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A dimensional property of Cartesian product

by

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Abstract. We show that the Cartesian product of three hereditarily infinite-dimensional compact metric spaces is never hereditarily infinite-dimensional. It is quite surprising that the proof of this fact (and this is the only proof known to the author) essentially relies on algebraic topology.

1. Introduction. Throughout this paper we assume that maps are continuous and spaces are separable metrizable. We recall that a compactum means a compact metric space. By the dimension $\dim X$ of a space X we mean the covering dimension.

An infinite-dimensional compactum X is said to be *hereditarily infinite-dimensional* if every (non-empty) closed subset of X is either 0-dimensional or infinite-dimensional. Hereditarily infinite-dimensional compacta were first constructed by Henderson [9]; for related results and simplified constructions see [17], [18], [15], [11], [12]. The main result of this paper is:

THEOREM 1.1. *Let $n > 0$ be an integer and X_i, $1 \leq i \leq n+2$, hereditarily infinite-dimensional compacta. Then the product $Z = \prod_{i=1}^{n+2} X_i$ contains an n-dimensional closed subset. In particular, the product of three hereditarily infinite-dimensional compacta is never hereditarily infinite-dimensional.*

Let us note that in general the compactum Z in Theorem 1.1 does not contain finite-dimensional subspaces of arbitrarily large dimension. Indeed, consider the Dydak–Walsh compactum X [7] having the following properties: $\dim X = \infty$, $\dim_{\mathbb{Z}} X = 2$ and $\dim_{\mathbb{Z}} X^n = n + 1$ for every positive integer n.

We recall that for an abelian group G the *cohomological dimension* $\dim_G X$ of a space X is the smallest integer n such that the Čech cohomology $H^{n+1}(X, A; G)$ vanishes for every closed subset A of X. Clearly

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$\dim_G X \leq \dim X$ for every abelian group G. By the classical result of Alexandroff, $\dim X = \dim_{\mathbb{Z}} X$ if X is finite-dimensional. Alexandroff's result was extended by Ancel [1] who showed that $\dim X = \dim_{\mathbb{Z}} X$ if X is a compact C-space. We recall that a space X is a *C-space* if for any infinite sequence $\mathcal{U}_i$ of open covers X there is an open cover $\mathcal{V}$ of X such that $\mathcal{V}$ splits into the union $\mathcal{V} = \bigcup_i \mathcal{V}_i$ of families $\mathcal{V}_i$ of disjoint sets such that $\mathcal{V}_i$ refines $\mathcal{U}_i$.

Thus the Dydak–Walsh compactum X is not a C-space. R. Pol [16] (see also [12]) showed that a compactum which is not a C-space contains a hereditarily infinite-dimensional closed subset. Hence, replacing X by its hereditarily infinite-dimensional closed subset, we may assume that X is hereditarily infinite-dimensional. Since $\dim_{\mathbb{Z}} X^{n+2} = n+3$, we deduce from Alexandroff's theorem that X^{n+2} does not contain finite dimensional subsets of dimension $> n+3$. Moreover, X^{n+2} does not contain compact subsets of dimension $n+3$. Indeed, if F is a finite dimensional closed subset of X^{n+2} then, since X is hereditarily infinite-dimensional, the projection $p : F \to X^{n+1}$ is 0-dimensional. By a result of Dranishnikov and Uspenskij [5] a 0-dimensional map of compacta cannot lower cohomological dimensions and hence $\dim F = \dim_{\mathbb{Z}} F \leq \dim_{\mathbb{Z}} X^{n+1} = n+2$.

This example together with Theorem 1.1 suggest

Problem 1.2. *Does the compactum Z in Theorem 1.1 always contain a closed subset of dimension $n+1$? of dimension $n+2$? a subset of dimension $n+1$? of dimensionn $+2$? of dimension $n+3$?*

Note that Theorem 1.1 implies that there are two hereditarily infinite-dimensional compacta whose product is not hereditarily infinite-dimensional. Indeed, let X_1, X_2 and X_3 be hereditarily infinite-dimensional compacta. If $X_1 \times X_2$ is hereditarily infinite-dimensional then, by Theorem 1.1, $(X_1 \times X_2) \times X_3$ is not hereditarily infinite dimensional. This observation motivates

Problem 1.3. *Do there exist two hereditarily infinite-dimensional compacta whose product is also hereditarily infinite-dimensional? Does there exist a hereditarily infinite-dimensional compactum whose square is hereditarily infinite-dimensional?*

It is quite surprising that the proof of Theorem 1.1 essentially relies on algebraic topology. It would be interesting to find an elementary direct proof of Theorem 1.1.

2. Proof of Theorem 1.1. Let us recall basic definitions and results in Extension Theory and Cohomological Dimension that will be used in the proof.

