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Quantum mechanics and nonabelian theta functions for the gauge group SU(2)

by

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Abstract. We propose a direction of study of nonabelian theta functions by establishing an analogy between the Weyl quantization of a one-dimensional particle and the metaplectic representation on the one hand, and the quantization of the moduli space of flat connections on a surface and the representation of the mapping class group on the space of nonabelian theta functions on the other. We exemplify this with the cases of classical theta functions and of the nonabelian theta functions for the gauge group SU(2). The emphasis of the paper is on this analogy and on the possibility of generalizing this approach to other gauge groups, and not on the results, of which some have appeared elsewhere.

1. Introduction. The paper outlines a study of the nonabelian theta functions that arise in Chern–Simons theory by adapting the method used by André Weil for studying classical theta functions [35]. The goal is to derive the constructs of Chern–Simons theory from quantum mechanics, as opposed to quantum field theory. We exemplify with the case of the gauge group SU(2). We envision two possible applications of our method: the generalization to other gauge groups, including noncompact ones, and the discovery of the analytical model for the quantization that corresponds to the quantum group quantization of the moduli space of flat SU(2)-connections on the torus. The latter is a long standing problem on which modest progress was made (see [2], [11], [23]).

In Weil's approach, classical theta functions come with an action of the finite Heisenberg group and a projective representation of the mapping class group. By analogy, the theory of nonabelian theta functions consists of:

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- the Hilbert space of nonabelian theta functions, namely the holomorphic sections of the Chern–Simons line bundle;
- an irreducible representation on the space of theta functions of the algebra generated by quantized Wilson lines (i.e. of the quantizations of traces of holonomies of simple closed curves);
- a projective representation of the mapping class group of the surface on the space of nonabelian theta functions.

The representation of the mapping class group intertwines the quantized Wilson lines; in this sense the two representations satisfy an exact Egorov identity.

Our prototype is the quantization of a one-dimensional particle. The paradigm is that *the quantum group quantization of the moduli space of flat* SU(2)*-connections on a surface and the Reshetikhin–Turaev representation of the mapping class group are the analogues of the Schrödinger representation of the Heisenberg group and of the metaplectic representation.* The Schrödinger representation arises from the quantization of the position and the momentum of a one-dimensional free particle, and is a consequence of a fundamental postulate in quantum mechanics. It is a unitary irreducible representation of the Heisenberg group, and the Stone–von Neumann theorem shows that it is unique. This uniqueness implies that linear changes of coordinates (which act as outer automorphisms of the Heisenberg group) are also quantizable, and their quantization yields an infinite-dimensional representation of the metaplectic group.

Weil [35] observed that a finite Heisenberg group acts on classical theta functions, and the action of the modular group is induced via a Stone–von Neumann theorem. Then it was noticed that classical theta functions, the action of the Heisenberg group, and of the modular group arise from the Weyl quantization of Jacobian varieties. As such, classical theta functions are the holomorphic sections of a line bundle over the moduli space of flat $u(1)$-connections on a surface; by analogy, the holomorphic sections of the similar line bundle over the moduli space of flat $\mathfrak{g}$-connections over a surface (where $\mathfrak{g}$ is the Lie algebra of a compact simple Lie group) were called nonabelian theta functions. Witten [37] placed nonabelian theta functions in the context of Chern–Simons theory, related them to the Jones polynomial [14] and conformal field theory, and gave new methods for studying them. We show how within Witten's theory one can find the nonabelian analogues of Weil's constructs.

The paper runs the parallel between the Schrödinger and metaplectic representations, the Weil representation of the Heisenberg group and the action of the modular group on theta functions, and the quantum group quantization of the moduli space of flat su(2)-connections on a surface and the Reshetikhin–Turaev representation. Among the features we mention the translation of the quantum group quantization of the moduli space into skein-theoretical language, and the derivation strictly from quantum mechanical considerations of the element Ω which is the building block of the Witten–Reshetikhin–Turaev invariants, and of the Reshethikhin–Turaev representation of the mapping class group. Our analogy suggests that any analytical model for the quantization of the moduli space of flat su(2)-connections should be similar to Weyl quantization. It also establishes a programme for studying Chern–Simons theory with general gauge group. For that reason, we present proofs of the results which could allow generalizations.

2. The prototype

2.1. The Schrödinger representation. In this section we briefly review the Schrödinger and the metaplectic representations. For details see [22].

Consider a particle in a 1-dimensional space. The phase space is $\mathbb{R}^2$, with coordinates the position x and the momentum y, symplectic form $\omega = dx \wedge dy$ and Poisson bracket $\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$. The symplectic form induces a nondegenerate antisymmetric bilinear form on $\mathbb{R}^2$, also denoted by ω, given by $\omega((x, y), (x', y')) = xy' - x'y$.

The Lie algebra of observables has a subalgebra generated by $Q(x, y) = x$, $P(x, y) = y$, and $E(x, y) = 1$, called the *Heisenberg Lie algebra.* Abstractly, this algebra is defined by $[Q, P] = E, [P, E] = [Q, E] = 0$.

It is a postulate of quantum mechanics that the quantization of the position, the momentum, and the constant functions is the representation of the Heisenberg Lie algebra on $L^2(\mathbb{R}, dx)$ defined by

$$Q \mapsto M_x, \qquad P \mapsto \frac{\hbar}{i}\frac{d}{dx}, \qquad E \mapsto i\hbar \,\mathrm{Id}\,.$$

Here M_x denotes the operator of multiplication by the variable: $\phi(x) \mapsto x\phi(x)$ and $\hbar = h/2\pi$ is the reduced Planck's constant. This is the Schrödinger representation of the Heisenberg Lie algebra.

By exponentiation one obtains the Schrödinger representation of the Heisenberg group $\mathbf{H}(\mathbb{R})$ with real entries:

$$\begin{aligned}\exp(x_0Q + y_0P + tE)\phi(x) &= e^{2\pi i(x_0Q+y_0P+tE)}\phi(x)\\ &= e^{2\pi i x_0 x + \pi i h x_0 y_0 + 2\pi i t}\phi(x + hy_0).\end{aligned}$$

Using this representation, Hermann Weyl gave a method for quantizing functions $f \in C^\infty(\mathbb{R}^2)$, using the Fourier transform

$$\hat{f}(\xi, \eta) = \iint f(x, y)\exp(-2\pi i x\xi - 2\pi i y\eta)\, dx\, dy$$

to define

$$\mathrm{Op}(f) = \iint \hat{f}(\xi,\eta)\exp(2\pi i(\xi Q + \eta P))\, d\xi\, d\eta,$$

where for $\exp(\xi Q + \eta P)$ he used the Schrödinger representation.

THEOREM (Stone–von Neumann). *The Schrödinger representation of the Heisenberg group is the unique irreducible unitary representation of this group such that* $\exp(tE)$ *acts as* $e^{2\pi i t}\,\mathrm{Id}$ *for all* $t \in \mathbb{R}$.

There are two other important realizations of the irreducible representation that this theorem characterizes. One comes from the quantization of the plane in a holomorphic polarization. The Hilbert space is the Bargmann space of holomorphic square integrable entire functions with respect to the measure $e^{-2\pi|\mathrm{Im}\, z|^2}\, dz\, d\bar{z}$, with the Heisenberg group acting by

$$\exp(x_0 Q + y_0 P + tE) f(z) = e^{\pi i h(y_0 + i x_0) + 2\pi i x_0 z + 2\pi i t} f(z + h(y_0 + i x_0)).$$

For the other realization, choose a Lagrangian subspace $\mathbf{L}$ of $\mathbb{R}P + \mathbb{R}Q$. Then $\exp(\mathbf{L} + \mathbb{R}E)$ is a maximal abelian subgroup of the Heisenberg group. Consider the character of this subgroup defined by $\chi_{\mathbf{L}}(\exp(l + tE)) = e^{2\pi i t}$, $l \in \mathbf{L}$. The Hilbert space of the quantization, $\mathcal{H}(\mathbf{L})$, is defined as the space of functions $\phi(u)$ on $\mathbf{H}(\mathbb{R})$ satisfying

$$\phi(uu') = \chi_L(u')^{-1}\phi(u) \quad \text{ for all } u' \in \exp(\mathbf{L} + \mathbb{R}E)$$

and such that $u \mapsto |\phi(u)|$ is a square-integrable function on the left equivalence classes modulo $\exp(\mathbf{L} + \mathbb{R}E)$. The representation of the Heisenberg group is given by

$$u_0\phi(u) = \phi(u_0^{-1}u).$$

Choosing an algebraic complement $\mathbf{L}'$ of $\mathbf{L}$ and writing $\mathbb{R}P + \mathbb{R}Q = \mathbf{L} + \mathbf{L}' = \mathbb{R} + \mathbb{R}$, $\mathcal{H}(\mathbf{L})$ is realized as $L^2(\mathbf{L}') \cong L^2(\mathbb{R})$. For $\mathbf{L} = \mathbb{R}P$ and $\mathbf{L}' = \mathbb{R}Q$, one gets the standard Schrödinger representation.

2.2. The metaplectic representation. By the Stone–von Neumann theorem, if we change coordinates by a linear symplectomorphism and then quantize, we get a unitarily equivalent representation of the Heisenberg group. Hence linear symplectomorphisms can be quantized by unitary operators. Schur's lemma implies that these operators are unique up to a multiplication by a constant. So we have a projective representation ρ of the linear symplectic group $\mathrm{SL}(2,\mathbb{R})$ on $L^2(\mathbb{R})$. This can be made into a true representation by passing to the double cover of $\mathrm{SL}(2,\mathbb{R})$, the metaplectic group $\mathrm{Mp}(2,\mathbb{R})$. The representation of the metaplectic group is known as the *metaplectic representation* or the Segal–Shale–Weil representation.

The fundamental symmetry of the Weyl quantization is

$$\mathrm{Op}(f \circ h^{-1}) = \rho(h)\,\mathrm{Op}(f)\rho(h)^{-1}$$

for every observable $f \in C^\infty(\mathbb{R}^2)$ and $h \in \mathrm{Mp}(2,\mathbb{R})$, where $\mathrm{Op}(f)$ is the operator associated to f through Weyl quantization. For other quantization models this relation holds only mod $O(\hbar)$ (*Egorov's theorem*). When satisfied with equality, as in our case, it is called an *exact* Egorov identity.

The metaplectic representation can be defined using the third version of the Schrödinger representation in §2.1, which identifies it as a Fourier transform (see [22]). Let h be a linear symplectomorphism of the plane, $\mathbf{L}_1$ a Lagrangian subspace of $\mathbb{R}P + \mathbb{R}Q$ and $\mathbf{L}_2 = h(\mathbf{L}_1)$. The quantization of h is $\rho(h) : \mathcal{H}(\mathbf{L}_1) \to \mathcal{H}(\mathbf{L}_2)$,

$$(\rho(h)\phi)(u) = \int_{\exp \mathbf{L}_2/\exp(\mathbf{L}_1\cap\mathbf{L}_2)} \phi(uu_2)\chi_{\mathbf{L}_2}(u_2)\,d\mu(u_2),$$

where $d\mu$ is the measure induced on the space of equivalence classes by the Haar measure on $\mathbf{H}(\mathbb{R})$.

For explicit formulas for $\rho(h)$ one needs to choose the algebraic complements $\mathbf{L}_1'$ and $\mathbf{L}_2'$ of $\mathbf{L}_1$ and $\mathbf{L}_2$ and unfold the isomorphism $L^2(\mathbf{L}') \cong L^2(\mathbb{R})$. For example, for

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

if we set $\mathbf{L}_1 = \mathbb{R}P$ with variable y and $L_2 = S(\mathbf{L}_1) = \mathbb{R}Q$ with variable x and $\mathbf{L}_1' = \mathbf{L}_2$ and $\mathbf{L}_2' = S(\mathbf{L}_1') = \mathbf{L}_1$, then

$$\rho(S)f(x) = \int_{\mathbb{R}} f(y)e^{-2\pi ixy}\,dy$$

is the usual Fourier transform, which establishes the unitary equivalence between the position and the momentum representations. Similarly, for

$$T_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

if we set $\mathbf{L}_1 = \mathbf{L}_2 = \mathbb{R}P =$, $\mathbf{L}_1' = \mathbb{R}Q$, and $\mathbf{L}_2' = \mathbb{R}(P+Q)$, then

$$\rho(T_a)f(x) = e^{2\pi ix^2a}f(x).$$

The cocycle of the projective representation of the symplectic group is

$$c_L(h',h) = e^{-\frac{i\pi}{4}\boldsymbol{\tau}(\mathbf{L},h'(\mathbf{L}),h'\circ h(\mathbf{L}))},$$

where $\boldsymbol{\tau}$ is the Maslov index. This means that

$$\rho(h'h) = c_{\mathbf{L}}(h',h)\rho(h')\rho(h)$$

for $h, h' \in \mathrm{SL}(2,\mathbb{R})$.

