

Longer chains of idempotents in βG

by

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Abstract. Given idempotents e and f in a semigroup, $e \leq f$ if and only if $e = fe = ef$. We show that if G is a countable discrete group, p is a right cancelable element of $G^* = \beta G \setminus G$, and λ is a countable ordinal, then there is a strictly decreasing chain $\langle q_\sigma \rangle_{\sigma < \lambda}$ of idempotents in C_p , the smallest compact subsemigroup of G^* with p as a member. We also show that if S is any infinite subsemigroup of a countable group, then any nonminimal idempotent in S^* is the largest element of such a strictly decreasing chain of idempotents. (It had been an open question whether there was a strictly decreasing chain $\langle q_\sigma \rangle_{\sigma < \omega+1}$ in \mathbb{N}^* .) As other corollaries we show that if S is an infinite right cancellative and weakly left cancellative discrete semigroup, then βS contains a decreasing chain of idempotents of reverse order type λ for every countable ordinal λ and that if S is an infinite cancellative semigroup then the set $U(S)$ of uniform ultrafilters contains such decreasing chains.

1. Introduction. A semigroup (S, \cdot) with a topology is *right topological* if for each $x \in S$ the function $\rho_x : S \rightarrow S$ is continuous, where for $y \in S$, $\rho_x(y) = y \cdot x$. In [1, Lemma 1], R. Ellis proved that any compact Hausdorff right topological semigroup contains an idempotent.

If (S, \cdot) is an infinite discrete semigroup, there is a unique extension of the operation to βS making $(\beta S, \cdot)$ a right topological semigroup with S contained in its topological center. (The *topological center* of a right topological semigroup is the set of points x such that λ_x is continuous, where $\lambda_x(y) = x \cdot y$.) The existence of idempotents in βS , especially idempotents in certain subsemigroups of βS , has provided the easiest, and often the first, proof of many results in Ramsey Theory. See [4, Part III] for a multitude of examples of this phenomenon.

As a compact right topological semigroup, βS has a smallest two-sided ideal, $K(\beta S)$, which is the union of all of the minimal left ideals of βS and is also the union of all of the minimal right ideals of βS . The intersection of a minimal left ideal and a minimal right ideal of βS is a group, and

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any two such groups are isomorphic. Any left ideal contains a minimal left ideal, which is compact, and any right ideal contains a minimal right ideal. Idempotents in $K(\beta S)$ are exactly the idempotents that are minimal with respect to the ordering defined in the abstract.

We take the points of βS to be the ultrafilters on S , identifying the principal ultrafilters with the points of S , and thus pretend that $S \subseteq \beta S$. Given $A \subseteq S$, we define the closure \overline{A} to be $\{p \in \beta S : A \in p\}$. We write $A^* = \overline{A} \setminus S$. Given $p, q \in \beta S$, we write $A \in pq$ if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : xy \in A\}$. (So, for example, in the semigroup (\mathbb{N}, \cdot) , if A is the set of odd positive integers, then $2^{-1}A = \emptyset$.) We let $U(S)$ be the set of uniform ultrafilters on S . Thus for $p \in \beta S$, $p \in U(S)$ if and only if for every $A \in p$, $|A| = |S|$. We take \mathbb{N} to be the set of positive integers. The first infinite ordinal ω is the set of nonnegative integers. See [4, Part I] for an elementary introduction to the topology and algebra of βS , and see the notes at the end of the chapters for the original references.

In [3] it was shown that any nonminimal idempotent in $(\beta\mathbb{N}, +)$ is part of an infinite decreasing chain of idempotents. That is, if q is a nonminimal idempotent, then there is a sequence of idempotents $\langle q_n \rangle_{n < \omega}$ such that $q_0 = q$ and for each $n \in \omega$, $q_{n+1} < q_n$. We shall show in this paper that the sequence can be extended to $\langle q_\sigma \rangle_{\sigma < \lambda}$ for any countable ordinal λ .

A fundamental tool in our proofs is an analysis of the structure of the smallest compact subsemigroup of $\beta\mathbb{N}$ containing a given member of $\beta\mathbb{N}$.

DEFINITION 1.1. Let S be a compact Hausdorff right topological semigroup and let $p \in S$. Then

$$C_p = \bigcap \{T : T \text{ is a compact subsemigroup of } S \text{ and } p \in T\}.$$

Section 2 consists of preliminary results. In Section 3 we will prove our main theorem dealing with decreasing chains of idempotents in C_p and derive from that several corollaries, including those mentioned in the abstract.

Most of the results in Section 3 deal with cancellative semigroups. In Section 4 we extend some of these results to left cancellative semigroups S which have a right cancelable element in S^* .

Besides the ordering \leq of idempotents in a semigroup, there are transitive and reflexive relations \leq_L and \leq_R defined by $e \leq_L f$ if $ef = e$, and $e \leq_R f$ if $fe = e$. We write $e <_L f$ when $e \leq_L f$ and it is not the case that $f \leq_L e$. Similarly we write $e <_R f$ when $e \leq_R f$ and it is not the case that $f \leq_R e$. Of course $e \leq f$ if and only if both $e \leq_L f$ and $e \leq_R f$. In [6] it was shown that given any ordinal λ with $|\lambda| \leq \mathfrak{c}$, there exist chains $\langle q_\sigma \rangle_{\sigma < \lambda}$ of idempotents in $\beta\mathbb{N}$ such that $q_\sigma <_L q_\tau$ whenever $\tau < \sigma < \lambda$ and $q_{\sigma+1} < q_\sigma$ for all σ with $\sigma + 1 < \lambda$. In Section 5 we extend this result by showing that for each nonminimal idempotent q in $\beta\mathbb{N}$, there is such a chain with $q_0 = q$.

2. Preliminary results. Given a semigroup (S, \cdot) with identity we will denote the identity of S by 1. Unless otherwise specified, we take the operation on \mathbb{N} and \mathbb{Z} to be addition.

Some of our proofs depend on the existence of elements in the closure of the set of idempotents in a given minimal right ideal of $\beta\mathbb{N}$ that are right cancelable in $\beta\mathbb{N}$. The following lemma guarantees their existence.

LEMMA 2.1. *Let R be a minimal right ideal of $\beta\mathbb{N}$. There is an injective sequence $\langle q_n \rangle_{n=1}^\infty$ of idempotents in R such that, if p is an accumulation point of $\langle q_n \rangle_{n=1}^\infty$, then $p \notin \mathbb{Z}^* + \mathbb{Z}^*$. In particular any accumulation point of $\langle q_n \rangle_{n=1}^\infty$ is right cancelable in $\beta\mathbb{Z}$.*

Proof. This is [5, Lemma 3.8]. ■

DEFINITION 2.2. If S is a semigroup, then $E(S)$ is the set of idempotents in S .

The following lemma is well known among aficionados. In its proof we use, for the first of many times in this paper, the fact that if p is an idempotent in a semigroup S , then p is a right identity for Sp and a left identity for pS . (If $q = ap$, then $qp = app = ap = q$.)

LEMMA 2.3. *Let S and T be compact Hausdorff right topological semigroups, let $h : S \rightarrow T$ be a continuous surjective homomorphism.*

- (1) *If q_1 is an idempotent in T , then there exists $p_1 \in E(S)$ such that $h(p_1) = q_1$.*
- (2) *If q_1 and q_2 are idempotents in T such that $q_2 < q_1$ and $p_1 \in E(S)$ such that $h(p_1) = q_1$, then there exists $p_2 \in E(S)$ such that $h(p_2) = q_2$ and $p_2 < p_1$.*

Proof. (1) Since $h^{-1}[\{q_1\}]$ is a compact subsemigroup of S , it has an idempotent.

(2) Assume that $p_1 \in E(S)$ is such that $h(p_1) = q_1$. If $x \in h^{-1}[\{q_2\}]$, then $h(xp_1) = q_2q_1 = q_2$ so $xp_1 \in h^{-1}[\{q_2\}] \cap Sp_1$ and consequently $h^{-1}[\{q_2\}] \cap Sp_1$ is a compact subsemigroup of S . Pick an idempotent $g \in h^{-1}[\{q_2\}] \cap Sp_1$ and let $p_2 = p_1g$. Then $gp_1 = g$ so $p_2p_1 = p_2$, $p_2p_2 = p_1gp_1g = p_1gg = p_1g = p_2$, and $p_1p_2 = p_1p_1g = p_1g = p_2$. ■

LEMMA 2.4. *Let S be a compact Hausdorff right topological semigroup and let R be a minimal right ideal of S . If $x \in \text{cl } E(R)$, then for all $u \in C_x$ and all $v \in R$, $uv = v$.*

Proof. If $p \in E(R)$, then $R = pS$, so for all $v \in R$, $pv = v$. Thus, given $v \in R$, ρ_v is constantly equal to v on $E(R)$ so $xv = v$. Hence

$$\{u \in S : (\forall v \in R)(uv = v)\}$$

is a compact subsemigroup of S with x as a member which therefore contains C_x . ■

Of course, in any semigroup S , an element x is *right cancelable* if ρ_x is injective.