The extension dimension of a space X is said to be dominated by a CW-complex K, written $\text{e-dim}\, X \le K$, if every map $f : A \to K$ from a closed subset A of X extends over X. Note that the property $\text{e-dim}\, X \le K$ depends only on the homotopy type of K. The covering and cohomological dimensions can be characterized by the following extension properties: $\dim X \le n$ if and only if the extension dimension of X is dominated by the n-dimensional sphere S^n and $\dim_G X \le n$ if and only if the extension dimension of X is dominated by the Eilenberg–Mac Lane complex $K(G,n)$. The extension dimension shares many properties of covering dimension. For example: if $\text{e-dim}\, X \le K$ then for every $A \subset X$ we have $\text{e-dim}\, A \le K$, and if X is a countable union of closed subsets whose extension dimension is dominated by K then $\text{e-dim}\, X \le K$. In the proof of Theorem 1.1 we will also use the following facts.

THEOREM 2.1 ([14]). *Let K be a countable CW-complex and A a subspace of a compactum X such that* $\text{e-dim}\, A \le K$. *Then there is a G_δ-set $A' \subset X$ such that $A \subset A'$ and* $\text{e-dim}\, A' \le K$.

THEOREM 2.2 ([3]). *Let K and L be countable CW-complexes and X a compactum such that* $\text{e-dim}\, X \le K * L$. *Then X decomposes into subspaces $X = A \cup B$ such that* $\text{e-dim}\, A \le K$ *and* $\text{e-dim}\, B \le L$.

THEOREM 2.3 ([13]). *Let $f : X \to Y$ be a map of compacta and let K and L be countable CW-complexes such that* $\text{e-dim}\, Y \le K$ *and* $\text{e-dim}\, f^{-1}(y) \le L$ *for every $y \in Y$. Then* $\text{e-dim}\, X \le K * L$. *In particular, if for a compactum Z we have* $\text{e-dim}\, Z \le L$ *then* $\text{e-dim}\, Y \times Z \le K * L$.

THEOREM 2.4 ([2], [4]). *Assume that for a compactum X and a CW-complex K we have* $\text{e-dim}\, X \le K$. *Then* $\dim_{H_n(K)} X \le n$ *for every $n \ge 1$.*

By $\mathbb{Z}_p$ we denote the p-cyclic group and by $\mathbb{Z}_{p^\infty} = \operatorname{dirlim} \mathbb{Z}_{p^k}$ the p-adic circle.

THEOREM 2.5 ([10], [4]). *Let p be a prime and X and Y compacta. Then* $\dim_{\mathbb{Z}_p} X \times Y = \dim_{\mathbb{Z}_p} X + \dim_{\mathbb{Z}_p} Y$.

THEOREM 2.6 ([6]). *Let p be a prime and $X = A \cup B$ a decomposition of a compactum X. Then* $\dim_{\mathbb{Z}_p} X \le \dim_{\mathbb{Z}_p} A + \dim_{\mathbb{Z}_p} B + 1$.

For an abelian group G we always assume that a Moore space $M(G,n)$ of type (G,n) is a CW-complex and $M(G,n)$ is simply connected if $n > 1$. Note that $M(G,n)$ is defined uniquely (up to homotopy equivalence) for $n > 1$ [8]. Recall that for CW-complexes K and L the join $K * L$ is homotopy equivalent to the suspension $\Sigma(K \wedge L) = S^0 * (K \wedge L)$ and $K * L$ is simply connected if at least one of the complexes K and L is connected. Then it follows from the Künneth formula that for distinct primes p and q:

(i) $M(\mathbb{Z}_p,1) * M(\mathbb{Z}_q,1)$ is contractible;
(ii) $M(\mathbb{Z}_{p^\infty},1) * M(\mathbb{Z}_q,1)$ is contractible;
(iii) $M(\mathbb{Z}_{p^\infty},1) * M(\mathbb{Z}_{q^\infty},1)$ is contractible;
(iv) $\Sigma^n M(\mathbb{Z}_p,1) = S^{n-1} * M(\mathbb{Z}_p,1)$ is a Moore space $M(\mathbb{Z}_p,n+1)$;
(v) $M(\mathbb{Z}_{p^\infty},1) * M(\mathbb{Z}_p,n)$ is a Moore space $M(\mathbb{Z}_p,n+3)$.

We say that a compactum X is *reducible* at a prime p if there is a non-zero-dimensional closed subset F of X with

$$\text{e-dim}\, F \le M(\mathbb{Z}_p,1) \quad \text{and} \quad \text{e-dim}\, F \le M(\mathbb{Z}_{p^\infty},1),$$

and we say that X is *irreducible* at p otherwise.