3. Classical theta functions

3.1. Classical theta functions from the quantization of the torus. For an extensive treatment of theta functions the reader can consult [24], [22], [25]. Here we only consider the simplest situation, that of theta functions on the Jacobian variety of a 2-dimensional complex torus $\mathbb{T}^2$. Our discussion is sketchy; details can be found, for all closed Riemann surfaces, in [12].

Given the complex torus and oriented simple closed curves a and b with algebraic intersection number 1, which define a canonical basis of $H_1(\mathbb{T}^2,\mathbb{R})$ (or equivalently of $\pi_1(\mathbb{T}^2)$), take a holomorphic 1-form ζ such that $\int_a \zeta = 1$. The complex number $\tau = \int_b \zeta$, which depends on the complex structure, has positive imaginary part. The *Jacobian variety* of $\mathbb{T}^2$, denoted $\mathcal{J}(\mathbb{T}^2)$, is a 2-dimensional torus with complex structure defined by τ (as an element in its Teichmüller space). Equivalently, $\mathcal{J}(\mathbb{T}^2) = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$. We introduce real coordinates (x, y) on $\mathcal{J}(\mathbb{T}^2)$ by setting $z = x + \tau y$. In these coordinates, $\mathcal{J}(\mathbb{T}^2)$ is the quotient of $\mathbb{R}^2$ by $\mathbb{Z}^2$. We endow $\mathcal{J}(\mathbb{T}^2)$ with the symplectic form $\omega = dx \wedge dy$, which is a generator of $H^2(\mathbb{T}^2,\mathbb{Z})$. With its complex structure and this symplectic form, $\mathcal{J}(\mathbb{T}^2)$ is a Kähler manifold.

Classical theta functions and the action of the Heisenberg group can be obtained by applying Weyl quantization to $\mathcal{J}(\mathbb{T}^2)$ in the holomorphic polarization. Theta functions are obtained by geometric quantization. We start by setting Planck's constant $h = 1/N$, with N a positive *even* integer.

The Hilbert space of the quantization consists of the *classical theta functions*, which are the holomorphic sections of a line bundle over the Jacobian variety. This line bundle is the tensor product of a line bundle of curvature $-2\pi i N\omega$ and a half-density. By pulling back the line bundle to $\mathbb{C}$, we can view these sections as entire functions with some periodicity. The line bundle with curvature $2\pi i N\omega$ is unique up to tensoring with a flat bundle. Choosing the latter appropriately, we can ensure that the periodicity conditions are

$$f(z + m + n\tau) = e^{-2\pi i N(\tau n^2+2nz)} f(z).$$

An orthonormal basis of the space of classical theta functions is given by the *theta series*

$$\theta_j^\tau(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i N[\tau(j/N+n)^2/2+z(j/N+n)]}, \qquad j = 0, 1, \ldots, N-1. \tag{3.1}$$

It is convenient to extend this definition to all indices j by the periodicity condition $\theta_{j+N}^\tau(z) = \theta_j^\tau(z)$, namely to take indices modulo N.

Let us turn to the operators. The only exponentials on the plane that are doubly periodic, and therefore give rise to functions on the torus, are

$$f(x, y) = \exp(2\pi i(mx + ny)), \qquad m, n \in \mathbb{Z}.$$

Since the torus is a quotient of the plane by a discrete group, we can apply the Weyl quantization procedure. In the complex polarization, Weyl quantization is defined as follows (see [6]): A fundamental domain of the torus is the unit square $[0,1]\times[0,1]$ (this is done in the (x,y) coordinates, in the complex plane it is actually a parallelogram). The value of a theta function is completely determined by its values on this unit square. The Hilbert space of classical theta functions can be isometrically embedded into $L^2([0,1]\times[0,1])$ with the inner product

$$\langle f,g\rangle = (-iN(\tau-\bar\tau))^{1/2}\int_0^1\int_0^1 f(x,y)\,\overline{g(x,y)}\,e^{iN(\tau-\bar\tau)\pi y^2}\,dx\,dy.$$

For a proof of the following result see [12].

Proposition 3.1. *The Weyl quantization of the exponentials in the momentum representation is given by*

$$\mathrm{Op}(e^{2\pi i(px+qy)})\theta_j^\tau(z) = e^{-\pi ipq/N-2\pi ijq/N}\theta_{j+p}^\tau(z).$$

The Weyl quantization of the exponentials gives rise to the Schrödinger representation of the Heisenberg group $\mathbf{H}(\mathbb{Z})$ with integer entries with multiplication

$$(p,q,k)(p',q',k') = (p+p',q+q',k+k'+(pq'-qp')).$$

The proposition implies that

$$(p,q,k)\mapsto \text{the Weyl quantization of } e^{\pi ik/N}\exp 2\pi i(px+qy)$$

is a group morphism. This is the Schrödinger representation.

The Schrödinger representation of $\mathbf{H}(\mathbb{Z})$ is far from faithful. Because of this we factor it out by its kernel. The kernel is the subgroup consisting of the elements of the form $(p,q,k)^N$, with k even [12]. Let $\mathbf{H}(\mathbb{Z}_N)$ be the finite Heisenberg group obtained by factoring $\mathbf{H}(\mathbb{Z})$ by this subgroup, and let $\exp(pP+qQ+kE)$ be the image of (p,q,k) in it.

The following is an analogue of the Stone–von Neumann theorem.

Theorem 3.2. *The Schrödinger representation of $\mathbf{H}(\mathbb{Z}_N)$ is the* unique *irreducible unitary representation of this group with the property that* $\exp(kE)$ *acts as* $e^{\pi ik/N}\,\mathrm{Id}$ *for all $k\in\mathbb{Z}$.*

The Schrödinger representation of the finite Heisenberg group can be extended by linearity to a representation of the group algebra with coefficients in $\mathbb{C}$ of the finite Heisenberg group, $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N)]$. Since the elements of $\exp(\mathbb{Z}E)$ act as multiplications by constants, this is in fact a representation of the algebra $\mathcal{A}_N$ obtained by factoring $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N)]$ by the relations $\exp(kE)-e^{\pi ik/N}$ for all $k\in\mathbb{Z}$. By abuse of language, we call this the Schrödinger representation as well. The Schrödinger representation of $\mathcal{A}_N$ defines the quantizations of trigonometric polynomials on the torus.

Proposition 3.3.

(a) *The algebra of Weyl quantizations of trigonometric polynomials contains all linear operators on the space of theta functions.*

(b) *The Schrödinger representation of $\mathcal{A}_N$ on theta functions is faithful.*

Proof. For (a) see [12]. Part (b) follows from the fact that $\exp(pP+qQ)$, $p, q = 0, 1, \ldots, N-1$, form a basis of $\mathcal{A}_N$ as a vector space. ∎

As explained in [12], the Schrödinger representation can be described as the left regular action of the group algebra of the finite Heisenberg group on a quotient of itself. The construction is like for the Schrödinger representation in the abstract setting in §2.2.

3.2. Classical theta functions from a topological perspective. In [12] the theory of classical theta functions was shown to admit a reformulation in purely topological language, by interpreting topologically the representation-theoretical model. Let us recall some of the facts.

Let M be a smooth oriented compact 3-manifold. A framed link in M is a smooth embedding of a disjoint union of circles, with the framing of each link component defined by a vector field orthogonal to it. We can view the framed link as an embedding of several annuli, each having a specified boundary component (which is the actual link component). We draw all diagrams in the blackboard framing, so that the framing is parallel to the plane of the paper.

Consider the free $\mathbb{C}[t, t^{-1}]$-module with basis the set of isotopy classes of framed oriented links in M, including the empty link $\emptyset$. Factor it by all equalities of the form shown in Figure 1. In each diagram, the two links are identical except for an embedded ball in M, inside of which they look as shown. Thus in a link we can smoothen a crossing provided that we add a coefficient of t or t^{-1}, and trivial link components can be ignored. These are called *skein relations.* The skein relations are considered for all possible embeddings of a ball. When strands are joined, framings should agree. The result of the factorization is the *linking number skein module of* M, denoted $\mathcal{L}_t(M)$. These skein modules were first introduced by Przytycki in [26].

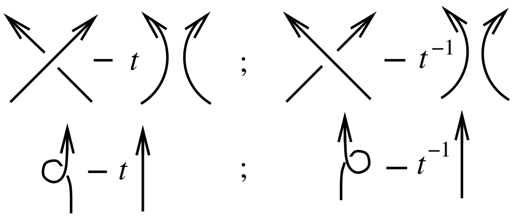

Fig. 1

If $M = S^3$, then each link L is, as an element of $\mathcal{L}_t(S^3)$, equivalent to the empty link with the coefficient equal to t raised to the sum of the linking numbers of ordered pairs of components and the writhes of the components, hence the name.

For a fixed even positive integer N we define the *reduced* linking number skein module of M, denoted $\widetilde{\mathcal{L}}_t(M)$, as the quotient of $\mathcal{L}_t(M)$ by $t = e^{i\pi/N}$ and $\gamma^N = \emptyset$ for every framed link component γ, where γ^N denotes N parallel copies of γ. As a rule, in a skein module t is a free variable, while in a reduced skein module it is a root of unity.

If $M = \mathbb{T}^2 \times [0,1]$, the topological operation of gluing a cylinder on top of another induces a multiplication in $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$ turning $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$ into an algebra, the *linking number skein algebra* of the cylinder over the torus. The multiplication descends to $\widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$. We explicate its structure.

For p and q coprime integers, orient the curve (p,q) by the vector from the origin to the point (p,q), and frame it so that the annulus is parallel to the torus. Call this the zero framing, or the *blackboard framing.* Any other framing of the curve (p,q) can be represented by an integer k, where $|k|$ is the number of full twists that are inserted on this curve, with k positive if the twists are positive, and k negative otherwise. In $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$, (p,q) with framing k is equivalent to $t^k(p,q)$.

If p and q are not coprime and n is their greatest common divisor, let $(p,q) = (p/n, q/n)^n$. Finally, $\emptyset = (0,0)$ is the empty link, the multiplicative identity of $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$.

Theorem 3.4 ([12]). *The algebra $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$ is isomorphic to the group algebra $\mathbb{C}[\mathbf{H}(\mathbb{Z})]$, with the isomorphism induced by*

$$t^k(p,q) \mapsto (p,q,k).$$

This map descends to an isomorphism between $\widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$ and the algebra $\mathcal{A}_N$ of Weyl quantizations of trigonometric polynomials.

Identifying the group algebra of the Heisenberg group with integer entries with $\mathbb{C}_t[U^{\pm 1}, V^{\pm 1}]$, we conclude that the linking number skein algebra of the cylinder over the torus is isomorphic to the ring of trigonometric polynomials in the noncommutative torus.

Let us look at the skein module of the solid torus $\mathcal{L}_t(S^1 \times \mathbb{D}^2)$. Let α be the curve that is the core of the solid torus, with a certain choice of orientation and framing. The reduced linking number skein module $\widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$ has basis α^j, $j = 0, 1, \ldots, N-1$.

Let h_0 be a homeomorphism of the torus to the boundary of the solid torus that maps the first generator of the fundamental group to a curve isotopic to α (a *longitude*) and the second generator to the curve on the boundary of the solid torus that bounds a disk in the solid torus (a *merid-*

ian). The operation of gluing $\mathbb{T}^2 \times [0,1]$ to the boundary of $S^1 \times \mathbb{D}^2$ via h_0 induces a left action of $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$ onto $\mathcal{L}_t(S^1 \times \mathbb{D}^2)$. This descends to a left action of $\widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$ onto $\widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$.

Observe that $\mathcal{L}_t(S^1 \times \mathbb{D}^2)$ and $\widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$ are quotients of $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$ respectively $\mathcal{L}_t(S^1 \times \mathbb{D}^2)$, with two framed curves equivalent on the torus if they are isotopic in the solid torus.