DEFINITION 2.5. Let S be an infinite semigroup with identity and let $p \in S^*$.

- (a) p is *weakly right cancelable* if there is no $q \in \beta S \setminus \{1\}$ such that $p = qp$.
- (b) p is *thin* if there is a function $M : S \rightarrow p$ such that $xM(x) \cap yM(y) = \emptyset$ whenever x and y are distinct members of S .
- (c) p is *strongly discrete* if p is thin, $p \in U(S)$, and for each $x \in S$, the restriction of λ_x to $M(x)$ is injective.

Note that if $p = qp$, then $1p = 1qp$, so if p is right cancelable, then it is weakly right cancelable.

LEMMA 2.6. *Let S be a countable semigroup with identity and let $p \in S^*$. If p is right cancelable in βS , then p is thin. If in addition S is left cancellative, then p is strongly discrete.*

Proof. By [4, Theorem 8.7] we see that for all $A \subseteq S$, there exists $B \subseteq S$ such that $A = \{x \in S : x^{-1}B \in p\}$. Enumerate S as $\langle x_n \rangle_{n=1}^\infty$. For each $n \in \mathbb{N}$, pick $B_n \subseteq S$ such that $\{x_n\} = \{x \in S : x^{-1}B_n \in p\}$. Let $M(x_1) = x_1^{-1}B_1$. For $n > 1$, let $M(x_n) = x_n^{-1}B_n \setminus \bigcup_{t=1}^{n-1} x_n^{-1}B_t$. If $t < n$, then $x_tM(x_t) \subseteq B_t$ and $x_nM(x_n) \cap B_t = \emptyset$. ■

LEMMA 2.7. *Let S be an infinite semigroup with identity and let $p \in S^*$.*

- (1) *If p is thin, then p is right cancelable in βS .*
- (2) *If p is weakly right cancelable and S is a countable group, then p is strongly discrete.*

Proof. (1) Let $M : S \rightarrow p$ be as guaranteed by the definition of thin. Let q and r be distinct members of βS and pick $Q \in q$ and $R \in r$ such that $Q \cap R = \emptyset$. Let $A = \bigcup_{x \in Q} xM(x)$ and let $B = \bigcup_{x \in R} xM(x)$. Then $A \in qp$, $B \in rp$, and $A \cap B = \emptyset$.

(2) By [4, Theorem 8.18], p is right cancelable in βS so Lemma 2.6 applies. ■

DEFINITION 2.8. Let κ be an infinite cardinal.

- (a) $H_\kappa = \bigoplus_{\sigma < \kappa} \mathbb{Z}_2$.
- (b) For $\gamma < \kappa$, let $H_{\kappa,\gamma} = \{x \in H_\kappa : (\forall \sigma < \gamma)(x(\sigma) = 0)\}$.
- (c) $\mathbb{H}_\kappa = \bigcap_{\gamma < \kappa} \text{cl}_{\beta H_\kappa}(H_{\kappa,\gamma} \setminus \{0\})$.

The set $\mathbb{H} \subseteq \beta \mathbb{N}$ is defined by $\mathbb{H} = \bigcap_{n \in \mathbb{N}} \text{cl}(2^n \mathbb{N})$. By [4, Theorem 6.27], \mathbb{H} is topologically and algebraically isomorphic to \mathbb{H}_ω . (When we say that

sets in right topological semigroups are “topologically and algebraically isomorphic” we mean that there is a function taking one to the other which is both an isomorphism and a homeomorphism.)

LEMMA 2.9. *Let κ be an infinite cardinal and let $p \in \mathbb{H}_\kappa$. The following statements are equivalent:*

- (1) *p is right cancelable in βH_κ .*
- (2) *p is weakly right cancelable.*
- (3) *p is thin.*

Proof. That (1) implies (2) is trivial and that (3) implies (1) follows from Lemma 2.7(1).

To see that (2) implies (3), assume that p is weakly right cancelable. We note first that $H_\kappa + p$ is discrete. Indeed, if $a \in H_\kappa$ and

$$a + p \in \text{cl}\{b + p : b \in H_\kappa \setminus \{a\}\} = (\beta H_\kappa \setminus \{a\}) + p,$$

then pick $q \in \beta H_\kappa \setminus \{a\}$ such that $a + p = q + p$. Then $p = -a + q + p$, and $-a + q \neq 0$. (If $q \in H_\kappa$ this is immediate, and if $q \in H_\kappa^*$, then $-a + q \in H_\kappa^*$ by [4, Corollary 4.33].)

The rest of the argument may be taken verbatim from the proof that (5) implies (6) in [8, Theorem 11.2]. ■

LEMMA 2.10. *Let S be an infinite semigroup with identity and let $p \in S^*$ be strongly discrete. There is a compact subsemigroup T_p of βS with $p \in T_p$ such that*

- (1) *for all $x \in \beta S \setminus (T_p \cup \{1\})$, $(xT_p) \cap T_p = \emptyset$, and*
- (2) *there is a continuous homomorphism $\pi_p : T_p \rightarrow \beta\mathbb{N}$ with $\pi_p(p) = 1$.*

Proof. Let $\mathcal{T}[p]$ be the largest topology on S with respect to which p converges to 1 and λ_a is continuous for each $a \in S$. Let

$$T_p = \{q \in S^* : q \text{ converges to } 1 \text{ with respect to } \mathcal{T}[p]\}.$$

By [8, Lemma 7.1], T_p is a compact subsemigroup of S^* . To verify conclusion (1) suppose we have $x \in \beta S \setminus (T_p \cup \{1\})$ and $q \in T_p$ such that $xq \in T_p$. By [8, Theorem 4.18], $\mathcal{T}[p]$ is zero-dimensional and Hausdorff. Since $x \notin T_p \cup \{1\}$, x does not converge to 1 with respect to $\mathcal{T}[p]$ and thus there is a neighborhood U of 1 such that $U \not\subseteq x$. Since $\mathcal{T}[p]$ is zero-dimensional, we may assume U is clopen with respect to $\mathcal{T}[p]$. Let $W = S \setminus U$. Then $W \in x$. We claim that $\text{cl}_{\beta S}(W)q \subseteq \text{cl}_{\beta S}(W)$, for which it suffices that $Wq \subseteq \text{cl}_{\beta S}(W)$, so let $a \in W$. Then W is a neighborhood of $a = \lambda_a(1)$ and λ_a is continuous with respect to $\mathcal{T}[p]$, so pick a neighborhood V of 1 such that $aV \subseteq W$. Then $V \in q$ and $V \subseteq a^{-1}W$ so $a^{-1}W \in q$ and thus $aq \in \text{cl}_{\beta S}(W)$ as claimed. We thus infer that $xq \in \text{cl}_{\beta S}(W)$ so $U \not\subseteq xq$ and thus $xq \notin T_p$.

Conclusion (2) holds by [8, Theorem 7.29]. ■

Notice that, since T_p is a compact subsemigroup of βS and $p \in T_p$, we have $C_p \subseteq T_p$.

DEFINITION 2.11. Let S be an infinite semigroup with identity and let $p \in S^*$ be strongly discrete. Then h_p is the restriction of π_p to C_p .

If $n \in \mathbb{N}$ and q is the sum of p with itself n times, then $h_p(p) = n$. Therefore $\mathbb{N} \subseteq h_p[C_p]$ and consequently $h_p[C_p] = \beta\mathbb{N}$. Observe also that the function h_p is completely determined by the fact that $h_p(p) = 1$. To see this, let $g : C_p \rightarrow \beta\mathbb{N}$ be a continuous homomorphism with $g(p) = 1$. Then $\{x \in C_p : g(x) = h_p(x)\}$ is a compact subsemigroup of C_p with p as a member and is therefore equal to C_p .

