Covering the plane with sprays

by

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Abstract. For any three noncollinear points \(c_0, c_1, c_2 \in \mathbb{R}^2\), there are sprays \(S_0, S_1, S_2\) centered at \(c_0, c_1, c_2\) that cover \(\mathbb{R}^2\). This improves the result of de la Vega in which \(c_0, c_1, c_2\) were required to be the vertices of an equilateral triangle.

Given a point \(c\) in the plane \(\mathbb{R}^2\) and a subset \(S \subseteq \mathbb{R}^2\), we say (following [2]) that \(S\) is a spray centered at \(c\) if whenever \(C \subseteq \mathbb{R}^2\) is a circle centered at \(c\), then \(S \cap C\) is finite. It was noted in [2] that if \(2^{|c_0|} \leq |\mathbb{R}_{n-2}|\), then the plane can be covered by \(n\) sprays. In fact, if \(c_0, c_1, \ldots, c_{n-1} \in \mathbb{R}^2\) are \(n\) distinct points and \(2^{|c_0|} \leq |\mathbb{R}_{n-2}|\), then the plane can be covered by \(n\) sprays centered at \(c_0, c_1, \ldots, c_{n-1}\). De la Vega [3] proved that if \(c_0, c_1, c_2 \in \mathbb{R}^2\) are distinct, collinear points, then the Continuum Hypothesis (CH) is equivalent to the existence of a covering of the plane by three sprays centered at \(c_0, c_1, c_2\). (See Theorem 7 for an extension of this result to \(n\) collinear points.) On the other hand, without using any additional set-theoretic hypothesis, he proved the very interesting result that the plane can be covered by three sprays whose centers are the vertices of an equilateral triangle. In the proof of this he used a computer algebra system (such as Maple) to show that a certain fourth degree polynomial in two variables is irreducible over \(\mathbb{C}\). This proved to be an obstacle to extending his proof to arbitrary triangles. To quote from [3]:

We suspect that the same remains true for any triangle (as long as the \(c_i\)'s don’t lie on the same line) and we have checked a couple of examples, but we have not found a reasonable way to prove it simultaneously because in one step of the proof we require a computer algebra system to check for the irreducibility of a certain polynomial.

The main purpose of this note is to confirm this suspicion by presenting a proof of the following theorem. This proof is self-contained without reliance on a computer algebra system.

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Theorem 1. If \( c_0, c_1, c_2 \in \mathbb{R}^2 \) are any three noncollinear points, then there are three sprays \( S_0, S_1, S_2 \subseteq \mathbb{R}^2 \) centered at \( c_0, c_1, c_2 \), respectively, that cover \( \mathbb{R}^2 \).

We make use of some terminology from [3]. Let \( X \) be an arbitrary set, and let \( E_0, E_1, E_2 \) be equivalence relations on \( X \). If \( x \in X \) and \( i < 3 \), then \( [x]_i \) is the equivalence class of \( E_i \) to which \( x \) belongs. A coloring \( \chi : X \to 3 = \{0, 1, 2\} \) is acceptable for \( \langle E_0, E_1, E_2 \rangle \) if whenever \( x \in X \) and \( i < 3 \), then \( [x]_i \cap \chi^{-1}(i) \) is finite. De la Vega [3] gave a characterization of those \( \langle E_0, E_1, E_2 \rangle \) for which there is an acceptable coloring.

Definition 2 (de la Vega [3 Def. 3.1]). Let \( E_0, E_1, E_2 \) be equivalence relations on a set \( X \). Then \( \langle E_0, E_1, E_2 \rangle \) is twisted if, whenever \( M, N \) are elementary substructures of the universe, \( \{i, j, k\} = \{0, 1, 2\}, a \in X, E_0, E_1, E_2 \subseteq M \cap N \) and \( N \in M \), then the set
\[
\{ x \in [a]_i : [x]_j \in M \setminus N, [x]_k \in N \setminus M \}
\]
is finite.

Since the phrase elementary substructure of the universe is not formalizable, we must reinterpret it in some formalizable way. One way would be to consider a sufficiently large yet specific \( n < \omega \), and then agree that elementary really means \( \Sigma_n \)-elementary. Alternatively, and the way that it is done in [3], is to agree that the universe is the set \( H(\theta) \) of sets of hereditary cardinality less than \( \theta \), where \( \theta = (2^{|X|} + \aleph_0) + (1) \).

