(\mathcal{C}) \setminus \mathrm{acl}(\emptyset)$ and thus by the Baire category theorem for the Stone topology of $\theta(\mathcal{C}) \setminus \mathrm{acl}(\emptyset)$ we get the required formula $\theta^* \in L$ and $\emptyset$-definable functions f, g as in (1).

To prove (2), assume that p is minimal. Then, by induction on n, we easily find that for every countable set A the number of (complete) types of realizations of p^n over A is countable. Thus by (1), for every countable set A the number of complete types over A extending $f[\theta^*(\mathcal{C})]$ is countable. Therefore $f[\theta^*(\mathcal{C})]$ has ordinal Morley rank.

We will be using the following two important facts. The first one is Buechler's dichotomy for minimal types (see [P1, Corollary 3.3]).

FACT 5.9. *Let T be superstable and let $p \in S(A)$ be a minimal type. Then either p is 1-based or* $\mathrm{RM}(p) = 1$.

The second fact is Wagner's result [W] on analysis in 1-based types in simple theories (it generalizes previous results of Hrushovski and Chazidakis).

FACT 5.10. *Let T be any simple theory and work with hyperimaginaries. Assume $p \in S(A)$ is analyzable in an A-invariant family of 1-based types. Then p is 1-based.*

THEOREM 5.11. *Let T be a countable theory with nfcp. Let $p \in S(\emptyset)$ be minimal. Then either*

(1) *every type $q \in S(A)$, with A countable, that is internal in p is essentially 1-based by means of the τ^f-topologies, or*
(2) *there is an infinite definable 1-based group of finite D-rank that is p-internal, or*
(3) *there exists a strongly minimal definable set that is p-internal.*

Proof. Assume (1) is false. By Theorem 5.3, there exists a weakly minimal formula $\theta(x, b)$ that is p-generated and in particular p-internal (in the stable case an invariant set is p-internal iff it is p-generated). First, assume $\theta(\mathcal{C}, b) \subseteq \mathrm{acl}(p(\mathcal{C}) \cup b)$. Then by Lemma 5.8, there exists a strongly minimal formula $\phi \in L(\mathcal{C})$ that is p-internal (even generated over $p^{\mathcal{C}}$). Thus, we may assume $\theta(\mathcal{C}, b) \not\subseteq \mathrm{acl}(p^{\mathcal{C}} \cup b)$. Let $a \in \theta(\mathcal{C}, b) \setminus \mathrm{acl}(p^{\mathcal{C}} \cup b)$. Let $q = \mathrm{tp}(a/\mathrm{acl}(b))$ and let $\Gamma = \mathrm{Aut}(q^{\mathcal{C}}/p^{\mathcal{C}} \cup \mathrm{acl}(b))$. We will be using the following fact [S2, Theorem 2.9], with its proof, which for simplicity we state for a special case. In the following, for a set S, possibly large, we let $\mathrm{DCL}(S)$ be the set of all elements in $\mathcal{C}$ that are fixed by any automorphism that fixes S pointwise; we say that a set V is *controlled by B over S*, if $V \subseteq \mathrm{DCL}(B \cup S)$.

FACT 5.12. *Let T be any simple theory. Let Q be a stably embedded type-definable set over $\emptyset$ and let $q \in S(\emptyset)$. Suppose there exists a subset B of $\mathrm{DCL}(q^{\mathcal{C}} \cup Q)$ with $\mathrm{tp}(B) \vdash \mathrm{Lstp}(B)$ such that $q^{\mathcal{C}}$ is controlled by B over Q. Then $\Gamma = \mathrm{Aut}(q^{\mathcal{C}}/Q)$ is type-definable with its action on $q^{\mathcal{C}}$ over $\emptyset$.*

REMARK 5.13. It is well known that in a stable theory if q is Q-internal then there is always a set B of realizations of q such that $q(\mathcal{C}) \subseteq \mathrm{dcl}(Q, B)$, in particular, q is controlled by B over Q; if q is stationary then B can be taken to be a finite initial segment of a Morley sequence of q and clearly $\mathrm{tp}(B) \vdash \mathrm{Lstp}(B)$.

Now, Γ in Fact 5.12 can be interpreted in the following way. As Q is a type-definable stably embedded set, there exists a partial type $\Sigma_Q(Y, Y')$ expressing that Y, Y' are Q-conjugate, for $Y, Y' \models \mathrm{tp}(B)$. Now, let $\Gamma_{B^2/Q}(Y, Y')$ be the type expressing that $\mathrm{tp}(Y) = \mathrm{tp}(Y') = \mathrm{tp}(B)$ and $\Sigma_Q(Y, Y')$. Now, by definition, $\sigma \in \Gamma = \mathrm{Aut}(q^{\mathcal{C}}/Q)$ iff σ is the restriction to $q^{\mathcal{C}}$ of some automorphism of $\mathcal{C}$ that fixes Q pointwise. As q is controlled by $B \subseteq \mathrm{DCL}(q^{\mathcal{C}} \cup Q)$ over Q, it is not hard to show (see proof of [S2, Theorem 2.9]) that Γ can be interpreted as $\Gamma_{B^2/Q}/E$ for a certain $\emptyset$-definable equivalence relation E.

By Remark 5.13 and the fact that $q(x) \vdash \theta(x, b)$, there is an infinite type-definable group G over $\mathrm{acl}(b)$ that is isomorphic to Γ such that for some $\mathrm{acl}(b)$-definable equivalence relation E and some $n < \omega$, we have $G \subseteq \theta(\mathcal{C}, b)^n/E$. Now, by stability of T, G is an intersection of definable groups over $\mathrm{acl}(b)$ [H1, Theorem 2]. By compactness, there is an infinite $\mathrm{acl}(b)$-definable group G_0 that is p-internal and has finite D-rank. By Fact 5.9 and Remark 5.7(2)(i) applied to the induced structure G_0^* on G_0 over $\mathrm{acl}(b)$, every minimal type r in G_0^* is either 1-based or of Morley rank 1. Thus if (3) fails, then any such r is 1-based in G_0^* by Remark 5.7(2)(iii) and stability of T. As G_0^* has finite SU-rank, we conclude, when working in G_0^*, that every non-algebraic type is non-orthogonal to a minimal type, and therefore any type in G_0^* is analyzable in 1-based types. By Fact 5.10, G_0^* is 1-based. By Remark 5.7(2)(ii), G_0 is 1-based.

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