LEMMA 2.12. *Let S be an infinite semigroup with identity and let $p \in S^*$ be strongly discrete. If $x \in C_p$ and $h_p(x)$ is right cancelable in $\beta\mathbb{N}$, then x is weakly right cancelable.*

Proof. Suppose not and pick $u \in \beta S \setminus \{1\}$ such that $x = ux$. Then $x \in (uT_p) \cap T_p$, so by Lemma 2.10(1), $u \in T_p \cup \{1\}$ and since $u \neq 1$, $u \in T_p$. Thus $h_p(x) = \pi_p(u) + \pi_p(x) = \pi_p(u) + h_p(x)$. By [4, Corollary 8.2] (since $1 + h_p(x) = 1 + \pi_p(u) + h_p(x)$) we have $\pi_p(u) \in \mathbb{N}^*$. But then, by [4, Theorem 8.18], $h_p(x)$ is not right cancelable in $\beta\mathbb{N}$. ■

LEMMA 2.13. *Let S be an infinite semigroup with identity, let $p \in S^*$ be strongly discrete, and let q be an idempotent in C_p such that $h_p(q) \notin K(\beta\mathbb{N})$. There exists $s \in C_pq$ which is weakly right cancelable such that $h_p[C_s] \cap K(\beta\mathbb{N}) = \emptyset$.*

Proof. By [4, Theorem 6.56], choose $y \in \beta\mathbb{N}$ such that $y + h_p(q)$ is right cancelable in $\beta\mathbb{Z}$. Pick $x \in C_p$ such that $h_p(x) = y$ and let $s = xq$. Then $h_p(s) = y + h_p(q)$, so by Lemma 2.12, s is weakly right cancelable. By [4, Theorem 8.57], $C_{h_p(s)} \cap K(\beta\mathbb{Z}) = \emptyset$. By [4, Exercise 4.3.8], $K(\beta\mathbb{Z}) = K(\beta\mathbb{N}) \cup -K(\beta\mathbb{N})$, so $C_{h_p(s)} \cap K(\beta\mathbb{N}) = \emptyset$. Since $h_p^{-1}[C_{h_p(s)}]$ is a compact subsemigroup containing s , we have $h_p[C_s] \subseteq C_{h_p(s)}$. ■

LEMMA 2.14. *Let S be an infinite semigroup with identity, let $p \in S^*$ be strongly discrete, and let R be a minimal right ideal of C_p . There exists $s \in \text{cl } E(R)$ which is weakly right cancelable.*

Proof. By [4, Exercise 1.7.3], $h_p[R]$ is a minimal right ideal of $\beta\mathbb{N}$, so by Lemma 2.1, there is an injective sequence $\langle q_n \rangle_{n=1}^\infty$ of idempotents in $h_p[R]$ all of whose limit points are right cancelable in $\beta\mathbb{Z}$. We claim that for each $n \in \mathbb{N}$ there is an idempotent $u_n \in R$ such that $h_p(u_n) = q_n$. To see this, let $n \in \mathbb{N}$. Then $\beta\mathbb{N} + q_n$ is a minimal left ideal of $\beta\mathbb{N}$, so $h_p^{-1}[\beta\mathbb{N} + q_n]$ is a left ideal of C_p which contains a minimal left ideal L . Let u_n be the identity of $R \cap L$. Then $h_p(u_n)$ is an idempotent in $h_p[R] \cap (\beta\mathbb{N} + q_n)$, whose only idempotent is q_n . Let s be a limit point of $\langle u_n \rangle_{n=1}^\infty$. Then $h_p(s)$ is a limit

point of $\langle q_n \rangle_{n=1}^\infty$ so it is right cancelable in $\beta\mathbb{Z}$. Thus by Lemma 2.12, s is weakly right cancelable in βS . ■

LEMMA 2.15. *Let S be an infinite semigroup with identity, let q be an idempotent in S^* , let $s \in \beta Sq$ be strongly discrete, let λ be an ordinal, and let $\langle u_\sigma \rangle_{\sigma < \lambda}$ be a strictly decreasing sequence of idempotents in C_s . For each $\sigma < \lambda$, let $v_\sigma = qu_\sigma$. Then $\langle v_\sigma \rangle_{\sigma < \lambda}$ is a strictly decreasing sequence of idempotents with $v_0 \leq q$.*

Proof. Note that $C_s \subseteq \beta Sq$, so for each $\sigma < \lambda$, $u_\sigma = u_\sigma q$. Given $\sigma < \lambda$, we have $v_\sigma v_\sigma = qu_\sigma qu_\sigma = qu_\sigma u_\sigma = qu_\sigma = v_\sigma$. Now let $\sigma < \tau < \lambda$. Then $v_\sigma v_\tau = qu_\sigma qu_\tau = qu_\sigma u_\tau = qu_\tau = v_\tau$ and $v_\tau v_\sigma = qu_\tau qu_\sigma = qu_\tau u_\sigma = qu_\tau = v_\tau$, so $v_\tau \leq v_\sigma$. We claim that $v_\sigma \neq v_\tau$. Suppose instead $v_\sigma = v_\tau$. Now $C_s s \cup \{s\}$ is a compact semigroup with s as a member, so $C_s \subseteq C_s s \cup \{s\} \subseteq C_s$ hence $C_s = C_s s \cup \{s\}$. Since u_σ and u_τ are idempotents, neither is equal to s , so pick x_σ and x_τ in C_s such that $u_\sigma = x_\sigma s$ and $u_\tau = x_\tau s$. Since $s \in \beta Sq$, pick $r \in \beta S$ such that $s = rq$. Now $qx_\tau rq = qx_\tau s = qu_\tau = v_\tau = v_\sigma = qx_\sigma rq$, so $x_\sigma r qx_\tau rq = x_\sigma r qx_\sigma rq$. That is, $u_\sigma u_\tau = u_\sigma u_\sigma$, so $u_\tau = u_\sigma$, a contradiction. ■

3. Long strictly decreasing chains

DEFINITION 3.1. Let S be an infinite semigroup with identity and let $p \in S^*$. Then p is *hereditarily strongly discrete* if p is strongly discrete and every $s \in C_p$ which is weakly right cancelable is also strongly discrete.

Notice that by Lemma 2.7(2), if S is a countable group and $p \in S^*$ is strongly discrete, then p is hereditarily strongly discrete.

LEMMA 3.2. *Let κ be an infinite cardinal and let $p \in \mathbb{H}_\kappa$ be strongly discrete. Then $C_p \subseteq U(H_\kappa)$ and p is hereditarily strongly discrete.*

Proof. Trivially \mathbb{H}_κ is a compact subsemigroup of βH_κ , and by [4, Lemma 6.34.3], $U(H_\kappa)$ is a compact subsemigroup (in fact an ideal) of βH_κ . Since $p \in U(H_\kappa)$ by the definition of strongly discrete, we infer that $\mathbb{H}_\kappa \cap U(H_\kappa)$ is a compact semigroup with p as a member, so $C_p \subseteq \mathbb{H}_\kappa \cap U(H_\kappa)$.

To see that p is hereditarily strongly discrete, let $s \in C_p$ be weakly right cancelable. By Lemma 2.9, s is thin. But also $C_p \subseteq \mathbb{H}_\kappa \cap U(H_\kappa)$, so $s \in U(H_\kappa)$, and (since H_κ is cancellative) s is strongly discrete. ■

DEFINITION 3.3. Let S be an infinite semigroup with identity and let λ be an ordinal. $P(\lambda)$ is the following statement: *Given any hereditarily strongly discrete $p \in S^*$ and any idempotent $q \in C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$, there is a strictly decreasing chain $\langle q_\sigma \rangle_{\sigma < \lambda}$ of idempotents in $C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$ such that $q_0 = q$.*

LEMMA 3.4. *Let S be an infinite semigroup with identity and let $\lambda > 0$ be an ordinal. Then $P(\lambda) \Rightarrow P(\lambda + 1)$.*

Proof. Assume $P(\lambda)$. Let $p \in S^*$ be hereditarily strongly discrete and let q be an idempotent in $C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$. By Lemma 2.13 pick $s \in C_p q$ which is weakly right cancelable in βS such that $h_p[C_s] \cap K(\beta\mathbb{N}) = \emptyset$. Since $s \in C_p$, s is strongly discrete. Let R be a minimal right ideal of C_s . By Lemma 2.14 pick $t \in \text{cl} E(R)$ which is weakly right cancelable. Note that $C_t \subseteq C_s \subseteq \beta S q$. By Lemma 2.4 we deduce that for all $u \in C_t$ and all $v \in R$, $uv = v$. Pick any idempotent w in $C_t \setminus h_t^{-1}[K(\beta\mathbb{N})]$ and choose a strictly decreasing chain $\langle u_\sigma \rangle_{\sigma < \lambda}$ of idempotents in $C_t \setminus h_t^{-1}[K(\beta\mathbb{N})]$ with $u_0 = w$. (We will not use the fact that $u_0 = w$.)

If $\sigma < \tau < \lambda$, then $u_\tau = u_\tau u_\sigma \in C_s u_\sigma$, so $\langle C_s u_\sigma \rangle_{\sigma < \lambda}$ is a nested sequence of closed left ideals of C_s , hence we may pick a minimal left ideal L of C_s with $L \subseteq \bigcap_{\sigma < \lambda} C_s u_\sigma$. Let u_λ be the identity of $R \cap L$. Let $\sigma < \lambda$. We have $u_\lambda \in C_s u_\sigma$, so $u_\lambda = u_\lambda u_\sigma$. Also $u_\sigma \in C_t$ and $u_\lambda \in R$, so $u_\lambda = u_\sigma u_\lambda$ and thus $u_\lambda \leq u_\sigma$. We need to show that $u_\sigma \neq u_\lambda$. (Of course, if λ is a limit ordinal, this is immediate. But our proof does not depend on λ being a successor.) We shall show that $u_\sigma \notin R$. So suppose instead that $u_\sigma \in R$. Then $u_\sigma \in C_t \cap R$, so $\emptyset \neq C_t \cap R \subseteq C_t \cap K(C_s)$, and thus by [4, Theorem 1.65], $K(C_t) = C_t \cap K(C_s)$, so $u_\sigma \in K(C_t)$. But then $h_t(u_\sigma) \in K(\beta\mathbb{N})$, a contradiction. We have thus established that $\langle u_\sigma \rangle_{\sigma \leq \lambda}$ is a strictly decreasing chain of idempotents in C_s . Recall that $s \in C_p q$, so pick $r \in C_p$ such that $s = r q$.