Theorem 3 (de la Vega [3 Th. 3.8]). Suppose \( E_0, E_1, E_2 \) are equivalence relations on \( X \). Then the following are equivalent:

1. \( \langle E_0, E_1, E_2 \rangle \) is twisted.
2. There is an acceptable coloring for \( \langle E_0, E_1, E_2 \rangle \).

It should be remarked that the statement of Theorem 3 in [3] required that \( X = \mathbb{R}^2 \), but the proof there applies equally to any \( X \).

Consider points \( c_0, c_1, c_2 \in \mathbb{R}^2 \). Let \( X = \mathbb{R}^2 \), and then let \( E_i \) be the equivalence relation on \( \mathbb{R}^2 \) such that \( [x]_i \) is a circle centered at \( c_i \). That is, \( y \in [x]_i \) iff \( \|y - c_i\|^2 = \|x - c_i\|^2 \). Clearly, there is a coloring of \( \mathbb{R}^2 \) acceptable for \( \langle E_0, E_1, E_2 \rangle \) iff there are sprays centered at \( c_0, c_1, c_2 \) covering \( \mathbb{R}^2 \). Thus, to prove Theorem 1, it would suffice to prove that if \( c_0, c_1, c_2 \) are noncollinear, then \( \langle E_0, E_1, E_2 \rangle \) is twisted. But, instead, we will consider a slightly different, but very closely related triple of equivalence relations.

\(^{(1)}\) The most common definition in the literature of the hereditary cardinality of a set \( x \) seems to be the cardinality of its transitive closure \( TC(x) \). Sometimes, although less frequently, the phrase “\( x \) has hereditary cardinality less than \( \theta \)” is rendered as “\( |x| < \theta \)” and, for all \( y \in TC(x), |y| < \theta \). For us, where \( \theta \) is regular, these two definitions are equivalent.
Now suppose $c_0, c_1, c_2 \in \mathbb{R}^2$ are distinct points. It is obvious that if $r_0, r_1, r_2 \in \mathbb{R}$ are such that there is a point $x \in \mathbb{R}^2$ with $r_i$ being the square of the radius of the circle $[x]_i$, then $r_0, r_1, r_2 \geq 0$ and $\sqrt{r_i} + \sqrt{r_j} \geq \|c_i - c_j\| \text{ whenever } i < j < 3$. Conversely, we can get a quadratic polynomial $q(u, v, w) \in \mathbb{R}[u, v, w]$ such that if $r_0, r_1, r_2 \in \mathbb{R}$ satisfy these obvious conditions, then $q(r_0, r_1, r_2) = 0$ iff there is a point $x \in \mathbb{R}^2$ such that $r_i$ is the square of the radius of the circle $[x]_i$. If we let

$$
(1) \quad d_0 = \|c_1 - c_2\|^2, \quad d_1 = \|c_2 - c_0\|^2, \quad d_2 = \|c_0 - c_1\|^2,
$$

then

$$
(2) \quad q(u, v, w) = d_0u^2 + d_1v^2 + d_2w^2 + (d_0 - d_1 - d_2)vw + (d_1 - d_2 - d_0)uw + (d_2 - d_0 - d_1)dv
$$

is such a polynomial (2). For example, in (3), $c_0, c_1, c_2$ were chosen to be the vertices of a carefully selected equilateral triangle with sides of length 2, and then the polynomial

$$
u^2 + v^2 + w^2 - vw - uw - uv - 4u - 4v - 4w + 16$$

was obtained.

Now, suppose that $\mathbb{F}$ is any field and that $p(u, v, w) \in \mathbb{F}[u, v, w]$. Let $X \subseteq \mathbb{F}^3$ be its zero-set, and for each $i < 3$, define the equivalence relation $E_i$ on $X$ by: if $(a_0, a_1, a_2), (b_0, b_1, b_2) \in X$, then

$$
(3) \quad \langle a_0, a_1, a_2 \rangle E_i \langle b_0, b_1, b_2 \rangle \iff a_i = b_i.
$$

Then we say that the coloring $\chi : X \rightarrow 3$ is acceptable for $p(u, v, w)$ if it is acceptable for $\langle E_0, E_1, E_2 \rangle$, and we say that $p(u, v, w)$ is twisted if $\langle E_0, E_1, E_2 \rangle$ is twisted.