THEOREM 3.5 ([12]). *There is an isomorphism that intertwines the action of the algebra of Weyl quantizations of trigonometric polynomials on the space of theta functions and the representation of $\widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$ onto $\widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$, and which maps the theta series $\theta_j^\tau(z)$ to α^j for all $j = 0, 1, \dots, N-1$.*

REMARK 3.6. The choice of generators of $\pi_1(\mathbb{T}^2)$ completely determines the homeomorphism h_0, allowing us to identify the Hilbert space of the quantization with the vector space with basis $\alpha^0 = \emptyset, \alpha, \dots, \alpha^{N-1}$. As we have seen above, these basis elements are the theta series.

3.3. The discrete Fourier transform for classical theta functions from a topological viewpoint. The symmetries of classical theta functions are an instance of the Fourier transform. We put them in a topological perspective (see [12]). An element

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \tag{3.2}$$

defines an action of the mapping class group on the Weyl quantizations of exponentials given by

$$h \cdot \exp(pP + qQ + kE) = \exp[(ap+bq)P + (cp+dq)Q + kE].$$

This action is easy to describe in the skein-theoretical setting, it just maps every framed link γ on the torus to $h(\gamma)$.

THEOREM 3.7. *There is a projective representation ρ of the mapping class group of the torus on the space of theta functions that satisfies the exact Egorov identity*

$$h \cdot \exp(pP + qQ + kE) = \rho(h) \exp(pP + qQ + kE) \rho(h)^{-1}.$$

Moreover, for every h, $\rho(h)$ is unique up to multiplication by a constant.

Proof. We will exhibit two proofs of this well-known result, to which we will refer when discussing nonabelian Chern–Simons theory.

Proof 1. The map that associates to $\exp(pP + qQ + kE)$ the operator that acts on theta functions as

$$\theta_j^\tau \mapsto \exp[(ap+bq)P + (cp+dq)Q + kE]\theta_j^\tau$$

is also a unitary irreducible representation of the finite Heisenberg group which maps $\exp(kE)$ to multiplication by $e^{i\pi/N}$. By the Stone–von Neumann theorem, this representation is unitarily equivalent to the Schrödinger representation. This proves the existence of $\rho(h)$ satisfying the exact Egorov identity. By Schur's lemma, the map $\rho(h)$ is unique up to multiplication by a constant. Hence, if h and h' are two elements of the mapping class group, then $\rho(h' \circ h)$ is a constant multiple of $\rho(h')\rho(h)$. It follows that ρ defines a projective representation.

Proof 2. The map $\exp(pQ + qQ + kE) \to h \exp(pP + qQ + kE)$ extends to an automorphism of the algebra $\mathbb{C}[\mathbf{H}(\mathbb{Z})]$. Because the ideal by which we factor to obtain $\mathcal{A}_N$ is invariant under the action of the mapping class group, this automorphism induces an automorphism $\Phi : \mathcal{A}_N \to \mathcal{A}_N$, which maps each scalar multiple of the identity to itself. Since, by Proposition 3.3, $\mathcal{A}_N$ is the algebra of all linear operators on the N-dimensional space of theta functions, Φ is inner [33], meaning that there is $\rho(h) : \mathcal{A}_N \to \mathcal{A}_N$ such that $\Phi(x) = \rho(h)x\rho(h)^{-1}$. In particular

$$h \cdot \exp(pP + qQ + kE) = \rho(h) \exp(pP + qQ + kE)\rho(h)^{-1}.$$

The Schrödinger representation of $\mathcal{A}_N$ is obviously irreducible, so again we apply Schur's lemma and conclude that $\rho(h)$ is unique up to multiplication by a constant, and $h \mapsto \rho(h)$ is a projective representation. ∎

The representation ρ is the well-known action of the modular group given by discrete Fourier transforms.

As a consequence of Proposition 3.3, for any element h of the mapping class group, the linear map $\rho(h)$ is in $\widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$, hence it can be represented by a skein $\mathcal{F}(h)$. This skein satisfies

$$h(\sigma)\mathcal{F}(h) = \mathcal{F}(h)\sigma$$

for all $\sigma \in \widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$. Moreover $\mathcal{F}(h)$ is unique up to multiplication by a constant. We recall the formula for $\mathcal{F}(h)$ derived in [12].

Every 3-dimensional manifold is the boundary of a 4-dimensional manifold obtained by adding 2-handles $\mathbb{D}^2 \times \mathbb{D}^2$ to a 4-dimensional ball along the solid tori $\mathbb{D}^2 \times S^1$. On the boundary S^3 of the ball, the operation of adding handles gives rise to surgery on a framed link. Thus any given 3-dimensional manifold can be obtained as follows: Start with a suitable framed link $L \subset S^3$. Take a regular neighborhood of L made out of disjoint solid tori, each with a framing curve on the boundary such that the core of the solid torus and this curve determine the framing of the corresponding link component. Remove these tori, then glue them back in so that meridians are glued to framing curves in a suitable way. The result is the desired 3-dimensional manifold.

Sliding one 2-handle over another corresponds to sliding one link component along another using a Kirby band-sum move [18]. A *slide* of K_1 along K, denoted by $K_1 \# K$, is obtained as by cutting open the two knots and then joining the ends along the opposite sides of an embedded rectangle. The band sum is not unique.

An element h of the mapping class group of the torus can also be described by surgery along a framed link L in the cylinder over the torus. Surgery still yields a cylinder over the torus, but the homeomorphism to the original cylinder is identity on $\mathbb{T}^2 \times \{0\}$ and h on $\mathbb{T}^2 \times \{1\}$.

We introduce the element

$$\Omega_{\mathrm{U}(1)} = N^{-1/2} \sum_{j=0}^{N-1} \alpha^j \in \widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2).$$

The index stands for U(1) Chern–Simons theory (see §4.1). There is a well-known analogue for the group SU(2), to be discussed in §6.1. For a framed link L we denote by $\Omega_{\mathrm{U}(1)}(L)$ the skein obtained by replacing every link component by $\Omega_{\mathrm{U}(1)}$ such that α becomes the framing.

Theorem 3.8 ([12]). *Let h be an element of the mapping class group of the torus obtained by performing surgery on a framed link L_h in $\mathbb{T}^2 \times [0, 1]$. The discrete Fourier transform $\rho(h) : \widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2) \to \widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$ is given by*

$$\rho(h)\beta = \Omega_{\mathrm{U}(1)}(L_h)\beta.$$

Remark 3.9. This result was proved using the exact Egorov identity. For a framed curve γ on the torus, $h(\gamma)$ is obtained by sliding γ along the components of L_h. The exact Egorov identity for $\Omega_{\mathrm{U}(1)}(L_h)$ means that we are allowed to perform slides in the cylinder over the torus along curves colored by $\Omega_{\mathrm{U}(1)}$. This points to a surgery formula for U(1)-quantum invariants of 3-manifolds [12].

Like for the metaplectic representation, the representation of the mapping class group can be made into a true representation by passing to an extension of the mapping class group of the torus. While a $\mathbb{Z}_2$-extension would suffice, we consider a $\mathbb{Z}$-extension instead, in order to show the similarity with the nonabelian theta functions.

Let $\mathbf{L}$ be a subspace of $H_1(\mathbb{T}^2, \mathbb{R})$ spanned by a simple closed curve. Define the $\mathbb{Z}$-extension of the mapping class group of the torus by the multiplication rule on $\mathrm{SL}(2, \mathbb{Z}) \times \mathbb{Z}$,

$$(h', n') \circ (h, n) = (h' \circ h, n + n' - \boldsymbol{\tau}(\mathbf{L}, h'(\mathbf{L}), h' \circ h(\mathbf{L})),$$

where $\boldsymbol{\tau}$ is the Maslov index [22]. Standard results in the theory of theta functions show that the projective representation of the mapping class group of the torus lifts to a true representation of this group.

4. Nonabelian theta functions from geometric considerations

4.1. Nonabelian theta functions from geometric quantization. Let G be a compact simple Lie group, $\mathfrak{g}$ its Lie algebra, and Σ_g a closed oriented surface of genus $g \geq 1$. The *moduli space of* $\mathfrak{g}$*-connections* on Σ_g is the quotient of the affine space of all $\mathfrak{g}$-connections on Σ_g (or rather on the trivial principal G-bundle P on Σ_g) by the group $\mathcal{G}$ of gauge transformations $A \mapsto \phi^{-1}A\phi + \phi^{-1}d\phi$, with $\phi : \Sigma_g \to G$ a smooth function. The space of all connections has a symplectic 2-form given by

$$\omega(A, B) = -\int_{\Sigma_g} \operatorname{tr}(A \wedge B),$$

where A and B are connection forms in its tangent space. The group of gauge transformations acts on the space of connections in a Hamiltonian fashion, with moment map the curvature. The moduli space of *flat* $\mathfrak{g}$-connections,

$$\mathcal{M}_g = \{A \mid A \text{ a flat } \mathfrak{g}\text{-connection}\}/\mathcal{G},$$

arises as the symplectic reduction of the space of connections modulo gauge transformations. This space is the same as the character variety of G-representations of the fundamental group of Σ_g. It is an affine algebraic set over the reals, and its smooth part is a symplectic manifold. If Σ_g is a Riemann surface, then $\mathcal{M}_g$ can be identified with the moduli space of semistable holomorphic G-bundles over Σ_g.

Each curve γ on the surface and irreducible representation V of G define a classical observable on $\mathcal{M}_g$, $W_{\gamma,V}(A) = \operatorname{tr}_V \operatorname{hol}_\gamma(A)$, called a *Wilson line*, by taking the trace of the holonomy of the connection along γ in the representation V. Wilson lines are regular functions on $\mathcal{M}_g$. For $G = \mathrm{SU}(2)$ let the Wilson line for the n-dimensional irreducible representation be $W_{\gamma,n}$, with $W_\gamma = W_{\gamma,2}$. The W_γ's span the algebra of regular functions on $\mathcal{M}_g$.

The form ω induces a Poisson bracket, which for $G = \mathrm{SU}(2)$ was found by Goldman [13] to be

$$\{W_\alpha, W_\beta\} = \frac{1}{2} \sum_{x \in \alpha\cap\beta} \operatorname{sgn}(x)(W_{\alpha\beta_x^{-1}} - W_{\alpha\beta_x})$$

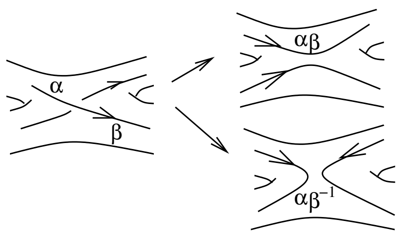

Fig. 2

where $\alpha\beta_x$ and $\alpha\beta_x^{-1}$ are computed as elements of the fundamental group with base point x (see Figure 2), and $\mathrm{sgn}(x)$ is the signature of the crossing: positive if the frame given by the tangent vectors to α and β is positively oriented with respect to the orientation of Σ_g, and negative otherwise.

The space $\mathcal{M}_g$, or rather the smooth part of it, is quantized in the direction of Goldman's Poisson bracket as follows. Set Planck's constant $h = 1/N$, with N an even positive integer.

The Hilbert space is obtained using the method of geometric quantization as the space of sections of a line bundle over $\mathcal{M}_g$, which is the tensor product of a line bundle with curvature $-2\pi i N\omega$ and a half-density [31]. The half-density is a square root of the canonical line bundle. Endow the surface with a complex structure, which induces a complex structure on $\mathcal{M}_g$ as follows. The tangent space to $\mathcal{M}_g$ at a nonsingular point A is the first cohomology group $H^1_A(\Sigma_g, \mathrm{ad}\, P)$ of the complex of $\mathfrak{g}$-valued forms

$$\Omega^0(\Sigma_g, \mathrm{ad}\, P) \xrightarrow{d_A} \Omega^1(\Sigma_g, \mathrm{ad}\, P) \xrightarrow{d_A} \Omega^2(\Sigma_g, \mathrm{ad}\, P).$$

Here P denotes the trivial principal G-bundle over Σ_g. Each complex structure on Σ_g induces a Hodge $*$-operator on the space of connections on Σ_g, hence a $*$-operator on $H^1_A(\Sigma_g, \mathrm{ad}\ P)$. The complex structure on $\mathcal{M}_g$ is $I : H^1_A(\Sigma_g, \mathrm{ad}\ P) \to H^1_A(\Sigma_g, \mathrm{ad}\ P)$, $IB = -*B$. For more details see [16]. The complex structure turns the smooth part of $\mathcal{M}_g$ into a complex manifold. Since the moduli space has a complex structure, one can then perform quantization in a Kähler polarization, so that the Hilbert space consists of the holomorphic sections of the prequantization line bundle. These are the *nonabelian theta functions.*

The analogue of the group algebra of the finite Heisenberg group is the algebra of operators quantizing Wilson lines. They arise in the theory of the Jones polynomial [14] as outlined by Witten [37], being defined via quantum field theory as integral operators with kernel

$$\langle A_1 | \mathrm{Op}(W_{\gamma,n}) | A_2 \rangle = \int_{\mathcal{M}_{A_1 A_2}} e^{iNL(A)} W_{\gamma,n}(A) \mathcal{D}A,$$

where A_1, A_2 are conjugacy classes of flat connections on Σ_g modulo the gauge group, A is a conjugacy class under the action of the gauge group on $\Sigma_g \times [0,1]$ such that $A_{\Sigma_g \times \{0\}} = A_1$ and $A_{\Sigma_g \times \{1\}} = A_2$, and

$$L(A) = \frac{1}{4\pi} \int_{\Sigma_g \times [0,1]} \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

is the Chern–Simons Lagrangian. The Feynman path integral defining the operator does not have a rigorous mathematical formulation, being thought of as an average of the Wilson line computed over all connections that interpolate between A_1 and A_2.