Now for each $\sigma \leq \lambda$, let $v_\sigma = q u_\sigma$. By Lemma 2.15 we see that $\langle v_\sigma \rangle_{\sigma \leq \lambda}$ is a strictly decreasing chain of idempotents in C_p .

Suppose there is some $\sigma \leq \lambda$ such that $h_p(v_\sigma) \in K(\beta\mathbb{N})$. Then $s u_\sigma \in C_s$ and $s u_\sigma = r q u_\sigma = r v_\sigma$, so $h_p(r v_\sigma) = h_p(r) + h_p(v_\sigma) \in K(\beta\mathbb{N})$, contradicting the fact that $h_p[C_s] \cap K(\beta\mathbb{N}) = \emptyset$.

Note that $v_0 \leq q$. If $v_0 = q$, then we are done, so assume $v_0 < q$ and replace v_0 by q . ■

The following is the main result of this paper. Recall that h_p is surjective and therefore there are idempotents in $C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$.

THEOREM 3.5. *Let S be an infinite semigroup with identity and let $\lambda > 0$ be a countable ordinal. Given any hereditarily strongly discrete $p \in S^*$ and any idempotent $q \in C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$, there is a strictly decreasing chain $\langle q_\sigma \rangle_{\sigma < \lambda}$ of idempotents in $C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$ such that $q_0 = q$.*

Proof. We prove by induction that $P(\lambda)$ holds. If $\lambda = 1$, let $q_0 = q$. So assume that $\lambda > 1$ and $P(\alpha)$ holds for all α with $0 < \alpha < \lambda$. If λ is a successor, then $P(\lambda)$ holds by Lemma 3.4, so assume that λ is a limit ordinal. Pick a strictly increasing sequence $\langle \alpha_n \rangle_{n < \omega}$ of ordinals with $\alpha_0 > 0$ such that $\lambda = \sup\{\alpha_n : n < \omega\}$. Let $p \in S^*$ be hereditarily strongly discrete and let q be an idempotent in $C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$. By $P(\alpha_0 + 1)$ pick a strictly decreasing chain $\langle q_\sigma \rangle_{\sigma \leq \alpha_0}$ in $C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$ such that $q_0 = q$.

Now let $n < \omega$ and assume we have chosen $\langle q_\sigma \rangle_{\sigma \leq \alpha_n}$. Let δ be the ordinal such that $\alpha_{n+1} = \alpha_n + \delta$. By $P(\delta+1)$, pick a strictly decreasing chain $\langle r_\sigma \rangle_{\sigma \leq \delta}$ in $C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$ such that $r_0 = q_{\alpha_n}$. For $0 < \tau \leq \delta$, let $q_{\alpha_n+\tau} = r_\tau$. ■

COROLLARY 3.6. *Let G be a countably infinite group and let $\lambda > 0$ be a countable ordinal. Given any $p \in G^*$ which is right cancelable in βG and any idempotent $q \in C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$, there is a strictly decreasing chain of idempotents $\langle q_\sigma \rangle_{\sigma < \lambda}$ in $C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$ such that $q_0 = q$.*

Proof. Let $p \in G^*$ be right cancelable in βG . As we have already remarked, p is hereditarily strongly discrete. So Theorem 3.5 applies. ■

COROLLARY 3.7. *Let κ be an infinite cardinal and let $\lambda > 0$ be a countable ordinal. Given any strongly discrete $p \in \mathbb{H}_\kappa$, and any idempotent $q \in C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$, there is a strictly decreasing chain of idempotents $\langle q_\sigma \rangle_{\sigma < \lambda}$ in $C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$ such that $q_0 = q$. This chain of idempotents is contained in $U(H_\kappa) \cap \mathbb{H}_\kappa$.*

Proof. Let $p \in \mathbb{H}_\kappa$ be strongly discrete. By Lemma 3.2, p is hereditarily strongly discrete. So Theorem 3.5 applies. Since $C_p \subseteq U(H_\kappa) \cap \mathbb{H}_\kappa$, the final conclusion holds. ■

Of course minimal idempotents by definition do not have any idempotents below them. We now show that for any subsemigroup of a countably infinite group, any nonminimal idempotent has long chains of idempotents below it.

COROLLARY 3.8. *Let G be a countably infinite group, let $\lambda > 0$ be a countable ordinal, and let S be an infinite subsemigroup of G . For every nonminimal idempotent $q \in S^*$, there is a strictly decreasing chain $\langle v_\sigma \rangle_{\sigma < \lambda}$ of idempotents in S^* such that $v_0 = q$.*

Proof. By [4, Theorem 6.56] pick $r \in S^*$ such that $s = rq$ is right cancelable in βG . By Lemma 2.7(2), s is strongly discrete. Since $\beta\mathbb{N}$ has nonminimal idempotents and $h_s[C_s] = \beta\mathbb{N}$, by Lemma 2.3 there is an idempotent $t \in C_s \setminus h_s^{-1}[K(\beta\mathbb{N})]$. By Corollary 3.6 pick a strictly decreasing sequence $\langle u_\sigma \rangle_{\sigma < \alpha}$ of idempotents in C_s . For each $\sigma < \lambda$, let $v_\sigma = qu_\sigma$. By Lemma 2.15, $\langle v_\sigma \rangle_{\sigma < \alpha}$ is a strictly decreasing sequence of idempotents with $v_0 \leq q$. If $v_0 \neq q$, replace v_0 by q . ■

LEMMA 3.9. *There is a strongly discrete $p \in \mathbb{H}_\kappa$.*

Proof. For each $\alpha < \kappa$ let a_α be the characteristic function of $\{\alpha\}$, and for each $\gamma < \kappa$ let

$$A_\gamma = \{a_\alpha : \gamma \leq \alpha < \kappa\}.$$

By [4, Corollary 3.14] pick $p \in U(H_\kappa)$ such that $\{A_\gamma : \gamma < \kappa\} \subseteq p$. Since for

each γ , $A_\gamma \subseteq H_{\kappa,\gamma}$, we have $p \in \mathbb{H}_\kappa$. For $x \in H_\kappa \setminus \{0\}$, let

$$\phi(x) = \max\{\sigma : x(\sigma) \neq 0\}$$

and let $\phi(0) = -1$. Define $M : H_\kappa \rightarrow p$ by $M(x) = A_{\phi(x)+1}$. Now let $x \neq y$ in H_κ . We claim that $(x + M(x)) \cap (y + M(y)) = \emptyset$. Suppose $z = x + a_\alpha = y + a_\delta$ where $a_\alpha \in M(x)$ and $a_\delta \in M(y)$. Then $\alpha = \phi(z) = \delta$, so $x = y$, a contradiction. ■

COROLLARY 3.10. *Let S be an infinite cancellative semigroup with identity and let λ be a countable ordinal. There is a strictly decreasing chain $\langle v_\sigma \rangle_{\sigma < \lambda}$ of idempotents in $U(S)$.*

Proof. Let $\kappa = |S|$. By [6, Theorem 2.7], S^* contains a copy of \mathbb{H}_κ . The proof of that theorem produces a subset T of S and a bijective function $\theta : T \rightarrow H_\kappa$ with continuous extension $\tilde{\theta} : \text{cl}_{\beta S} T \rightarrow \beta H_\kappa$. The restriction of $\tilde{\theta}$ to $\tilde{\theta}^{-1}[\mathbb{H}_\kappa]$ is a homeomorphism and an isomorphism. And $\tilde{\theta}[U(T)] = U(H_\kappa)$.

By Lemma 3.9 pick a strongly discrete member p of \mathbb{H}_κ . Since $\beta\mathbb{N}$ has nonminimal idempotents and $h_p[C_p] = \beta\mathbb{N}$, by Lemma 2.3 there is an idempotent $q \in C_p \setminus h_p^{-1}[K(\beta\mathbb{N})]$, so by Corollary 3.7 the copy of \mathbb{H}_κ contains a strictly decreasing chain $\langle v_\sigma \rangle_{\sigma < \lambda}$ of idempotents which is contained in $U(S)$. ■

The following, which answers [6, Question 3.19], is not an immediate consequence of Corollary 3.6 because there are points $p \in \mathbb{N}^*$ that are right cancelable in $\beta\mathbb{N}$ but not right cancelable in $\beta\mathbb{Z}$. (See [4, Example 8.29].)