Clearly, if $c_0, c_1, c_2 \in \mathbb{R}^2$, $q(u, v, w) \in \mathbb{R}[u, v, w]$ is as in (1) and (2), and there is an acceptable coloring for $q(u, v, w)$, then $\mathbb{R}^2$ can be covered by three sprays centered at $c_0, c_1, c_2$. Furthermore, $\mathbb{R}^2$ can be covered by three sprays centered at $c_0, c_1, c_2$ iff $q(u, v, w)$ is twisted. Thus, the following lemma suffices to prove Theorem 1.

**Lemma 4.** Suppose $c_0, c_1, c_2 \in \mathbb{R}^2$ are noncollinear and $q(u, v, w) \in \mathbb{R}[u, v, w]$ is as in (1) and (2). Then $q(u, v, w)$ is twisted.

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(3) Suppose we consider $c_0, c_1, c_2$ to be in $\mathbb{R}^3$ rather than $\mathbb{R}^2$, but still define $d_0, d_1, d_2$ as in (1). Then let $c_3 \in \mathbb{R}^3$ and let $u = \|c_3 - c_0\|^2, v = \|c_3 - c_1\|^2$ and $w = \|c_3 - c_2\|^2$. Let $V$ be the volume of the tetrahedron whose vertices are $c_0, c_1, c_2, c_3$. Then $q(u, v, w) = -144V^2$. This can be verified using the $5 \times 5$ Cayley–Menger determinant $D = 288V^2$, which is Tartaglia’s 3-dimensional generalization of Heron’s formula for the area of a triangle.
It will be convenient to work in the complex field $\mathbb{C}$ rather than $\mathbb{R}$. We will consider the polynomial $q(u, v, w)$ to be over $\mathbb{C}$, and then prove that even then it is twisted. Clearly, this implies that it is twisted as a polynomial over $\mathbb{R}$ (or use Theorem 3). Thus, our goal is to prove the following lemma.

**Lemma 5.** Suppose $c_0, c_1, c_2 \in \mathbb{R}^2$ are noncollinear and $q(u, v, w) \in \mathbb{C}[u, v, w]$ is as in (1) and (2). Then $q(u, v, w)$ is twisted.

Most of the rest of this paper is devoted to proving Lemma 5.

**Lemma 5.1.** Let $c_0, c_1, c_2 \in \mathbb{C}^2$ and assume that (1) and (2) hold. The following are equivalent:

(a) $c_0, c_1, c_2$ are collinear.

(b) $q(u, v, w)$ is reducible (or is the zero polynomial).

(c) $q(u, v, w)$ is the square of a polynomial.

(d) $d_0^2 + d_1^2 + d_2^2 = 2d_1d_2 + 2d_0d_1 + 2d_0d_2$.

**Proof.** (a)$\iff$(c): Let $s_0, s_1, s_2 \in \mathbb{C}$ be such that $s_i^2 = d_i$ for $i < 3$. Suppose that $c_0, c_1, c_2$ are collinear, so we can assume that $s_0 + s_1 + s_2 = 0$, in which case $q(u, v, w) = (s_0u + s_1v + s_2w + s_0s_1s_2)^2$. Conversely, suppose that $q(u, v, w)$ is a square, so we can assume that $q(u, v, w) = (s_0u + s_1v + s_2w + s_0s_1s_2)^2$. Looking at the coefficients of $vw, uw, uv$, we get

\[
\begin{pmatrix}
1 & -1 & -1 \\
-1 & 1 & -1 \\
-1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
d_0 \\
d_1 \\
d_2
\end{pmatrix} =
\begin{pmatrix}
2s_1s_2 \\
2s_0s_2 \\
2s_0s_1
\end{pmatrix}.
\]

Thus, in terms of $s_0, s_1, s_2$, there are unique $d_0, d_1, d_2$, and we have seen that $d_0 = s_0^2, d_1 = s_1^2, d_2 = s_2^2$ works. Thus, $s_0^2 - s_1^2 - s_2^2 = 2s_1s_2$, so $s_0^2 = (s_1 + s_2)^2$, and then either $s_0 - s_1 - s_2 = 0$ or $s_0 + s_1 + s_2 = 0$, implying that $c_0, c_1, c_2$ are collinear.

(c)$\iff$(b): The zero-set $X$ can be parametrized by $X = \{(||c_0 - x||^2, ||c_1 - x||^2, ||c_2 - x||^2) \in \mathbb{C}^3 : x \in \mathbb{C}^2\}$, so it is a variety. Then, since the polynomial $q(u, v, w)$ is at most a quadratic, it is either irreducible or the square of a polynomial.