The skein-theoretic approach to classical theta functions outlined in §3.2 can be motivated by the Chern–Simons–Witten field theory point of view.

Wilson lines can be quantized either by one of the classical methods for quantizing the torus, or by using the Feynman path integrals as above. The Feynman path integral approach allows localizations of computations to small balls, in which a single crossing shows up. Witten [37] has explained that in each such ball skein relations hold, in this case the skein relations from Figure 1, of the linking number. As such, the path integral quantization gives rise to the skein-theoretic model.

On the other hand, Witten's quantization is symmetric under the action of the mapping class group of the torus, a property shared by Weyl quantization. And indeed, we have seen in §3.2 that Weyl quantization and the skein-theoretic quantization are the same. The relevance of Weyl quantization to Chern–Simons theory was first pointed out in [11] for the gauge group SU(2). For the gauge group U(1), it was noticed in [2].

4.2. The Weyl quantization of the moduli space of flat SU(2)-connections on the torus. The moduli space $\mathcal{M}_1$ of flat SU(2)-connections on the torus, called the *pillow case*, is the quotient of the torus

$$\{(e^{2\pi ix}, e^{2\pi iy}) \mid x, y \in \mathbb{R}\}$$

by the "antipodal" map $x \mapsto -y$, $y \mapsto -y$. It is the quotient of $\mathbb{R}^2$ by horizontal and vertical integer translations and by the symmetry σ with respect to the origin. Except for four singularities, $\mathcal{M}_1$ is a symplectic manifold, with symplectic form $\omega = 2\pi i dx \wedge dy$ and associated Poisson bracket

$$\{f, g\} = \frac{1}{2\pi i}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right).$$

The Weyl quantization of $\mathcal{M}_1$ in the complex polarization has been described in [11] for one particular complex structure. We do it now in general. Again Planck's constant is the reciprocal of an even integer, $\hbar = 1/N = 1/2r$.

The tangent space at an arbitrary point on the pillow case is spanned by the vectors $\partial/\partial x$ and $\partial/\partial y$. In the formalism of §4.1, these vectors are identified respectively with the cohomology classes of the su(2)-valued 1-forms

$$i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dx \quad \text{and} \quad i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dy.$$

It follows that a complex structure on the original torus induces exactly the same complex structure on the pillow case. So we can think of the pillow case as the quotient of the complex plane by the group generated by $\mathbb{Z} + \mathbb{Z}\tau$ ($\operatorname{Im}\tau > 0$) and the symmetry σ with respect to the origin. As before, we denote by (x, y) the coordinates in the basis $(1, \tau)$ and by $z = x + \tau y$

the complex variable. A fundamental domain for the group action in the (x,y)-coordinates is $\mathcal{D}=[0,1/2]\times[0,1]$.

As seen in [11], a holomorphic line bundle $\mathcal{L}_1$ with curvature $4\pi ir dx\wedge dy$ on the pillow case is defined by the cocycle $\Lambda_1:\mathbb{R}^2\times\mathbb{Z}^2\to\mathbb{C}^*$,

$$\begin{aligned}\Lambda_1((x,y),(m,n)) &= e^{4\pi ir(\tau n^2/2-2n(x+\tau y))}=e^{4\pi ir(\tau n^2/2-2nz)},\\ \Lambda_1((x,y),\sigma) &= 1.\end{aligned}$$

The square root of the canonical form is no longer the trivial line bundle, since for example the form dz is not defined globally on the pillow case. The obstruction for dz to be globally defined can be incorporated in a line bundle $\mathcal{L}_2$ defined by the cocycle $\Lambda_2:\mathbb{R}^2\times\mathbb{Z}^2\to\mathbb{C}^*$,

$$\Lambda_2((x,y),(m,n))=1,\qquad \Lambda_2((x,y),\sigma)=-1.$$

This line bundle can then be taken as the half-density.

The line bundle of the quantization is therefore $\mathcal{L}_1\otimes\mathcal{L}_2$, defined by the cocycle $\Lambda_1\Lambda_2$. The Hilbert space $\mathcal{H}_r(\mathbb{T}^2)$ of nonabelian theta functions on the torus consists of the holomorphic sections of this line bundle, which consists of the odd theta functions (this was discovered in [3]).

To specify a basis of $\mathcal{H}_r(\mathbb{T}^2)$ we need a pair of generators of the fundamental group. The complex structure and generators of $\pi_1(\mathbb{T}^2)$ determine a point in the Teichmüller space of the torus, specified by the complex number τ mentioned before. The orthonormal basis of the Hilbert space is

$$\zeta_j^\tau(z)=(\theta_j^\tau(z)-\theta_{-j}^\tau(z)),\qquad j=1,\dots,r-1,$$

where $\theta_j^\tau(z)$ are the theta series from §3.1. The definition of $\zeta_j^\tau(z)$ is extended to all indices by $\zeta_{j+2r}^\tau(z)=\zeta_j^\tau(z)$, $\zeta_0^\tau(z)=0$, and $\zeta_{r-j}^\tau(z)=-\zeta_{r+j}^\tau(z)$.

The space $\mathcal{H}_r(\mathbb{T}^2)$ can be embedded isometrically into $L^2(\mathcal{D})$, with the inner product

$$\langle f,g\rangle=2(-2ir(\tau-\bar\tau))^{1/2}\iint_{\mathcal{D}}f(x,y)\overline{g(x,y)}e^{-2\pi ir(\tau-\bar\tau)y^2}\,dx\,dy.$$

We can apply the Weyl quantization procedure. If p and q are coprime integers, then the Wilson line of the curve (p,q) of slope p/q on the torus for the 2-dimensional irreducible representation is

$$W_{(p,q)}(x,y)=\frac{\sin 4\pi(px+qy)}{\sin 2\pi(px+qy)}=2\cos 2\pi(px+qy),$$

when viewing the pillow case as a quotient of the plane (because the character of the 2-dimensional irreducible representation is $\sin 2x/\sin x$). If p and q are arbitrary integers, then $f(x,y)=2\cos 2\pi(px+qy)$ is a linear combination of Wilson lines. Indeed, if $n=\gcd(p,q)$ then

$$2\cos 2\pi(px+qy)=\frac{\sin\left[2\pi(n+1)\left(\frac{p}{n}x+\frac{q}{n}y\right)\right]}{\sin 2\pi\left(\frac{p}{n}x+\frac{q}{n}y\right)}-\frac{\sin\left[2\pi(n-1)\left(\frac{p}{n}x+\frac{q}{n}y\right)\right]}{\sin 2\pi\left(\frac{p}{n}x+\frac{q}{n}y\right)},$$

so $2\cos 2\pi(px+qy) = W_{\gamma,n+1} - W_{\gamma,n-1}$ where γ is the curve of slope p/q on the torus. This formula also shows that Wilson lines are linear combinations of cosines, so it suffices to understand the quantization of the cosines. Because

$$2\cos 2\pi(px+qy) = e^{2\pi i(px+qy)} + e^{-2\pi i(px+qy)},$$

the Weyl quantization of cosines can be obtained from the Schrödinger representation of the quantum observables that are invariant under the map $\exp P \mapsto \exp(-P)$ and $\exp Q \mapsto \exp(-Q)$, and restrict to odd theta functions. We obtain

$$\mathrm{Op}(2\cos 2\pi(px+qy))\zeta_j^\tau(z) = e^{-\frac{\pi i}{2r}pq}(e^{\frac{\pi i}{r}qj}\zeta_{j-p}^\tau(z) + e^{-\frac{\pi i}{r}qj}\zeta_{j+p}^\tau(z)).$$

In particular the ζ_j^τ's are the eigenvectors of $\mathrm{Op}(2\cos 2\pi y)$, corresponding to the holonomy along the curve $(0,1)$ on the torus. This shows that they are correctly identified as the analogues of the theta series.

5. Nonabelian theta functions from quantum groups

5.1. A review of the quantum group $U_\hbar(\mathrm{sl}(2,\mathbb{C}))$. For the gauge group SU(2), Reshetikhin and Turaev [28] constructed rigorously, using quantum groups, a topological quantum field theory that realizes Witten's programme. Within this theory, for each surface there is a vector space, an algebra of quantized Wilson lines, and a projective finite-dimensional representation of the mapping class group, the Reshetikhin–Turaev representation. Quantum group quantization has the advantage over geometric quantization that it does not depend on additional structures, such as the polarization.

Set $\hbar = \frac{1}{N} = \frac{1}{2r}$, and furthermore $r > 1$. Let $t = e^{\frac{i\pi}{2r}}$ and, for an integer n, let

$$[n] = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}} = \frac{\sin\frac{n\pi}{r}}{\sin\frac{\pi}{r}},$$

called a *quantized integer*.

The quantum group associated to SU(2), denoted $U_\hbar(\mathrm{sl}(2,\mathbb{C}))$, is obtained by passing to the complexification $\mathrm{SL}(2,\mathbb{C})$ of SU(2), taking the universal enveloping algebra of its Lie algebra, then deforming this algebra with respect to $\hbar$. It is the Hopf algebra over $\mathbb{C}$ with generators X, Y, K, K^{-1} subject to the relations

$$KK^{-1} = K^{-1}K = 1, \qquad KX = t^2XK, \qquad KY = t^{-2}YK,$$
$$XY - YX = \frac{K^2 - K^{-2}}{t^2 - t^{-2}}.$$

At the root of unity, namely when $N = 2r$, with r an integer, one has the additional factorization relations $X^r = Y^r = 0$, $K^{4r} = 1$ ([1]).

As opposed to SU(2), at roots of unity $U_\hbar(\mathrm{sl}(2,\mathbb{C}))$ has only $4r-4$ irreducible representations, among which we select one family $V^1, V^2, \ldots, V^{r-1}$ (for details see [28] or [19]). For each k, the space V^k has basis e_j, $j = -k_0, \ldots, k_0-1, k_0$, where $k_0 = (k-1)/2$, and the quantum group acts on it by

$$Xe_j = [k_0+j+1]e_{j+1}, \qquad Ye_j = [k_0-j+1]e_{j-1}, \qquad Ke_j = t^{2j}e_j.$$

The highest weight vector of this representation is e_{k_0}; it spans the kernel of X, is a cyclic vector for Y, and an eigenvector of K.

The Hopf algebra structure of $U_\hbar(\mathrm{sl}(2,\mathbb{C}))$ makes duals and tensor products of representations be representations themselves. Moreover, there is a (nonnatural) isomorphism of representations $D : V^{k*} \to V^k$. A Clebsch–Gordan theorem holds,

$$V^m \otimes V^n = \bigoplus_p V^p \oplus B,$$

where p runs through all indices such that $m+n+p$ is odd and $|m-n|+1 \le p \le \min(m+n-1, 2r-1-m-n)$ and B is a representation that is ignored because it has no effect on computations.

A corollary of the Clebsch–Gordan theorem is the following formula:

$$V^n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n-j}{j} (V^2)^{n-2j} = S_{n-1}(V^2) \qquad \text{for } n = 1, \ldots, r-1.$$

Here $S_n(x)$ is the *Chebyshev polynomial of the second kind* defined recursively by

$$S_{n+1}(x) = xS_n(x) - S_{n-1}(x), \qquad S_0(x) = 1, \qquad S_1(x) = x.$$

We define the restricted representation ring $R(U_\hbar(\mathrm{sl}(2,\mathbb{C})))$ as the ring generated by V^j, $j = 1, \ldots, r-1$, with multiplication $V^m \otimes V^n = \sum_p V^p$, where the sum is taken over all indices p that satisfy the conditions from the Clebsch–Gordan theorem.