COROLLARY 3.11. *Given any $p \in \mathbb{N}^*$ which is right cancelable in $\beta\mathbb{N}$ and any countable ordinal λ , there is a strictly decreasing chain $\langle q_\sigma \rangle_{\sigma < \lambda}$ of idempotents in C_p .*

Proof. By Theorem 4.10 below (or [4, Exercise 8.5.1(6)]) there is an element $q \in \text{cl}\{2^n : n \in \mathbb{N}\}$ such that C_q is isomorphic to C_p . By [4, Theorem 8.28], q is right cancelable in $\beta\mathbb{Z}$. By Lemma 2.6, q is strongly discrete, and so by Theorem 3.5, there is a strictly decreasing chain $\langle q_\sigma \rangle_{\sigma < \lambda}$ of idempotents in C_q . ■

Note that the number of decreasing chains headed by a given nonminimal q is vast. By [4, Theorem 6.56], there are 2^c choices of $r \in \beta\mathbb{N}$ for which $r + q$ is right cancelable in $\beta\mathbb{Z}$ and, for any two different choices r_1 and r_2 among these, the left ideals $\beta\mathbb{N} + r_1 + q$ and $\beta\mathbb{N} + r_2 + q$ are disjoint. So, in defining decreasing chains $\langle q_n \rangle_{n < \omega}$ with $q_0 = q$, one has 2^c choices for q_1 . For each of these, there are 2^c choices for q_2 , and so on. The chains defined by these choices never intersect, except at q .

We conclude this section by establishing that one can get long chains of idempotents while weakening the cancellation hypotheses on S . Given a set X , we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X .

DEFINITION 3.12. Let (S, \cdot) be a semigroup and let $\langle x_n \rangle_{n=1}^\infty$ be a sequence in S .

(a) Let $m \in \mathbb{N}$. Then

$$\text{FP}(\langle x_n \rangle_{n=m}^\infty) = \left\{ \prod_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N}) \text{ and } \min F \geq m \right\},$$

where the products are taken in increasing order of indices.

(b) The sequence $\langle x_n \rangle_{n=1}^\infty$ has *distinct finite products* if whenever F and H are distinct members of $\mathcal{P}_f(\mathbb{N})$, $\prod_{t \in F} x_t \neq \prod_{t \in H} x_t$.

LEMMA 3.13. *Let S be a semigroup and let $\langle x_n \rangle_{n=1}^\infty$ be a sequence in S with distinct finite products. Then $\bigcap_{m=1}^\infty \overline{\text{FP}}(\langle x_n \rangle_{n=m}^\infty)$ is topologically and algebraically isomorphic to \mathbb{H} .*

Proof. See [4, Theorem 6.27]. ■

Recall that semigroup S is *weakly left cancellative* provided that for each $a, b \in S$, $\{c \in S : ac = b\}$ is finite.

LEMMA 3.14. *Let S be an infinite right cancellative and weakly left cancellative semigroup. There is a sequence $\langle x_n \rangle_{n=1}^\infty$ in S which has distinct finite products.*

Proof. This is a consequence of [4, Lemma 6.31]. ■

COROLLARY 3.15. *Let S be an infinite right cancellative and weakly left cancellative semigroup and let $\lambda > 0$ be a countable ordinal. Then βS contains a decreasing chain $\langle q_\sigma \rangle_{\sigma < \lambda}$ of idempotents.*

Proof. By [4, Lemma 6.8] all of the idempotents in $\beta\mathbb{N}$ are in \mathbb{H} . By Lemmas 3.13 and 3.14, S contains a copy of \mathbb{H} . By Corollary 3.8, \mathbb{H} contains a decreasing chain $\langle q_\sigma \rangle_{\sigma < \lambda}$ of idempotents. ■

4. Countable left cancellative semigroups. In this section we show that some of our earlier results can be extended to countable semigroups S for which only left cancellation is assumed, provided there is a right cancelable element of S^* . Theorems 4.10 and 4.12 extend [4, Theorem 8.62] and [4, Theorem 8.57] respectively, wherein the hypothesis on S was that it is a countable group. We observe that any right cancelable element of S^* is a strongly discrete ultrafilter by Lemma 2.6, so that the results of [8, §4.3] apply to it. However, the results of this section have self-contained proofs.

There are left cancellative semigroups S for which there are no right cancelable elements in S^* . For example, if S is a right zero semigroup, then βS is also a right zero semigroup and it contains no right cancelable elements if $|S| > 1$. So we begin by pointing out that it is quite easy to find examples of countable left cancellative semigroups S for which S^* does contain right

cancelable elements. One example of a left cancellative semigroup which is not right cancellative but in which S^* has right cancelable elements is the ordinal $\omega \cdot \omega$ under ordinal addition. But that is not an especially interesting example, since the only right cancelable elements of S^* are in ω^* . (If $\{\sigma < \omega \cdot \omega : \omega \leq \sigma\} \in p$, then $1 + p = 2 + p$.)

THEOREM 4.1. *There exist a countable left cancellative semigroup S with identity which is not right cancellative and an infinite subset A of S such that every $p \in A^*$ is right cancelable in βS .*

Proof. Let S be the set of all strictly increasing $f : \mathbb{N} \rightarrow \mathbb{N}$ with the property that there exist $m \in \mathbb{N}$ and $k \in \omega$ such that for all $t \geq m$, $f(t) = k + t$. Let the operation on S be composition. It is routine to verify that S is closed under composition. Further, since its members are injective, S is left cancellative. It is not right cancellative (though it is weakly right cancellative). For example, let $f(t) = t + 1$ for all t , let $g(1) = 1$, $h(1) = 2$, and $g(t) = h(t) = t + 1$ for all $t > 1$. Then $g \circ f = h \circ f$.

For each $f \in S$, let $k(f)$ be the number such that $f(t) = k(f) + t$ for all sufficiently large t and let $m(f) = \min\{t : f(t) = k(f) + t\}$. For $n \in \mathbb{N}$, define $h_n \in S$ by

$$h_n(t) = \begin{cases} t & \text{if } t < n, \\ 1 + t & \text{if } t \geq n. \end{cases}$$

Let $A = \{h_n : n \in \mathbb{N}\}$ and let $p \in A^*$. To see that p is right cancelable in βS , let q and r be distinct members of βS and suppose that $q \circ p = r \circ p$. (As usual, we denote the operation in βS by the same symbol used to denote the operation on S .) Pick disjoint subsets B and C of S such that $B \in q$ and $C \in r$. By [4, Theorem 4.15], $\{f \circ h_n : f \in B \text{ and } n > m(f)\} \in q \circ p$ and $\{g \circ h_n : g \in C \text{ and } n > m(g)\} \in q \circ p$, so the intersection is nonempty. Pick $f \in B$, $g \in C$, $n > m(f)$, and $l > m(g)$ such that $f \circ h_n = g \circ h_l$. Now if $t \geq \max\{n, l\}$, then $f(h_n(t)) = f(t + 1) = k(f) + t + 1$ and $g(h_l(t)) = g(t + 1) = k(g) + t + 1$, so $k(f) = k(g)$. We claim next that $n = l$, so suppose instead without loss of generality that $n < l$. Then $f(h_n(n)) = f(n + 1) = k(f) + n + 1$ while $g(h_l(n)) = g(n) \leq k(g) + n$, a contradiction.

Now, if $t < n$, then $f(t) = f(h_n(t)) = g(h_n(t)) = g(t)$. And, if $t > n$, then $f(t) = f(h_n(t - 1)) = g(h_n(t - 1)) = g(t)$. Finally, $f(n) = k(f) + n = k(g) + n = g(n)$, so $f = g$, a contradiction. ■

For the rest of this section we will fix a countable left cancellative semigroup S with identity 1 and $p \in S^*$ which is right cancelable in βS . We begin a construction in S based on a similar construction for countable groups in [4, Section 8.5]. We assume that we have enumerated S as $\langle s_n \rangle_{n=1}^\infty$ with $s_1 = 1$. For $s, t \in S$ we write $s \prec t$ if and only if $s = s_i$, $t = s_j$, and $i < j$.

Of course, $s \preceq t$ means $s \prec t$ or $s = t$. When we write $\text{FP}(\{t \in S : t \preceq b\})$ we mean all products of the form $t_1 \cdots t_m$ such that $t_1 \prec \cdots \prec t_m \preceq b$.

DEFINITION 4.2. M is a *strongly discrete function* for p if $M : S \rightarrow p$, $1 \notin M(x)$ for any x , and whenever x and y are distinct members of S , then $xM(x) \cap yM(y) = \emptyset$.

Note that by Lemma 2.6, p is strongly discrete, so there exists a strongly discrete function for p . We do not fix a particular strongly discrete function for p because we will use two such functions in the proof of Theorem 4.12.

DEFINITION 4.3. Let M be a strongly discrete function for p , let $k \in \mathbb{N}$, and let $\langle b_1, \dots, b_k \rangle \in M(1)^k$. Then $x \in S$ is the *M -product* of $\langle b_1, \dots, b_k \rangle$ if

- (i) $x = b_1 \cdots b_k$,
- (ii) if $1 \leq i < j \leq k$, then $b_i \prec b_j$, and
- (iii) if $i \in \{2, \dots, k\}$ and $s \in \text{FP}(\{t \in S : t \preceq b_{i-1}\})$, then $b_i \in M(s)$ and $sb_i \neq 1$.