(a)$\iff$(d): This is straightforward by algebraic calculations. □

From now on until the end of the proof of Lemma 5, we fix noncollinear $c_0, c_1, c_2 \in \mathbb{R}^2$ and assume that (1) and (2) hold. However, now we will consider $q(u, v, w) \in \mathbb{C}[u, v, w]$. Notice that $d_0d_1d_2 \neq 0$. Actually, the following proof will work for all noncollinear $c_0, c_1, c_2 \in \mathbb{C}^2$ with $d_0d_1d_2 \neq 0$. If we are willing to handle some additional details, we could extend the proof to all noncollinear $c_0, c_1, c_2 \in \mathbb{C}^2$ allowing the possibility that $d_0d_1d_2 = 0$.

By Lemma 5.1, $q(u, v, w)$ is irreducible, but the next lemma shows that even more is true.
Lemma 5.2. Let \( w_0 \in \mathbb{C} \). Then \( q(u,v,w_0) \in \mathbb{C}[u,v] \) is irreducible iff \( w_0 \neq 0 \).

To make the statement a little simpler, we have stated only one-third of the truth in the previous lemma, since either of the other two variables could instead be fixed. By symmetry, the other two-thirds easily follows. It should be understood, here and elsewhere, that the variables may be permuted.

Proof. Without loss of generality, assume that \( c_2 = (0,0) \). Since \( d_0 \neq 0 \), \( q(u,v,w_0) \) is a nonlinear quadratic polynomial. Let \( X \subseteq \mathbb{C}^2 \) be the zero-set of \( q(u,v,w_0) \). Then \( X \) can be parametrized by \( X = \{ (\|c_0 - x\|^2, \|c_1 - x\|^2) : \|x\|^2 = w_0 \} \).

Suppose \( w_0 = 0 \). Then \( \{ x \in \mathbb{C}^2 : \|x\|^2 = 0 \} \) is the union of the two lines \( \ell_0 = \{ (x,ix) \in \mathbb{C}^2 : x \in \mathbb{C} \} \) and \( \ell_1 = \{ (x,-ix) \in \mathbb{C}^2 : x \in \mathbb{C} \} \). Then \( X = X_0 \cup X_1 \), where \( X_e = \{ (\|c_0 - x\|^2, \|c_1 - x\|^2) \in \mathbb{C}^2 : x \in \ell_e \} \) for \( e = 0,1 \), so that \( q(u,v,w_0) \) is reducible.

Suppose \( w_0 \neq 0 \). Then \( \{ x \in \mathbb{C}^2 : \|x\|^2 = w_0 \} \) is a variety, so also \( X \) is. Thus, either \( q(u,v,w_0) \) is irreducible or is the square of a linear polynomial. Suppose that \( q(u,v,w_0) \) is not irreducible and so \( q(u,v,w_0) = (au+bv+c)^2 \). Then \( a^2 = d_0 \) and \( b^2 = d_1 \). Also, \( 2ab \) is the coefficient of \( uv \) in \( q(u,v,w_0) \), and a calculation shows that this implies that the equality of Lemma 5.1(d) holds, contradicting that \( c_0,c_1,c_2 \) are not collinear.

Corollary 5.3. For every \( w_0 \in \mathbb{C} \), the equation
\[
q(u,v,w_0) = \frac{\partial q}{\partial u}(u,v,w_0) = 0
\]
has at least one and at most finitely many solutions \((u,v)\).

Proof. Since \( d_0 \neq 0 \), it follows that \( \frac{\partial q}{\partial u}(u,v,w_0) \) is not a constant polynomial, so \( u \) is a linear function of \( v \). Plugging that into \( q(u,v,w_0) \) results in a quadratic polynomial in which the coefficient of \( v^2 \) is nonzero (because of Lemma 5.1(d)). Thus, there is at least one solution \((u,v)\) and there are at most two.

Lemma 5.4. If \( u_0,v_0 \in \mathbb{C} \), then \( q(u_0,v_0,w) = 0 \) has at most finitely many solutions.

Proof. This is immediate since \( d_2 \neq 0 \).

Lemma 5.5. If \( 0 \neq w_0 \in \mathbb{C} \), then the equation
\[
q(u,v,w_0) = \frac{\partial q}{\partial u}(u,v,w_0) = \frac{\partial q}{\partial v}(u,v,w_0) = 0
\]
has no solution \((u,v)\).