PROPOSITION 5.1. *The restricted representation ring $R(U_\hbar(\mathrm{sl}(2,\mathbb{C})))$ is isomorphic to $\mathbb{C}[V^2]/S_{r-1}(V^2)$. If we define $V^n = S_{n-1}(V^2)$ in this ring for all $n \ge 0$, then $V^{r+n} = -V^{r-n}$, $V^r = 0$, and $V^{n+2r} = V^n$ for all $n > 0$.*

5.2. The quantum group quantization of the moduli space of flat SU(2)-connections on a surface of genus greater than 1. The quantization of the moduli space $\mathcal{M}_g$ of flat SU(2)-connections on a surface Σ_g uses *ribbon graphs and framed links* embedded in 3-dimensional manifolds.

([1]) In this case the quantum group is denoted by U_t in [28] and by $\mathcal{A}$ in [19].

A ribbon graph consists of the embeddings in the 3-dimensional manifold of finitely many connected components, each of which is homeomorphic to either an annulus or a tubular neighborhood of a planar trivalent graph in the plane. Intuitively, one can think of the edges as being ribbons, hence the name. When embedding the ribbon graph in a 3-dimensional manifold, the framings keep track of the twistings of edges. A framed link is a particular case of a ribbon graph. The link components and the edges of ribbon graphs are oriented. All ribbon graphs depicted below are taken with the "blackboard framing", meaning that the ribbon is in the plane of the paper.

With these conventions, let us quantize $\mathcal{M}_g$. The Hilbert space $\mathcal{H}_r(\Sigma_g)$ is defined by specifying a basis, the analogue of the theta series. To *specify a basis* of the space of theta functions we need a pair of generators of $\pi_1(\mathbb{T}^2)$; analogously here we need an *oriented rigid structure* on the surface. This is a collection of simple closed curves that decompose it into pairs of pants, together with "seams" that keep track of the twistings. The seams are simple closed curves that, when restricted to any pair of pants, give three nonintersecting arcs that connect pairwise the boundary components. An oriented rigid structure is one in which the decomposing curves are oriented. An example is shown in Figure 3(a), with decomposing curves drawn with continuous lines, and seams with dotted lines.

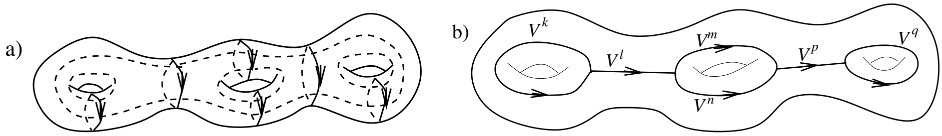

Fig. 3

Given an oriented rigid structure, map Σ_g to the boundary of a handlebody H_g so that the decomposition curves bound disks in H_g. The disks cut H_g into balls. Take the oriented framed trivalent graph that is the core of H_g, with a vertex at the center of each ball, an edge for each disk, and frame edges parallel to the region of the surface that lies between the seams. The disks are oriented by the decomposition curves on the boundary, and the orientation of the edges should agree with that of the disks.

The vectors forming a basis of $\mathcal{H}_r(\Sigma_g)$ consist of all colorings of this framed oriented trivalent graph by V^j's so that at each vertex the three indices satisfy the conditions from the Clebsch–Gordan theorem (the double inequality is invariant under permutations of m, n, p). Such a coloring is called *admissible*. For the rigid structure from Figure 3(a), a basis element is shown in Figure 3(b). The inner product $\langle \cdot, \cdot \rangle$ is defined so that these basis elements are orthogonal. That we can represent nonabelian theta functions as such graphs follows from the relation between theta functions and conformal field theory, [37].

The matrix of the operator $\mathrm{Op}(W_{\gamma,n})$ associated to the Wilson line $W_{\gamma,n}$: $A \mapsto \mathrm{tr}_{V^n} \mathrm{hol}_\gamma(A)$ is computed as follows. First, let $1 \le n \le r-1$. Place Σ_g in standard position in S^3 so that it bounds a genus g handlebody on each side. Draw a curve γ on Σ_g, frame it parallel to the surface, and color it by the representation V^n of $U_\hbar(\mathrm{sl}(2,\mathbb{C}))$. Add two basis elements e_p and e_q, viewed as admissible colorings by irreducible representation of the cores of the interior, respectively exterior, handlebodies (see Figure 4(a)). The oriented rigid structures on the boundaries of the two handlebodies should coincide.

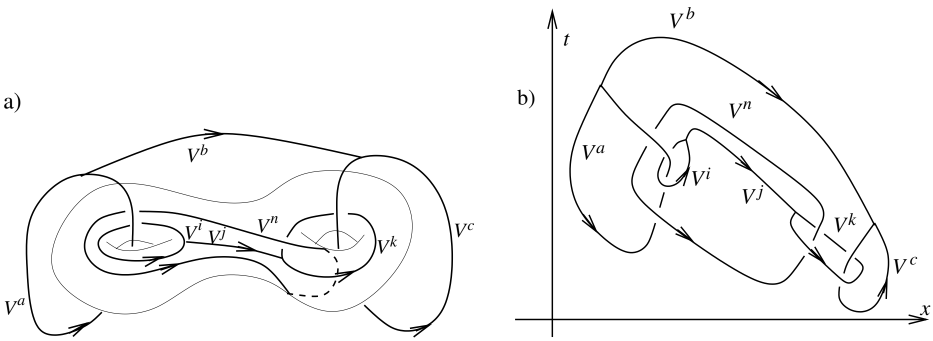

Fig. 4

Erase the surface to obtain an oriented tangled ribbon graph in S^3 whose edges are decorated by irreducible representations of $U_\hbar(\mathrm{sl}(2,\mathbb{C}))$ (Figure 4(b)). Project this graph onto a plane while keeping track of the crossings. Reshetikhin–Turaev theory [28] shows how to associate a number to this ribbon graph, which is independent of the particular projection; the Reshetikhin–Turaev invariant of the ribbon graph.

In short, the Reshetikhin–Turaev invariant is computed as follows. The ribbon graph is mapped by an isotopy to one whose projection can be cut by finitely many horizontal lines into slices, each containing one of the phenomena from Figure 5 and some vertical strands. To each horizontal line slicing the graph, associate the tensor product of the representations that color the crossing strands, when pointing downwards, or their duals, when pointing upwards. To the phenomena from Figure 5 associate, in order, the following operators:

- the flipped universal R-matrix $\check{R} : V^m \otimes V^n \to V^n \otimes V^m$ (obtained by composing the universal R-matrix with the flip $v \otimes w \mapsto w \otimes v$),
- the inverse $\check{R}^{-1}$ of $\check{R}$,

- the projection operator $\beta_p^{mn} : V^m \otimes V^n \to V^p$, whose existence and uniqueness is guaranteed by the Clebsch–Gordan theorem,
- the inclusion $\beta_{mn}^p : V^p \to V^m \otimes V^n$,
- the contraction $E : V^{n*} \otimes V^n \to \mathbb{C}$, $E(f \otimes x) = f(x)$,
- its dual $N : \mathbb{C} \to V^n \otimes V^{n*}$, $N(1) = \sum_j e_j \otimes e^j$,
- the isomorphism $D : V^{n*} \to V^n$,
- and its dual $D^* : V^{n*} \to V^{n**} = V^n$ (see [19, Lemma 3.18] for the precise identification of V^{n**} with V^n).

Compose the operators from the bottom to the top to obtain a linear map from $\mathbb{C}$ to $\mathbb{C}$, which must be of the form $z \mapsto \lambda z$. The number λ is the *Reshetikhin–Turaev invariant* of the ribbon graph. The blank coupons, i.e. the maps D, might be required in order to change the orientations of the three edges that meet at a vertex, to make them look as depicted in Figure 5.

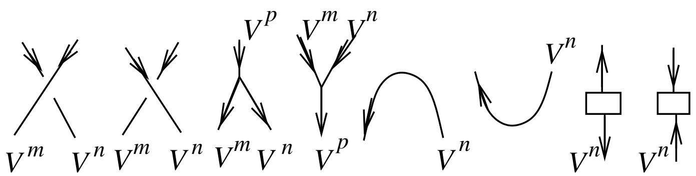

Fig. 5

Returning to the quantization of Wilson lines, the Reshetikhin–Turaev invariant of the graph is equal to $[\mathrm{Op}(W_{\gamma,n})e_p, e_q]$, where $[\cdot,\cdot]$ is the nondegenerate bilinear pairing on $\mathcal{H}_r(\Sigma_g)$ defined by erasing the curve colored by V^n in Figure 4(b). We think of it as being the p,q-entry of the matrix of the operator, although this is not quite true because the bilinear pairing is not the inner product. But because the pairing is nondegenerate (see Appendix), the above formula completely determines the operator associated to the Wilson line.

In view of Proposition 5.1, this definition of quantized Wilson lines is extended to arbitrary n by the conventions

$$\mathrm{Op}(W_{\gamma,r}) = 0, \quad \mathrm{Op}(W_{\gamma,r+n}) = -\mathrm{Op}(W_{\gamma,r-n}),$$
$$\mathrm{Op}(W_{\gamma,n+2r}) = -\mathrm{Op}(W_{\gamma,n}).$$

It was shown [1] that the quantization is in the direction of Goldman's Poisson bracket.

Remark 5.2. It is interesting that not all irreducible representations participate in the quantization of the moduli space. A similar phenomenon happens in the case of abelian Chern–Simons theory [10]. This is worth investigating.

5.3. Nonabelian theta functions from skein modules. We rephrase the construction from §5.2 in the language of skein modules. The goal is to express the quantum group quantization of Wilson lines as the left representation of a skein algebra on a quotient of itself, similar to the Schrödinger representation described as the left representation of the reduced linking number skein algebra of the cylinder over the torus on a quotient of itself (§3.2).

One usually associates to SU(2) Chern–Simons theory the skein modules of the Kauffman bracket. The Reshetikhin–Turaev topological quantum field theory has a Kauffman bracket analogue defined in [4]. However, the Kauffman bracket skein relations introduce sign discrepancies in the computation of the action of operators! Because Theorem 5.9 in §5.4 brings evidence that quantum group quantization is the nonabelian analogue of Weyl quantization, we define our modules by the skein relations found by Kirby and Melvin [19] for the Reshetikhin–Turaev version of the Jones polynomial.

First we replace oriented framed ribbon graphs colored by irreducible representations of $U_\hbar(\mathrm{sl}(2,\mathbb{C}))$ by formal sums of oriented framed links colored by the 2-dimensional irreducible representation. Two results are needed.

LEMMA 5.3. *For all $n = 3, 4, \ldots, r-1$ the identity from Figure* 6(a) *holds.*

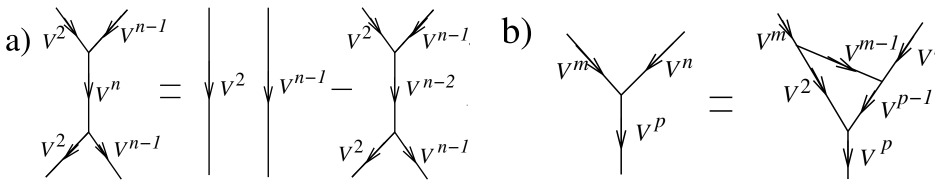

Fig. 6

Proof. This is a corollary of $V^n = V^2 \otimes V^{n-1} - V^{n-2}$. ∎

LEMMA 5.4. *For integers m, n, p satisfying $m+n+p$ odd and $|m-n|+1 \leq p \leq \min(m+n-1, 2r-1-m-n)$, the identity in Figure* 6(b) *holds.*

Proof. We assume familiarity with the proof of the quantum Clebsch–Gordan theorem in [28]. Set $m_0 = \frac{m-1}{2}$, $p_0 = \frac{p-1}{2}$. The morphism described by the diagram on the right is the composition of maps

$$V^p \xrightarrow{\beta^p_{2,p-1}} V^2 \otimes V^{p-1} \xrightarrow{1\otimes\beta^{p-1}_{mn}} V^2 \otimes (V^{m-1} \otimes V^n)$$
$$= (V^2 \otimes V^{m-1}) \otimes V^n \xrightarrow{\beta^{2,m-1}_m \otimes 1} V^m \otimes V^n.$$

By Schur's lemma and the quantum Clebsch–Gordan theorem, the composition is either zero or identity. To see that it is not the zero map, look at the highest weight vector $e_{p_0} \in V^p$. We have

$$e_{p_0} \mapsto e_{1/2} \otimes e_{p_0-1/2} \mapsto e_{1/2} \otimes \sum_{i+j=p_0} c_{ij} e_i \otimes e_j = \sum_{i+j=p_0} c_{ij} e_{1/2} \otimes e_i \otimes e_j.$$

The Clebsch–Gordan coefficients c_{ij} are nonzero, and in the sum there is a term $c_{m_0-1/2,j} e_{1/2} \otimes e_{m_0-1/2} \otimes e_j$.