We say that x is an *M -product* provided there is some $\langle b_1, \dots, b_k \rangle$ such that x is the M -product of $\langle b_1, \dots, b_k \rangle$. We note that we require an M -product to satisfy more stringent conditions than in [8, §4.3] wherein the only requirement on $\langle b_1, \dots, b_k \rangle$ was that for $t \in \{2, \dots, k\}$, $b_t \in M(b_1 \cdots b_{t-1})$. We need the stronger requirements for our proofs.

DEFINITION 4.4. Let M be a strongly discrete function for p . Then $T_M = \{x : x \text{ is an } M\text{-product}\}$.

LEMMA 4.5. Let M be a strongly discrete function for p . Assume that $s \in S \setminus \{1\}$, x is the M -product of $\langle b_1, \dots, b_m \rangle$, y is the M -product of $\langle c_1, \dots, c_n \rangle$, $sb_1 \neq 1$, $s \prec b_1$, $b_1 \in M(s)$, and $sx = y$. Then $n > m$ and, if $k = n - m$, then $s = c_1 \cdots c_k$ and, for $i \in \{1, \dots, m\}$, $b_i = c_{k+i}$.

Proof. Suppose the conclusion fails and pick a counterexample with $n + m$ as small as possible. If $m = n = 1$, then $sb_1 = c_1$, so $sM(s) \cap 1M(1) \neq \emptyset$ and thus $s = 1$, a contradiction.

Suppose next that $m = 2$ and $n = 1$. Let $u = sb_1$. Then

$$u \in \text{FP}(\{t \in S : t \preceq b_1\}),$$

so $b_2 \in M(u)$, and thus $uM(u) \cap 1M(1) \neq \emptyset$ and so $sb_1 = 1$, a contradiction.

Now assume that $m > 2$ and $n = 1$. Let $u = sb_1 \cdots b_{m-1}$. Then

$$uM(u) \cap 1M(1) \neq \emptyset,$$

so $u = 1$. But $sb_1 \cdots b_{m-2} \in \text{FP}(\{t \in S : t \preceq b_{m-2}\})$, so $(sb_1 \cdots b_{m-2})b_{m-1} \neq 1$.

We thus have $n > 1$. If $m = 1$, let $v = c_1 \cdots c_{n-1}$. Then

$$sM(s) \cap vM(v) \neq \emptyset,$$

so $s = v$, and by left cancellation, $b_1 = c_n$. Thus with $k = n - 1$ the conclusion of the lemma holds.

Finally assume that $m > 1$ and $n > 1$. Let $u = sb_1 \cdots b_{m-1}$ and $v = c_1 \cdots c_{n-1}$. Then $uM(u) \cap vM(v) \neq \emptyset$, so $u = v$. Let $k = (n - 1) - (m - 1) = n - m$. By the minimality of $n + m$, we have $s = c_1 \cdots c_k$ and, for $i \in \{1, \dots, m - 1\}$, $b_i = c_{k+i}$. By left cancellation, $b_m = c_n = c_{k+m}$. ■

LEMMA 4.6. *Let M be a strongly discrete function for p . Then M -products are unique.*

Proof. Assume that x is the M -product of $\langle d_1, \dots, d_m \rangle$ and x is the M -product of $\langle c_1, \dots, c_n \rangle$. We need to show that $m = n$ and, for $i \in \{1, \dots, n\}$, $d_i = c_i$. We may assume without loss of generality that $m \leq n$. Suppose first that $m = 1$ and $n > 1$ and let $v = c_1 \cdots c_{n-1}$. Then $d_1 \in 1M(1) \cap vM(v)$ so $v = 1$. If $n = 2$, then $v = c_1 \neq 1$ because $c_1 \in M(1)$. If $n > 2$, then $v \neq 1$ because $(c_1 \cdots c_{n-2})c_{n-1} \neq 1$.

Thus we have $m > 1$. Let $s = d_1$ and, for $i \in \{1, \dots, m - 1\}$, let $b_i = d_{1+i}$. Then by Lemma 4.5, if $k = n - (m - 1)$, we have $d_1 = c_1 \cdots c_k$ and, for $i \in \{1, \dots, m - 1\}$, $b_i = c_{k+i}$. If $k = 1$, this is the conclusion we are after, so suppose $k > 1$. Let $v = c_1 \cdots c_{k-1}$. Then $d_1 \in 1M(1) \cap vM(v)$, so $v = 1$. But this yields a contradiction just as in the previous paragraph. ■

DEFINITION 4.7. Let M be a strongly discrete function for p . Define $h_M : T_M \rightarrow \mathbb{N}$ and $\varphi_M : T_M \rightarrow \mathbb{N}$ as follows:

- (a) If x is the M -product of $\langle b_1, \dots, b_m \rangle$ then $h_M(x) = m$.
- (b) If x is the M -product of $\langle b_1, \dots, b_m \rangle$ and, for $j \in \{1, \dots, m\}$, $b_j = s_{t(j)}$, then $\varphi_M(x) = \sum_{j=1}^m 2^{t(j)}$.

By Lemma 4.6, h_M and φ_M are well defined. We denote by \widetilde{h}_M and $\widetilde{\varphi}_M$ the continuous extensions of these functions taking $\overline{T_M}$ to $\beta\mathbb{N}$.

DEFINITION 4.8. Let M be a strongly discrete function for p and let $n \in \mathbb{N}$.

- (a) $T_{M,n} = \{x : x \text{ is the } M\text{-product of } \langle b_1, \dots, b_m \rangle \text{ and for all } s \in \text{FP}(\langle s_i \rangle_{i=1}^n), s \prec b_1, sb_1 \neq 1, \text{ and } b_1 \in M(s)\}$.
- (b) $T_{M,\infty} = \bigcap_{n=1}^\infty \overline{T_{M,n}}$.

LEMMA 4.9. *Let M be a strongly discrete function for p . Then $T_{M,\infty}$ is a compact subsemigroup of S^* , $p \in T_{M,\infty}$, the restriction of \widetilde{h}_M to $T_{M,\infty}$ is a homomorphism with $\widetilde{h}_M(p) = 1$, and the restriction of $\widetilde{\varphi}_M$ to $T_{M,\infty}$ is a homomorphism.*

Proof. Since $M(1) \subseteq T_M$ and h_M is constantly equal to 1 on $M(1)$, we have $\widetilde{h}_M(p) = 1$. To see that $p \in T_{M,\infty}$ (and thus that $T_{M,\infty} \neq \emptyset$), let $n \in \mathbb{N}$. Then $\bigcap \{M(s) : s \in \text{FP}(\langle s_i \rangle_{i=1}^n)\} \in p$ and for each $s \in \text{FP}(\langle s_i \rangle_{i=1}^n)$,

$\{t \in S : t \preceq s\}$ is finite and $\{b \in S : sb = 1\}$ has at most one member. The remainder is a member of p which is contained in $T_{M,n}$.

To see that $T_{M,\infty}$ is a subsemigroup of S^* , the restriction of \widetilde{h}_M to $T_{M,\infty}$ is a homomorphism, and the restriction of $\widetilde{\varphi}_M$ to $T_{M,\infty}$ is a homomorphism, it suffices by [4, Theorems 4.20 and 4.21] to let $n \in \mathbb{N}$, let $x \in T_{M,n}$, and show that there is some $m \in \mathbb{N}$ such that for all $y \in T_{M,m}$, $xy \in T_{M,n}$, $h_M(xy) = h_M(x) + h_M(y)$, and $\varphi_M(xy) = \varphi_M(x) + \varphi_M(y)$.

Let $n \in \mathbb{N}$, let $x \in T_{M,n}$, and pick $\langle b_1, \dots, b_k \rangle \in M(1)^k$ such that x is the M -product of $\langle b_1, \dots, b_k \rangle$ and, for all $s \in FP(\langle s_i \rangle_{i=1}^k)$, we have $s \prec b_1$, $sb_1 \neq 1$, and $b_1 \in M(s)$. Pick $m \in \mathbb{N}$ such that $b_k = s_m$. Let $y \in T_{M,m}$ and pick $\langle c_1, \dots, c_l \rangle \in M(1)^l$ such that y is the M -product of $\langle c_1, \dots, c_l \rangle$ and, for all $s \in FP(\langle s_i \rangle_{i=1}^l)$, we have $s \prec c_1$, $sc_1 \neq 1$, and $c_1 \in M(s)$. Then xy is the M -product of $\langle b_1, \dots, b_k, c_1, \dots, c_l \rangle$, $xy \in T_{M,n}$, $h_M(xy) = h_M(x) + h_M(y)$, and $\varphi_M(xy) = \varphi_M(x) + \varphi_M(y)$. ■

For the statement of the following theorem, we remind the reader of our standing hypothesis about S and p .