Proof. If \( p(u,v) \in \mathbb{C}[u,v] \) is a nonlinear quadratic polynomial such that
\[
p(u,v) = \frac{\partial p}{\partial u}(u,v) = \frac{\partial p}{\partial v}(u,v) = 0
\]
has a solution, then \( p(u, v) \) is reducible. For, without loss of generality, assume that \((0, 0)\) is a solution so that \( p(u, v) = au^2 + buv + cv^2 \), which is reducible. For any \( w_0 \in \mathbb{C} \), it is clear that \( q(u, v, w_0) \) is a nonlinear quadratic polynomial. Thus, if the equation in the lemma has a solution, then \( q(u, v, w_0) \) is reducible. By Lemma 5.2, this happens only if \( w_0 = 0 \). □

The next lemma is the main algebraic fact used in the proof of Lemma 5.

**Lemma 5.6.** Suppose that \( w_0, w_1, w_2, w_3 \in \mathbb{C} \) are such that any three of them are algebraically independent over the coefficients of \( q(u, v, w) \). Then the equation

\[
q(u, v, w_0) = q(u, v', w_1) = q(u', v', w_2) = q(u', v, w_3) = 0
\]

has at most finitely many solutions \((u, u', v, v') \in \mathbb{C}^4\).

**Proof.** It suffices to assume that all coefficients of \( q(u, v, w) \) are algebraic, so we will do so. [Alternatively, we could reinterpret each use of the term algebraic in this proof to mean algebraic over the coefficients of \( q(u, v, w) \).] Suppose there are infinitely many solutions. It follows from Lemma 5.4 that there are infinitely many \( u_0 \in \mathbb{C} \) for which there are \( u_0', v_0, v_0' \in \mathbb{C} \) such that \((u_0, u_0', v_0, v_0')\) is a solution. Then, for all but possibly finitely many \( u_0 \in \mathbb{C} \) there are \( u_0', v_0, v_0' \) such that \((u_0, u_0', v_0, v_0')\) is a solution. Since \( d_0 \neq 0 \neq d_1 \), it follows that for every bounded \( A \subseteq \mathbb{C} \), there is a bounded \( B \subseteq \mathbb{C} \) such that whenever \((u_0, u_0', v_0, v_0')\) is a solution and \( u_0 \in A \), then \( u_0, u_0', v_0, v_0' \in B \). Thus, the set of all \( u_0 \) that extend to a solution is a closed subset of \( \mathbb{C} \), and therefore all of \( \mathbb{C} \). Moreover, since \( q(u, v, w) \) is irreducible, whenever \( q(u_0, v_0, w_0) = 0 \), then there are \( u_0', v_0' \) such that \((u_0, u_0', v_0, v_0')\) is a solution.

By Corollary 5.3, we can let \( u_0, v_0 \) be such that

\[
q(u_0, v_0, w_0) = \frac{\partial q}{\partial u}(u_0, v_0, w_0) = 0,
\]

and then obtain \( u_0', v_0' \) so that

\[
q(u_0, v_0', w_1) = q(u_0', v_0', w_2) = q(u_0', v_0, w_3) = 0.
\]

Corollary 5.3 and (4) imply that \( w_0 \) is algebraic over \( v_0 \), and then since \( w_0 \) is not algebraic, neither is \( v_0 \). Again by Corollary 5.3 and (4), \( u_0 \) and \( v_0 \) are algebraic over \( w_0 \).

From Lemma 5.4 and (5), \( w_3 \) is algebraic over \( \{u_0', v_0\} \). Since \( v_0 \) is algebraic over \( w_0 \) and \( w_3 \) is not algebraic over \( w_0 \), it cannot be that \( u_0' \) is algebraic. Similarly, \( v_0' \) is not algebraic.