PROPOSITION 2.7. *Let X be a hereditarily infinite-dimensional compactum. Then X is irreducible at at most one prime.*

Proof. Aiming at a contradiction assume X is irreducible at two distinct primes p and q. By (i) we have $\text{e-dim}\, X \le M(\mathbb{Z}_p,1)*M(\mathbb{Z}_q,1)$ and hence, by Theorem 2.2, the compactum X decomposes as $X = A\cup B$ with $\text{e-dim}\, A \le M(\mathbb{Z}_p,1)$ and $\text{e-dim}\, B \le M(\mathbb{Z}_q,1)$ and, by Theorem 2.1, we may assume that B is G_δ and A is σ-compact.

If $\dim A > 0$ then A contains a non-zero-dimensional compactum $F \subset A$ and clearly F is hereditarily infinite-dimensional and $\text{e-dim}\, F \le M(\mathbb{Z}_p,1)$.

If $\dim A \le 0$ then replacing A by a bigger 0-dimensional G_δ-subset of X we may assume that B is σ-compact. Since X is infinite-dimensional we have $\dim B > 0$ and hence B contains a non-zero-dimensional compactum $F \subset B$. Clearly F is hereditarily infinite-dimensional and $\text{e-dim}\, F \le M(\mathbb{Z}_q,1)$.

Thus without loss of generality we may assume that X contains a hereditarily infinite-dimensional compactum F with $\text{e-dim}\, F \le M(\mathbb{Z}_p,1)$. By (iii) we have $\text{e-dim}\, F \le M(\mathbb{Z}_{p^\infty},1) * M(\mathbb{Z}_{q^\infty},1)$. Then using the above reasoning we can replace F by a hereditarily infinite-dimensional closed subset of F and assume, in addition, that the extension dimension of F is dominated by at least one the complexes $M(\mathbb{Z}_{p^\infty},1)$ or $M(\mathbb{Z}_{q^\infty},1)$.

If $\text{e-dim}\, F \le M(\mathbb{Z}_{p^\infty},1)$ then X is reducible at p and we are done. If $\text{e-dim}\, F \le M(\mathbb{Z}_{q^\infty},1)$ then, by (ii), we have $\text{e-dim}\, F \le M(\mathbb{Z}_{p^\infty},1)*M(\mathbb{Z}_q,1)$ and once again by the reasoning described above one can replace F by its closed hereditarily infinite-dimensional subset with the extension dimension dominated by at least one of the complexes $M(\mathbb{Z}_{p^\infty},1)$ or $M(\mathbb{Z}_q,1)$. This implies that X is reducible at at least one of the primes p and q, and the proposition follows. ∎

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 2.7 there is a prime p such that every X_i is reducible at p. Hence for every i there is a hereditarily infinite-dimensional compactum $F_i \subset X_i$ such that $\text{e-dim}\, F_i \le M(\mathbb{Z}_p,1)$ and $\text{e-dim}\, F_i$

$\leq M(\mathbb{Z}_{p^\infty}, 1)$. Then, by Theorem 2.4, $\dim_{\mathbb{Z}_p} F_i \leq 1$ and since F_i is not 0-dimensional, we have $\dim_{\mathbb{Z}_p} F_i = 1$ and hence, by Theorem 2.5, $\dim_{\mathbb{Z}_p} F = n + 2$ for $F = F_1 \times \cdots \times F_{n+2}$.

On the other hand, by Theorem 2.3,

$$\text{e-dim}\, F \leq K = M(\mathbb{Z}_{p^\infty}, 1) * \cdots * M(\mathbb{Z}_{p^\infty}, 1) * M(\mathbb{Z}_p, 1)$$

(the join of $M(\mathbb{Z}_p, 1)$ and $n + 1$ copies of $M(\mathbb{Z}_{p^\infty}, 1)$). By (v) and (iv) we have $K = M(\mathbb{Z}_p, 3n+4) = S^{3n+2} * M(\mathbb{Z}_p, 1)$. Then, by Theorem 2.2, F splits into $F = A \cup B$ such that $\text{e-dim}\, A \leq M(\mathbb{Z}_p, 1)$ and B is finite-dimensional. In addition, we may assume by Theorem 2.1 that B is G_δ and A is σ-compact. Then, by Theorem 2.4, the property $\text{e-dim}\, A \leq M(\mathbb{Z}_p, 1)$ implies $\dim_{\mathbb{Z}_p} A \leq 1$. Again by Theorem 2.1, we can replace A by a bigger G_δ subset of F and assume that $\dim_{\mathbb{Z}_p} A \leq 1$ and B is finite-dimensional and σ-compact.

Then, by Theorem 2.6, we have

$$n + 2 = \dim_{\mathbb{Z}_p} F \leq \dim_{\mathbb{Z}_p} A + \dim_{\mathbb{Z}_p} B + 1 \leq \dim_{\mathbb{Z}_p} B + 2$$

and hence $\dim_{\mathbb{Z}_p} B \geq n$. Thus $\dim B \geq n$ and, since B is finite dimensional and σ-compact, B contains an n-dimensional compact subset. The theorem is proved. ■

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