THEOREM 4.10. *Let S be a countable left cancellative semigroup with identity and let $p \in S^*$ be right cancelable in βS . There is some $q \in \{2^n : n \in \mathbb{N}\}^*$ such that $C_q \subseteq \beta\mathbb{N}$ is topologically isomorphic to C_p .*

Proof. Since p is strongly discrete, pick a strongly discrete function M for p . By Lemma 4.6, φ_M is injective on T_M , so $\widetilde{\varphi}_M$ is injective on $T_{M,\infty}$. By Lemma 4.9, $C_p \subseteq T_{M,\infty}$ and $\widetilde{\varphi}_M$ is a homomorphism on C_p . Let $q = \widetilde{\varphi}_M(p)$. ■

LEMMA 4.11. *Let M be a strongly discrete function for p . If $x \in \beta S \setminus (T_{M,\infty} \cup \{1\})$, then $xT_{M,\infty} \cap T_{M,\infty} = \emptyset$.*

Proof. Let $x \in \beta S \setminus \{1\}$ and assume that $xT_{M,\infty} \cap T_{M,\infty} \neq \emptyset$. We shall show that $x \in T_{M,\infty}$. Pick $y \in T_{M,\infty}$ such that $xy \in T_{M,\infty}$. Let $n \in \mathbb{N}$. We shall show that $T_{M,n} \in x$.

Now $\{s \in S \setminus \{1\} : s^{-1}T_{M,n} \in y\} \in x$ so it suffices that

$$\{s \in S \setminus \{1\} : s^{-1}T_{M,n} \in y\} \subseteq T_{M,n}.$$

So let $s \in S \setminus \{1\}$ be such that $s^{-1}T_{M,n} \in y$. Pick $r \in \mathbb{N}$ such that $s = s_r$. Then $T_{M,r} \in y$, so pick $z \in s^{-1}T_{M,n} \cap T_{M,r}$. Pick $l \in \mathbb{N}$ and $\langle c_1, \dots, c_l \rangle \in M(1)^l$ such that $sb_1 \cdots b_m$ is the M -product of $\langle c_1, \dots, c_l \rangle$ and for all $t \in FP(\langle s_i \rangle_{i=1}^l)$, $t \prec c_1$, $tc_1 \neq 1$, and $c_1 \in M(t)$. By Lemma 4.5, $s = c_1 \cdots c_k$ for some $k \in \mathbb{N}$, so that $s \in T_{M,n}$ as required. ■

Again, we remind the reader of our standing hypotheses.

THEOREM 4.12. *Let S be a countable left cancellative semigroup with identity and let $p \in S^*$ be right cancelable in βS . Then $C_p \cap K(\beta S) = \emptyset$.*

Proof. Suppose instead that we have some $q \in C_p \cap K(\beta S)$. Pick a strongly discrete function $N : S \rightarrow p$. Now $S^* \cap \bigcap_{s \in S} \overline{N(s)}$ is a nonempty G_δ , so by [4, Theorem 3.36] pick $r \neq p$ in $S^* \cap \bigcap_{s \in S} \overline{N(s)}$ and pick $B \in r \setminus p$. For $s \in S$, let $M(s) = N(s) \setminus B$. Then M is a strongly discrete function for p .

Let R be the minimal right ideal of βS with $q \in R$ and pick a minimal left ideal L of βS such that $L \subseteq \beta Sr$. Let u be the identity of $L \cap R$. By [4, Corollary 4.33], $L \subseteq S^*$, so $u \in S^*$. Then $uq = q$ and $q \in T_{M,\infty}$ by Lemma 4.9. Thus, by Lemma 4.11, $u \in T_{M,\infty}$. In particular $T_M \in u$.

Now also $u \in \beta Sr$ and since r is strongly discrete (via the function N) and u is an idempotent, $r \neq u$, so there is some $s \in S \setminus \{1\}$ such that $T_M \in sr$. Pick $t \in s^{-1}T_M \cap B \cap N(s)$. Then $st \in T_M$, so pick $k \in \mathbb{N}$ and $\langle b_1, \dots, b_k \rangle \in M(1)^k$ such that st is the M -product of $\langle b_1, \dots, b_k \rangle$. If $k = 1$, then $st \in sN(s) \cap 1N(1)$, so $s = 1$, a contradiction. Thus $k > 1$. Let $v = b_1 \cdots b_{k-1}$. Then $st \in sN(s) \cap vN(v)$, so $s = v$ and by left cancellation, $t = b_k$. But then $t \in B$ and $b_k \in M(1) = N(1) \setminus B$, a contradiction. ■

5. Long $<_L$ -chains. Recall that for idempotents e and f in a semi-group, $e <_L f$ if $ef = e$ and it is not the case that $fe = f$. Equivalently, $e <_L f$ if and only if Se is a proper subset of Sf . Using a result from [6] we show here that one can get chains as long as possible below any nonminimal idempotent in $\beta\mathbb{N}$, provided that one weakens the strictly decreasing requirement at limit ordinals to the requirement that if $\sigma < \tau$, then $p_\tau <_L p_\sigma$. We will use the following extension of Lemma 2.3.

LEMMA 5.1. *Let S and T be compact Hausdorff right topological semi-groups, let $h : S \rightarrow T$ be a continuous surjective homomorphism, and let λ be an ordinal. Assume that $\langle u_\sigma \rangle_{\sigma < \lambda}$ is a chain of idempotents in T such that $u_\tau <_L u_\sigma$ whenever $\sigma < \tau < \lambda$ and $u_{\sigma+1} < u_\sigma$ whenever $\sigma + 1 < \lambda$. Then there is a chain $\langle p_\sigma \rangle_{\sigma < \lambda}$ of idempotents in S such that $p_\tau <_L p_\sigma$ whenever $\sigma < \tau < \lambda$, $p_{\sigma+1} < p_\sigma$ whenever $\sigma + 1 < \lambda$, and $h(p_\sigma) = u_\sigma$ for every $\sigma < \lambda$.*

Proof. Pick by Lemma 2.3 an idempotent $p_0 \in S$ such that $h(p_0) = u_0$. Assume that $0 < \gamma < \lambda$ and we have chosen $\langle p_\sigma \rangle_{\sigma < \gamma}$ as required.

CASE 1: $\gamma = \delta + 1$ for some δ . Pick by Lemma 2.3 an idempotent $p_\gamma \in S$ such that $h(p_\gamma) = u_\gamma$ and $p_\gamma < p_\delta$.

CASE 2: γ is a limit ordinal. We claim that for each $\sigma < \gamma$,

$$(S + p_\sigma) \cap h^{-1}[\{u_\gamma\}] \neq \emptyset.$$

To see this, pick $x \in h^{-1}[\{u_\gamma\}]$. Then $x + p_\sigma \in S + p_\sigma$ and $h(x + p_\sigma) = u_\gamma + u_\sigma = u_\gamma$. Consequently, $h^{-1}[\{u_\gamma\}] \cap \bigcap_{\sigma < \gamma} (S + p_\sigma)$ is a compact right topological semigroup, so pick $p_\gamma \in h^{-1}[\{u_\gamma\}] \cap \bigcap_{\sigma < \gamma} (S + p_\sigma)$. ■

THEOREM 5.2. *Let $\lambda > 0$ be an ordinal.*

- (1) *If there is a chain $\langle p_\sigma \rangle_{\sigma < \lambda}$ of idempotents in $\beta\mathbb{N}$ such that $p_\tau <_L p_\sigma$ whenever $\sigma < \tau < \lambda$, then $|\lambda| \leq \mathfrak{c}$.*
- (2) *If $|\lambda| \leq \mathfrak{c}$, then there is a chain $\langle p_\sigma \rangle_{\sigma < \lambda}$ of idempotents in $\beta\mathbb{N}$ such that $p_\tau <_L p_\sigma$ whenever $\sigma < \tau < \lambda$ and $p_{\sigma+1} < p_\sigma$ whenever $\sigma + 1 < \lambda$.*

Proof. See [6, Corollary 3.18]. ■

THEOREM 5.3. *Let q be a nonminimal idempotent in $\beta\mathbb{N}$ and let $\lambda > 0$ be an ordinal such that $|\lambda| \leq \mathfrak{c}$. Then there is a chain $\langle q_\sigma \rangle_{\sigma < \lambda}$ of idempotents in $\beta\mathbb{N}$ such that $q_0 = q$, $q_\tau <_L q_\sigma$ whenever $\sigma < \tau < \lambda$, and $q_{\sigma+1} < q_\sigma$ whenever $\sigma + 1 < \lambda$.*