Since \( w_0 \) is not algebraic, we see from Lemma 5.5 and (4) that

\[
\frac{\partial q}{\partial v}(u_0, v_0, w_0) \neq 0.
\]
We deduce from Corollary 5.3 and (5) that
\[
\frac{\partial q}{\partial u}(u_0', v_0, w_3) \neq 0,
\]
as otherwise \(w_3\) (and also \(u_0'\)) would be algebraic over \(v_0\) and thus algebraic over \(w_0\). Also, \(v_0'\) is algebraic over \(\{w_0, w_1\}\), and then since \(w_3\) is not algebraic over \(\{w_0, w_1\}\), Corollary 5.3 and (5) yield
\[
\frac{\partial q}{\partial v}(u_0, v_0', w_1) \neq 0.
\]
Similarly, since \(w_1\) is not algebraic, Corollary 5.3 and (5) imply that
\[
\frac{\partial q}{\partial u}(u_0, v_0', w_1) \neq 0,
\]
and, likewise, since \(w_2\) is not algebraic over \(\{w_0, w_1\}\) and neither of \(u_0', v_0'\) is algebraic, Corollary 5.3 and (5) yield
\[
\frac{\partial q}{\partial u}(u_0', v_0', w_2) \neq 0 \neq \frac{\partial q}{\partial v}(u_0, v_0', w_2).
\]
We can now invoke the Implicit Function Theorem, using (4)–(10), to get neighborhoods \(U, V, V'\) of \(u_0, v_0, v_0'\) and analytic functions \(f_0, f_1 : U \to \mathbb{C}, f_2 : V' \to \mathbb{C}\) and \(f_3 : V \to \mathbb{C}\) such that
\[
\begin{align*}
(a) & \quad f_0(u_0) = v_0, f_1(u_0) = v_0', f_2(v_0') = u_0' \text{ and } f_3(v_0) = u_0', \\
(b) & \quad q(u, f_0(u), w_0) = q(u, f_1(u), w_1) = 0 \text{ for all } u \in U, \\
(c) & \quad q(f_2(v'), v', w_2) = q(f_3(v), v, w_3) = 0 \text{ for all } v' \in V' \text{ and } v \in V, \\
(d) & \quad f_0'(u_0) = 0 \text{ and } f_1'(u_0) \neq 0 \neq f_2'(v_0'), \\
(e) & \quad f_3(f_0(u)) = f_2(f_1(u)) \text{ for all } u \in U.
\end{align*}
\]
Then (a), (e) and the Chain Rule imply that
\[
f_3'(v_0)f_0'(u_0) = f_2'(v_0')f_1'(u_0),
\]
contradicting (d). \(\blacksquare\)

**Proof of Lemma 5.** Let \(X \subseteq \mathbb{C}^3\) be the zero-set of \(q(u, v, w)\), and let \(E_0, E_1, E_2\) be the equivalence relations on \(X\) as defined in (3). Without loss of generality, in Definition 2, we will let \(i = 1, j = 0\) and \(k = 2\). Let \(v_0 \in \mathbb{C}\), and then define
\[
B = \{(u, w) \in \mathbb{C}^2 : q(u, v_0, w) = 0, u \in M \setminus N, w \in N \setminus M\}.
\]
Our goal is to show that \(B\) is finite, so, for a contradiction, assume that \(B\) is infinite. Let
\[
A = \{w \in \mathbb{C} : \langle u, w \rangle \in B \text{ for some } u \in \mathbb{C}\}.
\]
Then \(A \subseteq N \setminus M\) and, by Lemma 5.4, \(A\) is infinite.

At this point, we would like to get \(w_0, w_1 \in A\) that are algebraically independent over the coefficients of \(q(u, v, w)\) so as then to make use of
Lemma 5.6. But that may not be possible. One way around this problem is as follows. Enumerate all polynomials whose coefficients are in the field generated by the coefficients of \( q(u, v, w) \) and then say, for example, that \( w_0, w_1 \in \mathbb{C} \) are \( n \)-algebraically independent if they are not the zeroes of any of the first \( n \) of the enumerated polynomials. Then there is some sufficiently large \( n < \omega \) such that Lemma 5.6 is still true when we replace “algebraically independent over the coefficients of \( q(u, v, w) \)” with “\( n \)-algebraically independent”. Then there is some \( m < \omega \) such that it suffices to choose \( w_0, w_1 \in A \) to be \( m \)-algebraically independent.

Another, and simpler, way is just take an ultrapower of the universe so that everything is \( \aleph_0 \)-saturated. This is the approach we will take, so we assume that \( M \) and \( N \) are \( \aleph_0 \)-saturated.

Since \( A \) is infinite and definable, by \( \aleph_0 \)-saturation we can get \( w_0, w_1 \in A \) which are algebraically independent over the coefficients of \( q(u, v, w) \). Then let \( u_0, u_0' \) be such that \( \langle w_0, w_0 \rangle, \langle u_0, w_1 \rangle \in B \). Notice that \( u_0, u_0' \in M \setminus N \).