The inclusion $\beta^m_{2,m-1} : V^m \to V^2 \otimes V^{m-1}$ maps the highest weight vector $e_{m_0} \in V^m$ to $e_{1/2} \otimes e_{m_0-1/2}$, which is the product of the vectors of highest weights in V^2, respectively V^{m-1}. Hence if the $1/2, m_0 - 1/2$-component of a vector v written in the basis $e_i \otimes e_j$ of $V^2 \otimes V^{m-1}$ is nonzero, then $\beta^m_{2,m-1} v$ is nonzero in V^m.

In particular, the above sum maps to a nonzero vector in $V^m \otimes V^n$. Hence the diagram on the right of Figure 6(b) equals the inclusion $\beta^p_{mn} : V^p \to V^m \otimes V^n$, proving the identity. ∎

Proposition 5.5. *There is an algorithm for replacing each connected ribbon graph Γ in S^3 colored by irreducible representations of $U_\hbar(\mathrm{sl}(2,\mathbb{C}))$ by a sum of oriented framed links colored by V^2 that lie in an ϵ-neighborhood of the graph, such that if in any ribbon graph Γ' that has Γ as a connected component we replace Γ by this sum of links, we obtain a ribbon graph with the same Reshetikhin–Turaev invariant as Γ'.*

Proof. For framed knots, the property follows from the cabling formula given in [19, Theorem 4.15]; a knot colored by V^n is replaced by $S_{n-1}(V^2)$.

If the connected ribbon graph has vertices, then by using the isomorphism D to identify irreducible representations of $U_\hbar(\mathrm{sl}(2,\mathbb{C}))$, with their duals, we can obtain the identity from Figure 6(b) with the arrows reversed. Also, by taking the adjoint of the map described by the diagram, we can turn it upside down, meaning that we can write a similar identity for β^{mn}_p.

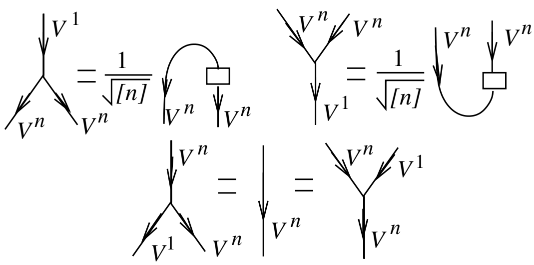

Fig. 7

Using the lemmas, the algorithm works as follows. First, use the identities in Figure 7 to remove all edges colored by V^1. Then apply repeatedly

Lemma 5.4 until at each vertex of the new ribbon graph at least one of the three edges is colored by V^2. Next, use Lemma 5.3 to obtain a sum of graphs with the property that, at each vertex, two of the three edges are colored by V^2 and one is colored by V^1. Then use the identities in Figure 7 for $n = 2$ to transform everything into a sum of framed links whose edges are colored by V^2. Each of the links in the sum has an even number of blank coupons (representing the isomorphism D or its dual) on each component. Cancel the coupons on each link component in pairs, adding a factor of -1 each time the two coupons are separated by an odd number of maxima on the link component. The result is a formal sum of framed links with components colored by V^2. ∎

Theorem 4.3 in [19] allows us to compute the Reshetikhin–Turaev invariant of a framed link whose components are colored by V^2 using skein relations. First, forget about the orientation of links. Next, if three framed links L, H, V in S^3 colored by V^2 coincide except in a ball where they look like in Figure 8, then their Reshetikhin–Turaev invariants, denoted by J_L, J_H, and J_V satisfy

$$J_L = tJ_H + t^{-1}J_V \quad \text{or} \quad J_L = \epsilon(tJ_H - t^{-1}J_V),$$

depending on whether the two crossing strands come from different components or not. Here ϵ is the sign of the crossing, obtained after orienting that link component (either orientation produces the same sign). Specifically, if the tangent vectors to the over and under strand form a positive frame then the sign is positive, otherwise it is negative. Additionally, if a link component bounds a disk inside a ball disjoint from the rest of the link, then it is replaced by a factor of $t^2 + t^{-2}$.

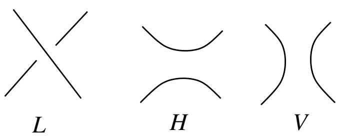

Fig. 8

This prompts us to introduce skein modules defined by these skein relations. Let for now t be an abstract variable, rather than the root of unity chosen at the beginning of §5.2. For an orientable 3-dimensional manifold M, consider the free $\mathbb{C}[t, t^{-1}]$ module with basis the isotopy classes of framed links in M including the empty link. Factor this module by the skein relations

$$L = tH + t^{-1}V \quad \text{or} \quad L = \epsilon(tH - t^{-1}V),$$

depending on whether the two crossing strands come from different components or not, where the links L, H, V are identical except in an embedded ball, in which they look as in Figure 8. The same convention for ϵ is used as before, with the orientation of the crossing decided inside the ball. Also replace any trivial link component that lies inside a ball disjoint from the rest of the link by a factor of t^2+t^{-2}. We call the result of the factorization the *Reshetikhin–Turaev skein module* and denote it by $\mathrm{RT}_t(M)$. Notice that $\mathrm{RT}_t(M)$ is isomorphic as a module to the Kauffman bracket skein module.

If $M=\Sigma_g\times[0,1]$, then the homeomorphism

$$\Sigma_g\times[0,1]\cup_{\Sigma_g}\Sigma_g\times[0,1]\approx\Sigma_g\times[0,1]$$

induces a multiplication on $R_t(\Sigma_g\times[0,1])$, turning it into an algebra, the *Reshetikhin–Turaev skein algebra.* This algebra is *not* canonically isomorphic to the Kauffman bracket skein algebra except in genus one. In higher genus the multiplication rules are different, as can be seen by examining the product of a separating and a nonseparating curve that intersect.

The operation of gluing $\Sigma_g\times[0,1]$ to the boundary of a genus g handlebody H_g by a homeomorphism of the surface induces an $\mathrm{RT}_t(\Sigma_g\times[0,1])$ module structure on $\mathrm{RT}_t(H_g)$. Moreover, by gluing H_g with the empty skein inside to $\Sigma_g\times[0,1]$, we see that $\mathrm{RT}_t(H_g)$ is the quotient of $\mathrm{RT}_t(\Sigma_g\times[0,1])$ obtained by identifying the skeins in $\Sigma_g\times[0,1]$ that are isotopic in H_g.

By Lemma 5.3 and the identities in Figure 7, the irreducible representations V^n can be represented by skeins. Explicitly, $V^n=S_{n-1}(V^2)=f^{n-1}$, where f^n is defined recursively in Figure 9. These are the Jones–Wenzl idempotents [15], [36].

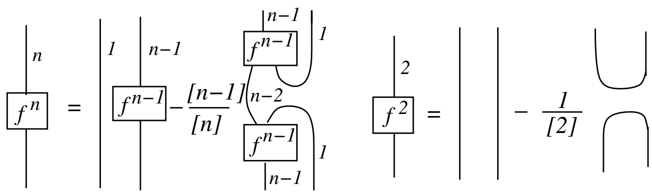

Fig. 9

The condition $S_{r-1}(V^2)=0$ translates to $f^{r-1}=0$. This yields the *reduced Reshetikhin–Turaev skein module* $\widetilde{\mathrm{RT}}_t(M)$, obtained by factoring $\mathrm{RT}_t(M)$ by $t=e^{i\pi/2r}$ and the skein relation $f^{r-1}=0$, taken in every embedded ball. The reduction is compatible with multiplication $\mathrm{RT}_t(\Sigma_g\times[0,1])$ and with its action on $\mathrm{RT}_t(H_g)$.

Proposition 5.6. *The quantum group quantization of the moduli space of flat* SU(2)*-connections on a surface Σ_g and of Wilson lines is isomorphic to the left action of $\widetilde{\mathrm{RT}}_t(\Sigma_g\times[0,1])$ on $\widetilde{\mathrm{RT}}_t(H_g)$.*

Proof. The proof is based on Propositions 5.5 and 5.1. Because each f^n involves n parallel strands, $\mathrm{RT}_t(H_g)$ is a free $\mathbb{C}[t,t^{-1}]$-module with basis the skeins obtained by

- replacing each edge of the core of H_g by a Jones–Wenzl idempotent in such a way that, if f^m, f^n, f^p meet at a vertex, then $m+n+p$ is even, $m+n \leq p$, $m+p \leq n$, $n+p \leq m$, and
- replacing the vertices by the unique collection of strands that lie in a disk neighborhood of the vertex and join the ends of the three Jones–Wenzl idempotents meeting there in such a way that there are no crossings.

By the Clebsch–Gordan theorem and Proposition 5.1, in $\widetilde{\mathrm{RT}}_t(H_g)$, only edges colored by f^n with $n \leq r-2$ need be considered, and if f^m, f^n, f^p meet at a vertex, then $m+1, n+1, p+1$ and their cyclic permutations must satisfy the double inequality from the Clebsch–Gordan theorem. Each element of this form comes from a basis element in the quantum group quantization. A detailed explanation of this can be found, for Kauffman bracket skein modules, in [21].

The computation in Figure 10, performed in the dotted annulus, shows that for a simple closed curve γ on the torus, $\mathrm{Op}(W_{\gamma,n})$ can be identified with the skein $S_{n-1}(\gamma) \in \widetilde{\mathrm{RT}}_t(\Sigma_g \times [0,1])$. We conclude that the action of quantum observables on the Hilbert space is modeled by the action of $\widetilde{\mathrm{RT}}_t(\Sigma_g \times [0,1])$ on $\widetilde{\mathrm{RT}}_t(H_g)$.

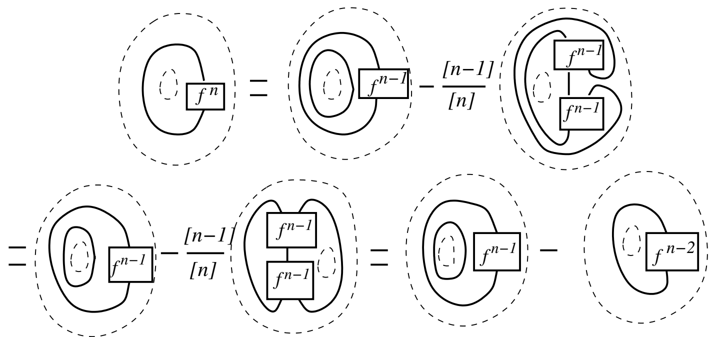

Fig. 10

To identify the two quantization models, we also have to prove that the skeins associated to admissible colorings of the core of the handlebody form a basis, namely that they are linearly independent in $\widetilde{\mathrm{RT}}_t(H_g)$.

The smooth part of $\mathcal{M}_g$ has real dimension $6g-6$, and is a completely integrable manifold in the Liouville sense. Indeed, the Wilson lines W_{α_i}, where α_i, $i=1,\ldots,3g-3$, are the curves in Figure 11(a), form a maximal set of Poisson commuting functions (meaning that $\{W_{\alpha_i}, W_{\alpha_j}\} = 0$). The quantum group quantization of the moduli space is thus a quantum inte-

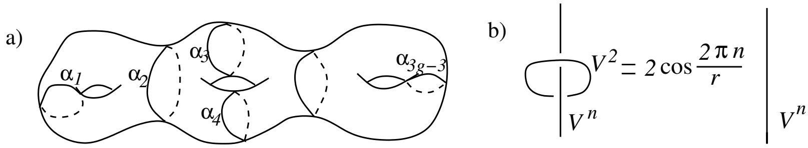

Fig. 11

grable system, with the operators $\mathrm{Op}(W_{\alpha_1}),\ldots,\mathrm{Op}(W_{\alpha_{3g-3}})$ satisfying the integrability condition.