Proof. Pick by Theorem 5.2 a chain $\langle u_\sigma \rangle_{\sigma < \lambda}$ of idempotents in $\beta\mathbb{N}$ such that $u_\tau <_L u_\sigma$ whenever $\sigma < \tau < \lambda$ and $u_{\sigma+1} < u_\sigma$ whenever $\sigma + 1 < \lambda$. By [4, Theorem 6.56] pick $r \in \beta\mathbb{N}$ such that $r + q$ is right cancelable in $\beta\mathbb{N}$ and let $p = r + q$. By Lemma 5.1 pick a chain $\langle v_\sigma \rangle_{\sigma < \lambda}$ of idempotents in C_p such that $v_\tau <_L v_\sigma$ whenever $\sigma < \tau < \lambda$, $v_{\sigma+1} < v_\sigma$ whenever $\sigma + 1 < \lambda$, and $h_p(v_\sigma) = u_\sigma$ for every $\sigma < \lambda$. For $\sigma < \lambda$, let $q_\sigma = q + v_\sigma$. Exactly as in the proof of Lemma 3.4 we see that $q_\tau <_L q_\sigma$ whenever $\sigma < \tau < \lambda$ and $q_{\sigma+1} < q_\sigma$ whenever $\sigma + 1 < \lambda$. If $q_0 = q$, then we are done. Otherwise replace q_0 by q . ■

Clearly, Theorem 5.2 implies that $\beta\mathbb{N}$ contains many infinite decreasing chains of principal left ideals. Whether $\beta\mathbb{N}$ contains any infinite increasing chain of semiprincipal left ideals, has been a tantalising open problem for several decades. (See the notes to Chapter 6 of [4] for a discussion of the history of this problem.) It was shown in [2, Corollary 1.8] that $\beta\mathbb{Z}$ contains strictly increasing chains of principal right ideals. (In [2] these were called left ideals because $\beta\mathbb{Z}$ was taken to be left topological rather than right topological.) We show here that one can get such increasing chains generated by idempotents. Recall that the statement that $p <_R q$ means that $p \leq_R q$ and it is not true that $q \leq_R p$, or equivalently that $p + \beta\mathbb{N} \not\subseteq q + \beta\mathbb{N}$.

THEOREM 5.4. *There is an infinite sequence $\langle p_n \rangle_{n < \omega}$ of idempotents in $\beta\mathbb{N}$ such that $p_n <_R p_{n+1}$ for every $n \in \omega$.*

Proof. We choose an infinite sequence $\langle E_n \rangle_{n < \omega}$ of infinite subsets of ω such that for each n , $E_{n+1} \subseteq E_n$ and $E_n \setminus E_{n+1}$ is infinite. Given $x \in \mathbb{N}$, let $\text{supp}(x)$ be the finite subset of ω such that $x = \sum_{t \in \text{supp}(x)} 2^t$. For each $n \in \omega$ we let

$$H_n = \bigcap_{k=0}^{\infty} \text{cl}_{\beta\mathbb{N}} \{x \in \mathbb{N} : \text{supp}(x) \subseteq E_n \text{ and } \min \text{supp}(x) > k\}.$$

Using [4, Theorem 4.20] we see that each H_n is a compact subsemigroup of \mathbb{H} and it is routine to verify that $H_n \setminus H_{n+1}$ is an ideal of H_n . For each

$n \in \mathbb{N}$ define $\mu_n : \omega \rightarrow \omega$ by $\mu_n(x) = \sum_{t \in E_n \cap \text{supp}(x)} 2^t$, where $\sum_{t \in \emptyset} 2^t = 0$. Using [4, Theorem 4.21] we see that for each $n \in \mathbb{N}$, the restriction of μ_n to \mathbb{H} is a homomorphism. Note also that for all $x \in \omega$ and all $m \leq n$, $\mu_m(\mu_n(x)) = \mu_n(x) = \mu_n(\mu_m(x))$, so $\widetilde{\mu}_m \circ \widetilde{\mu}_n = \widetilde{\mu}_n = \widetilde{\mu}_n \circ \widetilde{\mu}_m$.

Next we claim that if p is a minimal idempotent in H_0 and $m \in \mathbb{N}$, then $\widetilde{\mu}_m(p)$ is a minimal idempotent in H_m . To see this, let $A = \{x \in \mathbb{N} : \text{supp}(x) \cap E_m \neq \emptyset\}$ and let $B = \overline{A} \cap H_0$. Then B is an ideal of H_0 and $\widetilde{\mu}_m[B] = H_m$. Since B is an ideal of H_0 , $K(H_0) \subseteq B$ and therefore by [4, Theorem 1.65], $K(B) = K(H_0)$. Then by [4, Exercise 1.7.3], $\widetilde{\mu}_m[K(H_0)] = K(H_m)$ and so $\widetilde{\mu}_m(p) \in K(H_m)$ as required.

Now pick a minimal idempotent q of H_0 . Let $r_0 = q$ and inductively for $n \in \mathbb{N}$ let $r_n = \widetilde{\mu}_n(q) + r_{n-1}$. Then $r_n = \widetilde{\mu}_n(q) + \widetilde{\mu}_{n-1}(q) + \cdots + \widetilde{\mu}_1(q) + q$, so if $1 \leq m < n$, then

$$\begin{aligned} \widetilde{\mu}_m(r_n) &= \widetilde{\mu}_m(\widetilde{\mu}_n(q)) + \widetilde{\mu}_m(\widetilde{\mu}_{n-1}(q)) + \cdots + \widetilde{\mu}_m(\widetilde{\mu}_1(q)) + \widetilde{\mu}_m(q) \\ &= \widetilde{\mu}_n(q) + \widetilde{\mu}_{n-1}(q) + \cdots + \widetilde{\mu}_m(q) \end{aligned}$$

since $\widetilde{\mu}_m(\widetilde{\mu}_t(q)) = \widetilde{\mu}_m(q)$ if $t \leq m$ and $\widetilde{\mu}_m(q)$ is an idempotent.

Consequently if $m < n$ we have $\widetilde{\mu}_m(r_n) + r_{m-1} = r_n$. Now if $m < t \leq n$, then $\widetilde{\mu}_t(q) \in H_t \subseteq H_m$, so $\widetilde{\mu}_m(r_n) \in H_m + \widetilde{\mu}_m(q)$, which is a (compact) minimal left ideal of H_m . Let x be a cluster point of the sequence $\langle r_n \rangle_{n=1}^\infty$. Then for each $m \in \mathbb{N}$, $\widetilde{\mu}_m(x) \in H_m + \widetilde{\mu}_m(q) \subseteq K(H_m)$ and $\widetilde{\mu}_m(x) + r_{m-1} = x$. We then have $x \in \bigcap_{m=1}^\infty (\widetilde{\mu}_m(x) + H_0)$, so, being nonempty, $\bigcap_{m=1}^\infty (\widetilde{\mu}_m(x) + H_0)$ is a right ideal of H_0 ; hence by [4, Theorem 2.7] we may pick an idempotent $p_0 \in \bigcap_{m=1}^\infty (\widetilde{\mu}_m(x) + H_0)$ which is minimal in H_0 . Then for each $m \in \mathbb{N}$, $p_0 \leq_R \widetilde{\mu}_m(x)$ and so $\widetilde{\mu}_m(p_0) \leq_R \widetilde{\mu}_m(x)$. We have seen that $\widetilde{\mu}_m(p_0) \in K(H_m)$ and $\widetilde{\mu}_m(x) \in K(H_m)$, so $\widetilde{\mu}_m(p_0) + H_m = \widetilde{\mu}_m(x) + H_m$ and therefore $\widetilde{\mu}_m(x) \leq_R \widetilde{\mu}_m(p_0)$ and thus $p_0 \leq_R \widetilde{\mu}_m(p_0)$. Thus if $k < m$, $\widetilde{\mu}_k(p_0) \leq_R \widetilde{\mu}_k(\widetilde{\mu}_m(p_0)) = \widetilde{\mu}_m(p_0)$. For each $n \in \mathbb{N}$, let $p_n = \widetilde{\mu}_n(p_0)$. We then see that for each $n \in \omega$, $p_n \leq_R p_{n+1}$. To complete the proof we need to show that it is not the case that $p_{n+1} \leq_R p_n$. We have noted that $H_n \setminus H_{n+1}$ is an ideal of H_n , so $K(H_n) \cap H_{n+1} = \emptyset$. If we had $p_{n+1} = p_n + p_{n+1}$, we would have $p_{n+1} \in K(H_n)$. ■

We conclude by listing some open questions. In [9, Corollary 5] it was shown that if G is a countable discrete group, then Martin’s Axiom implies that there is an idempotent in $p \in G^*$ which is minimal and \leq_L -maximal. (That is, there does not exist q such that $p <_L q$.) In particular, p is minimal and maximal with respect to $<$.

QUESTIONS 5.5.

- (1) Is there a strictly decreasing chain $\langle q_\sigma \rangle_{\sigma < \omega_1}$ of idempotents in $\beta\mathbb{N}$?
- (2) Can one show in ZFC that there is a minimal idempotent q in $\beta\mathbb{N}$ which is also maximal?

- (3) Can it be shown in ZFC that maximal idempotents in $\beta\mathbb{N}$ exist?
 (4) Are all semigroups of the form C_q , where $q \in \{2^n : n \in \mathbb{N}\}^*$, isomorphic?

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