Define \( G \subseteq \mathbb{C}^2 \) to be the set of all \( \langle b_0, b_1 \rangle \) such that:

- \( \langle b_0, b_1 \rangle \in N \) and \( b_0 \neq b_1 \);
- there is \( v \) such that \( q(u_0, v, b_0) = q(u_0', v, b_1) = 0 \);
- \( b_0, b_1 \) are algebraically independent over the coefficients of \( q(u, v, w) \).

Since \( \langle w_0, w_1 \rangle \in G \), we see that \( G \neq \emptyset \). By elementarity, there is \( \langle b_0, b_1 \rangle \in M \) satisfying the first two of the above conditions and also such that \( b_0, b_1 \) are \( n \)-algebraically independent over the coefficients of \( q(u, v, w) \) for any \( n < \omega \). Then, by \( \aleph_0 \)-saturation, it follows that \( G \cap M \neq \emptyset \), so let \( \langle w_2, w_3 \rangle \in G \cap M \).

Clearly, \( w_2 \neq w_3 \), and also \( \{w_2, w_3\} \cap \{w_0, w_1\} = \emptyset \) since \( w_0, w_1 \in N \setminus M \) and \( w_2, w_3 \in M \). In fact, each 3-element subset of \( \{w_0, w_1, w_2, w_3\} \) is algebraically independent over the coefficients of \( q(u, v, w) \). Let \( v_0' \) be such that \( q(u_0, v_0', b_0) = q(u_0, v_0', b_1) = 0 \). Then, by Lemma 5.6, the system of equations

\[
q(u, v, w_0) = q(u', v, w_1) = q(u', v', w_2) = q(u, v', w_3) = 0
\]

has only finitely many solutions \((u, u', v, v')\). Since \( w_0, w_1, w_2, w_3 \in N \), all the solutions are in \( N \). In particular, \((u_0, u_0', v_0, v_0') \in N \), so that \( u_0 \in N \), which is a contradiction. 

Theorem 1, which deals with noncollinear points \( c_0, c_1, c_2 \in \mathbb{R}^2 \), is now proved. We next consider what happens with collinear points. The \( n = 3 \) case of Theorem 7 was proved by de la Vega [3] using Theorem 3. The proof of Theorem 7 that is given here is more direct.

We make the obvious generalizations of some of the definitions given right after Theorem 1. Suppose that \( X \) is an arbitrary set and \( E_0, E_1, \ldots, E_{n-1} \) are equivalence relations on \( X \). If \( x \in X \) and \( i < n \), then \([x]_i\) is the equivalence class of \( E_i \) to which \( x \) belongs. A coloring \( \chi : X \rightarrow n = \{0, 1, \ldots, n-1\} \)
is acceptable for \(\langle E_0, E_1, \ldots, E_{n-1}\rangle\) if whenever \(x \in X\) and \(i < n\), then \([x]_i \cap \chi^{-1}(i)\) is finite.

**Lemma 6.** Suppose that \(n < \omega\). Let \(l_0, l_1, \ldots, l_{n+1} \subseteq \mathbb{R}^2\) be distinct lines passing through the origin, and then let \(E_i\) be the equivalence relation on \(\mathbb{R}^2\) defined as follows: if \(x \in \mathbb{R}^2\), then \([x]_i\) is the line containing \(x\) that is parallel (or equal) to \(l_i\). If there is an acceptable coloring of \([0, 1]^2\) for
\[
\langle E_0 \cap [0, 1]^2, E_1 \cap [0, 1]^2, \ldots, E_{n+1} \cap [0, 1]^2\rangle,
\]
then \(2^{\mathbb{N}_0} \leq \aleph_n\).

A slightly weaker result was proved by Davies [1]. His result had the same conclusion but had the weaker hypothesis that there is an acceptable coloring of \(\mathbb{R}^2\) for \(\langle E_0, E_1, \ldots, E_{n+1}\rangle\). However, the proof in [1] can easily be applied to prove Lemma 6.

**Theorem 7.** Suppose that \(n < \omega\) and \(c_0, c_1, \ldots, c_{n+1} \in \mathbb{R}^2\) are \(n + 2\) distinct, collinear points. Then \(2^{\mathbb{N}_0} \leq \aleph_n\) iff \(\mathbb{R}^2\) can be covered by \(n + 2\) sprays \(S_0, S_1, \ldots, S_{n+1}\) centered at \(c_0, c_1, \ldots, c_{n+1}\), respectively.