The identity in Figure 11(b), which holds for any choice of orientation of strands, implies that the spectral decomposition of the commuting $(3g-3)$-tuple of self-adjoint operators $(\mathrm{Op}(W_{\alpha_1}),\ldots,\mathrm{Op}(W_{\alpha_{3g-3}}))$ has only 1-dimensional eigenspaces consisting precisely of the colorings of the edges following the given rule. Indeed, the basis elements are as described in §5.2 for the case where the curves that cut the surface into pairs of pants are $\alpha_1,\ldots,\alpha_{3g-3}$, and the identity in Figure 11(b) shows that the eigenvalues of an e_j with respect to the $3g-3$ quantized Wilson lines completely determine the colors of its edges. ■

Remark 5.7. Proposition 5.6 should be compared with Theorem 3.5. Again the algebra of quantized observables is a skein algebra, the space of nonabelian theta functions is a quotient of this algebra, and the factorization relation is of topological nature: it is defined by gluing the cylinder over the surface to a handlebody via a homeomorphism. The skein modules $\mathrm{RT}_t(\Sigma_g\times[0,1])$ and $\widetilde{\mathrm{RT}}_t(\Sigma_g\times[0,1])$ are the analogues, for the gauge group SU(2), of the algebras $\mathbb{C}[\mathbf{H}(\mathbb{Z})]$ and $\mathcal{A}_N$.

Since we have not yet proved that the pairing $[\cdot,\cdot]$ from §5.2 is nondegenerate, we take for the moment this representation of $\widetilde{\mathrm{RT}}_t(\Sigma_g\times[0,1])$ to be the quantum group quantization of the moduli space $\mathcal{M}_g$. Nondegeneracy is proved in Appendix.

The quantum group quantization is more natural than it seems. Quantum groups were introduced by Drinfeld as means of finding operators that satisfy the Yang–Baxter equation. They lead to the deformation quantization of $\mathcal{M}_g$ in [1]. This gives rise to the skein algebra of the surface, and by analogy with §3.2 we are led to consider the skein module of the handlebody. The basis consisting of admissible colorings of the core of the handlebody appears when looking at the spectral decomposition of the commuting operators in Proposition 5.6.

5.4. The quantum group quantization of the moduli space of flat SU(2)-connections on the torus. The quantum group quantization of $\mathcal{M}_1$ is a particular case of the construction in §5.2 (see [11]). A basis for the Hilbert space is specified by an oriented rigid structure on the torus.

The curves a and b define such a structure with a the seam and b the curve that cuts the torus into an annulus. Mapping the torus to the boundary of the solid torus to make b null-homologous and a the generator of the fundamental group, we get an orthonormal basis consisting of the vectors $V^1(\alpha), \ldots, V^{r-1}(\alpha)$—the colorings of the core α of the solid torus by the irreducible representations $V^1, \ldots, V^{r-1}$ of $U_\hbar(\mathrm{sl}(2,\mathbb{C}))$. These are the quantum group analogues of the ζ_j^τ's. Here, the orientation of the rigid structure, and hence of the core of the solid torus, is irrelevant: reversing the orientation gives the same results in computations (orientation of link components is irrelevant [37]).

The operator associated to the function $f(x,y) = 2\cos 2\pi(px+qy)$ is computed like for higher genus surfaces. The bilinear form on the Hilbert space comes from the Hopf link and is

$$[V^j(\alpha), V^k(\alpha)] = [jk], \qquad j,k = 1, \ldots, r-1.$$

The value of $[\mathrm{Op}(2\cos 2\pi(px+qy))V^j(\alpha), V^k(\alpha)]$ is equal to the Reshetikhin–Turaev invariant of the three-component colored framed link consisting of the curve of slope p/q on the torus embedded in standard position in S^3, colored by $V^{n+1} - V^{n-1}$ where $n = \gcd(p,q)$, the core of the solid torus that lies on one side of the torus, colored by V^j, and the core of the solid torus that lies on the other side, colored by V^k. Coloring the curve by $V^{n+1} - V^{n-1}$ is the same as coloring it by $T_n(V^2)$, where $T_n(x)$ is the Chebyshev polynomial of the first kind defined by $T_0(x) = 2$, $T_1(x) = x$, $T_{n+1}(x) = xT_n(x) - T_{n-1}(x)$ for $n \geq 1$. Again, the quantum group quantization is modeled as the action of the reduced Reshetikhin–Turaev skein algebra of the torus on the reduced Reshetikhin–Turaev skein module of the solid torus.

It was shown in [11] that the quantum group quantization of $\mathcal{M}_1$ is unitarily equivalent to Weyl quantization. However, that proof uses the Reshetikhin–Turaev representation of the mapping class group, and does not serve our purpose of showing *how* the Reshetikhin–Turaev representation arises from quantum mechanics. For that reason we give here a different proof of this result.

For $p,q \in \mathbb{Z}$, let $n = \gcd(p,q)$, and let $(p,q)_T = T_n((p/n, q/n)) \in \mathrm{RT}_t(\mathbb{T}^2 \times [0,1])$. The proof of the following result is identical to that of Theorem 4.1 in [7], which covers the case of the Kauffman bracket.

Theorem 5.8. *For any integers p, q, p', q' the following product-to-sum formula holds:*

$$(p,q)_T(p',q')_T = t^{\begin{vmatrix} p & q \\ p' & q' \end{vmatrix}}(p+p', q+q')_T + t^{-\begin{vmatrix} p & q \\ p' & q' \end{vmatrix}}(p-p', q-q')_T.$$

So the Reshetikhin–Turaev and the Kauffman bracket skein algebras of the torus are isomorphic.

THEOREM 5.9 ([11]). *The Weyl quantization and the quantum group quantization of the moduli space of flat* SU(2)*-connections on the torus are unitarily equivalent.*

Proof. We rephrase the quantum group quantization in terms of skein modules. The Hilbert space is $\widetilde{\mathrm{RT}}_t(S^1\times\mathbb{D}^2)$. Indeed, $\widetilde{\mathrm{RT}}_t(S^1\times\mathbb{D}^2)$ is spanned by the vectors $S_{j-1}(\alpha)$, $j=1,\dots,r-1$, and these are linearly independent because they are eigenvectors with different eigenvalues of the operator defined by $(0,1)$.

Considering the projection $\pi:\mathrm{RT}_t(\mathbb{T}^2\times[0,1])\to\mathrm{RT}_t(S^1\times\mathbb{D}^2)$ defined by attaching the cylinder over the torus to the solid torus by the homeomorphism h_0 from §3.2, and using Theorem 5.8, we find the recursion

$$\pi((p+1,q)_T)=t^{-q}\alpha\pi((p,q)_T)-t^{-2q}\pi((p-1,q)_T).$$

Also $\pi((0,q)_T)=t^{2q}+t^{-2q}$, and $\pi((1,q)_T)=t^{-2q}\alpha$. Solving the recursion we get

$$\pi((p,q)_T)=t^{-pq}(t^{2q}S_p(\alpha)-t^{-2q}S_{p-2}(\alpha)).$$

Again using Theorem 5.8 we have

$$\begin{aligned}(p,q)_TT_j(\alpha)&=\pi[(p,q)_T(j,0)_T]=\pi[t^{-jq}(p+j,q)_T+t^{jq}(p-j,q)_T]\\&=t^{-pq}[t^{-(2j-2)q}S_{p+j}(\alpha)+t^{(2j+2)q}S_{p-j}(\alpha)-t^{-(2j-2)q}S_{p+j-2}(\alpha)-t^{(2j-2)q}S_{p-j-2}(\alpha)].\end{aligned}$$

Since $T_n(x)=S_n(x)-S_{n-2}(x)$ for all n, we have

$$(p,q)_TS_{j-1}(\alpha)=t^{-pq}(t^{-2qj}S_{p+j-1}(\alpha)+t^{2qj}S_{p-j+1}(\alpha)),\ \text{for } j>0.$$

Reducing to the relative skein modules and using the equality $S_{j-1}(\alpha)=V^j(\alpha)$, we get

$$\begin{aligned}&\mathrm{Op}(2\cos 2\pi(px+qy))V^j(\alpha)\\&\qquad=e^{-\frac{\pi i}{2r}pq}(e^{\frac{\pi i}{r}qj}V^{j-p}(\alpha)+e^{-\frac{\pi i}{r}qj}V^{j+p}(\alpha)).\end{aligned}\tag{5.1}$$

This is the desired formula for the Weyl quantization of the pillow case from §4.2. ∎

5.5. A Stone–von Neumann theorem on the moduli space of flat SU(2)-connections on the torus. Weyl quantization yields a representation of $\widetilde{\mathrm{RT}}_t(\mathbb{T}\times[0,1])$ so that t acts as multiplication by $e^{i\pi/2r}$ and every simple closed curve on the torus acts as a self-adjoint operator. The last condition is necessary, since for the group SU(2) Wilson lines are real-valued, so their quantizations are self-adjoint operators. The algebra $\widetilde{\mathrm{RT}}_t(\mathbb{T}\times[0,1])$ is a nonabelian analogue of the group algebra of the finite Heisenberg group. A Stone–von Neumann theorem also holds in this case.

THEOREM 5.10. *The representation of the reduced Reshetikhin–Turaev skein algebra of the torus defined by the Weyl quantization of the moduli* *space of flat* su(2)*-connections on the torus is the* unique *irreducible representation of this algebra that maps simple closed curves to self-adjoint operators and t to multiplication by $e^{\pi i/2r}$. Moreover, the quantized Wilson lines span the algebra of all linear operators on the Hilbert space of the quantization.*

Proof. We prove irreducibility by showing that any vector is cyclic. Because the eigenspaces of each quantized Wilson line are 1-dimensional, in particular those of $\mathrm{Op}(2\cos 2\pi y)$, it suffices to check this property for the eigenvectors of this operator, namely for ζ_j^τ, $j = 1, \dots, r-1$. And since

$$\begin{aligned}\mathrm{Op}(2\cos 2\pi x)\zeta_j^\tau &= \zeta_{j-1}^\tau + \zeta_{j+1}^\tau,\\ \mathrm{Op}(2\cos 2\pi(x+y))\zeta_j^\tau &= t^{-1}(t^2\zeta_{j-1}^\tau + t^{-2}\zeta_{j+1}^\tau),\end{aligned}$$

by taking linear combinations we see that from ζ_j^τ we can generate both ζ_{j+1}^τ and ζ_{j-1}^τ. Repeating, we can generate the entire basis. This shows that ζ_j^τ is cyclic for each $j = 1, \dots, r-1$, hence the representation is irreducible.

To prove uniqueness, consider an irreducible representation of $\widetilde{\mathrm{RT}}_t(\mathbb{T}^2 \times [0,1])$ with the required properties. The condition $S_{r-1}(\gamma) = 0$ for any simple closed curve γ on the torus implies, by the spectral mapping theorem, that the eigenvalues of the operator associated to γ are among the numbers $2\cos\frac{k\pi}{r}$, $k = 1, \dots, r-1$.