**Proof.** As already noted, it was shown in [2] that if \(2^{\mathbb{N}_0} \leq \aleph_n\), then \(\mathbb{R}^2\) can be covered by sprays centered at \(c_0, c_1, \ldots, c_{n+1}\). We consider the converse, so let \(c_0, c_1, \ldots, c_{n+1} \in \mathbb{R}^2\) be distinct, collinear points, and suppose that \(S_0, S_1, \ldots, S_{n+1}\) are sprays centered at \(c_0, c_1, \ldots, c_{n+1}\), respectively, covering \(\mathbb{R}^2\). We might as well assume that \(c_0, c_1, \ldots, c_{n+1}\) are all on the \(x\)-axis and that \(c_i = (a_i, 0)\), where \(0 = a_0 < a_1 < \cdots < a_{n+1}\). For \(i, j \leq n + 1\), let
\[
s_{ij} = |a_i - a_j| = ||c_i - c_j||.
\]
Let \(X \subseteq \mathbb{R}^{n+2}\) be such that
\[
X = \{\|c_0 - x\|^2, \|c_1 - x\|^2, \ldots, \|c_{n+1} - x\|^2 \in \mathbb{R}^{n+2} : x \in \mathbb{R}^2\}.
\]
It easily follows, as in the proof of (a)\(\Leftrightarrow\)(c) of Lemma 5.1, that \((r_0, r_1, \ldots, r_{n+1}) \in X\) iff \(r_i \geq 0\) whenever \(i \leq n + 1\), \(\sqrt{r_i} - \sqrt{r_j} \geq s_{ij}\) whenever \(i < j \leq n + 1\), and \(s_{ij}r_k + s_{jk}r_i - s_{ik}r_j = s_{ij}s_{jk}s_{ik}\) whenever \(i < j < k \leq n + 1\). For this last equation, it suffices to restrict to \(i = 0\) and \(j = 1\), so the zero-set \(Z\) of these equations is a 2-dimensional plane in \(\mathbb{R}^{n+2}\).

Let \(\psi : X \rightarrow n + 2\) be such that \(\psi(r_0, r_1, \ldots, r_{n-1}) = i\), then there is \(x \in S_i\) such that
\[
(r_0, r_1, \ldots, r_{n+1}) = (\|c_0 - x\|^2, \|c_1 - x\|^2, \ldots, \|c_{n+1} - x\|^2).
\]
Then \(\psi\) has the property that if \(i \leq n + 1\) and \(H \subseteq \mathbb{R}^{n+2}\) is a hyperplane orthogonal to the \(i\)th coordinate axis, then \(\psi^{-1}(i) \cap H\) is finite.

Let \(f : \mathbb{R}^2 \rightarrow Z\) be an isometry such that \(f([0, 1]^2) \subseteq X\). Then there are distinct lines \(l_0, l_1, \ldots, l_{n+1} \subseteq \mathbb{R}^2\) such that whenever \(i \leq n + 1\) and \(H \subseteq \mathbb{R}^{n+2}\) is a hyperplane orthogonal to the \(i\)th coordinate axis, then \(f^{-1}(i) \cap H\) is parallel (or equal) to \(l_i\). Let \(E_i\) be defined as in Lemma 6. Then \(\psi f\) is an acceptable coloring of \([0, 1]^2\) for
\[
\langle E_0 \cap [0, 1]^2, E_1 \cap [0, 1]^2, \ldots, E_{n+1} \cap [0, 1]^2\rangle.
\]
Lemma 6 then implies that \(2^{\mathbb{N}_0} \leq \aleph_n\).
The proofs of Theorems 1 and 7 suggest a general problem. First, a
definition. If \(2 \leq n < \omega\) and \(X \subseteq \mathbb{R}^n\), then we say that a function \(\chi : X \rightarrow n\) is \emph{acceptable} if whenever \(i < n\) and \(H \subseteq \mathbb{R}^n\) is a hyperplane orthogonal to
the \(i\)th coordinate axis, then \(\chi^{-1}(i) \cap H\) is finite.

**Problem.** Assuming \(\mathsf{CH}\), determine those polynomials \(p(x_0, x_1, \ldots, x_{n-1}) \in \mathbb{R}[x_0, x_1, \ldots, x_{n-1}]\) whose zero-sets have acceptable colorings. Do the same assuming other restrictions on \(2^{\aleph_0}\).

**References**


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