For generators $X = (1,0)$, $Y = (0,1)$, and $Z = (1,1)$ of $\widetilde{\mathrm{RT}}_t(\mathbb{T}^2 \times [0,1])$ we write the relations

$$\begin{aligned}&tXY - t^{-1}YX = (t^2 - t^{-2})Z, \qquad tYZ - t^{-1}ZY = (t^2 - t^{-2})X,\\ &tZX - t^{-1}XZ = (t^2 - t^{-2})Y,\\ &t^2X^2 + t^{-2}Y^2 + t^2Z^2 - tXYZ - 2t^2 - 2t^{-2} = 0,\end{aligned}$$

by analogy with the presentation of the Kauffman bracket skein algebra of the torus found by Bullock and Przytycki [5]. In fact $\widetilde{\mathrm{RT}}_t(\mathbb{T}^2 \times [0,1])$ is generated by just X and Y, since we can substitute Z from the first equation. We thus have

$$\begin{aligned}&(t^2 + t^{-2})YXY - (XY^2 + Y^2X) = (t^4 + t^{-4} - 2)X,\\ &(t^2 + t^{-2})XYX - (YX^2 + X^2Y) = (t^4 + t^{-4} - 2)Y,\\ &(t^6 + t^{-2} - 2t^2)X^2 + (t^{-6} + t^2 - 2t^{-2})Y^2 + XYXY + YXYX\\ &\qquad\qquad - t^2YX^2Y - t^{-2}XY^2X = 2(t^6 + t^{-6} - t^2 - t^{-2}).\end{aligned} \tag{5.2}$$

On setting $t = e^{i\pi/2r}$ the first equation in (5.2) becomes

$$2\cos\frac{\pi}{r}\,YXY - (XY^2 + Y^2X) = 4\sin^2\frac{\pi}{r}\,Y.$$

Let v_k be an eigenvector of Y with eigenvalue $2\cos\frac{k\pi}{r}$ for some $k \in \{1, \dots, r-1\}$. We wish to generate a basis of the representation by acting repeatedly on v_k by X. For this, set $Xv_k = w$. The above relation yields

$$2\cos\frac{\pi}{r}\cdot 2\cos\frac{k\pi}{r}\,Yw - 4\cos^2\frac{k\pi}{r}\,w - Y^2w = 4\sin^2\frac{\pi}{r}\,w.$$

Rewrite this as

$$\left[Y^2 - 4\cos\frac{k\pi}{r}\cos\frac{\pi}{r}\,Y - 4\left(\sin^2\frac{\pi}{r} + \cos^2\frac{k\pi}{r}\right)\right]w = 0.$$

It follows that either $w = 0$ or w is in the kernel of the operator

$$Y^2 - 4\cos\frac{k\pi}{r}\cos\frac{\pi}{r}\,Y - 4\left(\sin^2\frac{\pi}{r} + \cos^2\frac{k\pi}{r}\right)\mathrm{Id}. \tag{5.3}$$

The second equation in (5.2) shows that if $Xv_k = w = 0$ then $Yv_k = 0$. This is impossible because of the third relation in (5.2). Hence $w \neq 0$, so w lies in the kernel of the operator (5.3). Note that if λ is an eigenvalue of Y which satisfies

$$\lambda^2 - 4\cos\frac{k\pi}{r}\cos\frac{\pi}{r}\lambda - 4\left(\sin^2\frac{\pi}{r} + 4\cos^2\frac{k\pi}{r}\right) = 0,$$

then necessarily $\lambda = 2\cos\frac{(k\pm 1)\pi}{r}$. It follows that

$$Xv_k = v_{k+1} + v_{k-1},$$

where $Yv_{k\pm 1} = 2\cos\frac{(k\pm 1)\pi}{4}v_{k\pm 1}$, and v_{k+1} and v_{k-1} are not simultaneously equal to zero. We wish to enforce v_k, v_{k+1}, and v_{k-1} to be elements of a basis. For that we need to check that $v_{k+1}, v_{k-1} \neq 0$, and understand Xv_{k+1} and Xv_{k-1}.

Set $Xv_{k+1} = \alpha v_k + v_{k+2}$ and $Xv_{k-1} = \beta v_k + v_{k-2}$, where $Yv_{k\pm 2} = 2\cos\frac{(k\pm 2)\pi}{r}v_{k\pm 2}$. It might be that the scalars α and β are zero. The vectors v_{k+2}, v_{k-2} might as well be zero; if they are not zero, then they are eigenvectors of Y, and their respective eigenvalues are as specified (which can be seen by repeating the above argument).

Applying both sides of the second equation in (5.2) to v_k and comparing the v_k coordinate of the results we obtain

$$\begin{aligned}\cos\frac{\pi}{r}\cos\frac{(k+1)\pi}{r}\,\alpha + \cos\frac{\pi}{r}\cos\frac{(k-1)\pi}{r}\,\beta - \cos\frac{k\pi}{r}\,(\alpha+\beta)&\\ = \cos\frac{2\pi}{r}\cos\frac{k\pi}{r} - \cos\frac{k\pi}{r}&.\end{aligned}$$

This is equivalent to

$$\left(\cos\frac{(k+2)\pi}{r} + \cos\frac{k\pi}{r}\right)(\alpha - 1) + \left(\cos\frac{(k-2)\pi}{r} + \cos\frac{k\pi}{r}\right)(\beta - 1) = 0,$$

that is, $\sin\frac{(k+1)\pi}{r}\,(\alpha - 1) + \sin\frac{(k-1)\pi}{r}\,(\beta - 1) = 0$. For further use, we write this as

$$(t^{4k+4}-1)(\alpha-1)+(t^{4k}-t^4)(\beta-1)=0. \tag{5.4}$$

Applying the two sides of the last equation in (5.2) to v_k and comparing the v_k coordinate of the results we obtain

$$\begin{aligned}
&(t^6+t^{-2}-2t^2)(\alpha+\beta)+(t^{-6}+t^2-2t^{-2})4\cos^2\frac{k\pi}{r}\\
&\quad+8\cos\frac{k\pi}{r}\cos\frac{(k+1)\pi}{r}\alpha+8\cos\frac{k\pi}{r}\cos\frac{(k-1)\pi}{r}\beta-4t^2\cos^2\frac{k\pi}{r}(\alpha+\beta)\\
&\quad-4t^{-2}\cos^2\frac{(k+1)\pi}{r}\alpha-4t^{-2}\cos^2\frac{(k-1)\pi}{r}\beta=2(t^6+t^{-6}-t^2-t^{-2}).
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
&(t^6+t^{-2}-2t^2)(\alpha+\beta)+4\cos\frac{(2k+1)\pi}{r}\alpha+4\cos\frac{\pi}{r}\alpha+4\cos\frac{(2k-1)\pi}{r}\beta\\
&\quad+4\cos\frac{\pi}{r}\beta-2t^2\cos\frac{2k\pi}{r}\alpha-2t^2\cos\frac{2k\pi}{r}\beta-2t^2\alpha-2t^2\beta\\
&\quad-2t^{-2}\cos\frac{(2k+2)\pi}{r}\alpha-2t^{-2}\alpha-2t^{-2}\cos\frac{(2k-2)\pi}{r}\beta-2t^{-2}\beta\\
&=2(t^6+t^{-6}-t^2-t^{-2})-2(t^{-6}+t^2-2t^{-2})-2(t^{-6}+t^2-2t^{-2})\cos\frac{2k\pi}{r}.
\end{aligned}$$

Using the fact that $t=\cos\frac{k\pi}{2r}+i\sin\frac{k\pi}{2r}$ we change this to

$$\begin{aligned}
&(t^{-4k-6}+t^{-4k+2}+2t^2-2t^{-4k-2}-t^6-t^{-2})(\alpha-1)\\
&\qquad+(t^{4k-6}+t^{4k+2}+2t^2-2t^{4k-2}-t^6-t^{-2})(\beta-1)=0.
\end{aligned}$$

Dividing through by $t^{-6}+t^2-2t^{-2}$ we obtain

$$(t^{-4k}-t^4)(\alpha-1)+(t^{4k}-t^4)(\beta-1)=0.$$

Combining this with (5.4), we obtain the system

$$\begin{aligned}
(t^{4k+4}-1)u+(t^{4k}-t^4)v&=0,\\
(t^{-4k}-t^4)u+(t^{4k}-t^4)v&=0
\end{aligned}$$

in the unknowns $u=\alpha-1$ and $v=\beta-1$. Recall that $t=e^{i\pi/2r}$.

The coefficient of v equals zero if and only if $k=1$, in which case we are forced to have $\beta=0$, because 0 is not an eigenvalue of Y. The coefficient of u in one of the equations is equal to zero if and only if $k=r-1$, in which case we are forced to have $\alpha=0$, because -1 is not an eigenvalue of Y.

In any other situation, by subtracting the equations we obtain

$$(t^4-t^{-4k})(t^{4k}+1)u=0.$$

This can happen only if $t^{4k}=-1$, namely if $2k=r$.

So, if $k \neq \frac{r}{2}$, then $Xv_k = v_{k+1} + v_{k-1}$ with v_{k+1} and v_{k-1} eigenvectors of Y with eigenvalues $2\cos\frac{(k+1)\pi}{r}$ respectively $2\cos\frac{(k-1)\pi}{r}$, and $Xv_{k\pm 1} = v_k + v_{k\pm 2}$, where $v_{k\pm 2}$ lie in the eigenspaces of Y of the eigenvalues $2\cos\frac{(k\pm 2)\pi}{r}$.

When $k = \frac{r}{2}$, one of v_{k+1} and v_{k-1} is nonzero, say v_{k+1}. Applying the above considerations to v_{k+1} we have $Xv_{k+1} = \alpha v_k + v_{k+2}$ and $X\alpha v_k = v_{k+1} + v'_{k-1}$, for some v'_{k-1} in the eigenspace of Y of the eigenvalue $2\cos\frac{(k-1)\pi}{r}$. Then on the one hand $Xv_k = v_{k+1} + v_{k-1}$ and on the other $\alpha X v_k = v_{k+1} + v'_{k-1}$. This shows that $\alpha = 1$, and because $(\alpha - 1) + (\beta - 1) = 0$, it follows that $\beta = 1$. A similar conclusion is reached if $v_{k-1} \neq 0$.

Repeating the argument we conclude that the irreducible representation, which must be the span of $X^m Y^n v_k$ for $m, n \geq 0$, has the basis $v_1, \ldots, v_{r-1}$, and X and Y act on these vectors by

$$Xv_j = v_{j+1} + v_{j-1}, \qquad Yv_j = 2\cos\frac{j\pi}{r}\, v_j$$

(here $v_0 = v_r = 0$). This is the representation defined by Weyl quantization of $\mathcal{M}_1$.

The fact that the algebra of all quantized Wilson lines consists of all linear operators on the Hilbert space follows from [9, Theorem 6.1]. ■

6. The Reshetikhin–Turaev representation as a Fourier transform

6.1. The Reshetikhin–Turaev representation of the mapping class group of the torus. In this section we deduce the existence of the Reshetikhin–Turaev representation for the torus from *quantum mechanical considerations*. The method shows how to derive the element that allows handle slides without a priori knowing it, and *should be generalizable to other gauge groups.*

There is an action of the mapping class group of the torus on the ring of functions on the pillow case, given by

$$h \cdot f(A) = f(h_*^{-1} A),$$

where $h_*^{-1}A$ denotes the pull-back of the connection A by h. In particular the Wilson line of a curve γ is mapped to the Wilson line of the curve $h(\gamma)$. The action of the mapping class group on functions on the pillow case induces an action on the quantum observables by

$$h \cdot \mathrm{Op}(f(A)) = \mathrm{Op}(f(h_*^{-1}A)),$$

which for Wilson lines is

$$h \cdot \mathrm{Op}(W_\gamma) = \mathrm{Op}(W_{h(\gamma)}).$$

THEOREM 6.1. *There exists a projective representation of the mapping class group of the torus that satisfies the exact Egorov identity*

$$\mathrm{Op}(W_{h(\gamma)}) = \rho(h)\,\mathrm{Op}(W_\gamma)\rho(h)^{-1}$$

with the quantum group quantization of Wilson lines. Moreover, $\rho(h)$ is unique up to multiplication by a constant.

Proof. We follow the first proof of Theorem 3.7. The bijective map $L \mapsto h(L)$ on the set of isotopy classes of framed links in the cylinder over the torus induces an automorphism of the free $\mathbb{C}[t,t^{-1}]$-module with basis these isotopy classes of links. Because this map leaves invariant the ideal defined by the skein relations (for crossings and for the $r-1$-st Jones–Wenzl idempotent), it defines an automorphism $\Phi : \widetilde{\mathrm{RT}}_t(\mathbb{T}^2\times[0,1]) \to \widetilde{\mathrm{RT}}_t(\mathbb{T}^2\times[0,1])$. The representation of $\widetilde{\mathrm{RT}}_t(\mathbb{T}^2\times[0,1])$ given by $V^j(\alpha)\mapsto \mathrm{Op}(W_{h(\gamma)})V^j(\alpha)$ is an irreducible representation of $\widetilde{\mathrm{RT}}_t(\mathbb{T}^2\times[0,1])$ which still maps t to multiplication by $e^{i\pi/2r}$ and simple closed curves to self-adjoint operators. In view of Theorem 5.10 this representation is equivalent to the standard representation. This proves the existence of the map $\rho(h)$ that satisfies the exact Egorov identity with quantizations of Wilson lines. Schur's lemma implies that $\rho(h)$ is unique up to multiplication by a constant and that ρ is a projective representation of the mapping class group. (A computational proof of the uniqueness can be found in [9].) ∎

Theorem 5.10 implies that for every h in the mapping class group, the map $\rho(h)$ is multiplication by a skein $\mathcal{F}(h)\in\widetilde{\mathrm{RT}}_t(\mathbb{T}^2\times[0,1])$. Let us explain the algorithm of finding $\mathcal{F}(h)$ (this was applied to abelian Chern–Simons in [12], and could be applied for other gauge groups as well). We start with the case of the positive Dehn twist T along the curve $(0,1)$. Since the twist leaves the curve $(0,1)$ invariant,

$$\mathcal{F}(T)(0,1)V^k(\alpha) = (0,1)\mathcal{F}(T)V^k(\alpha) \qquad \text{for all } k.$$

And because the eigenspaces of $\mathrm{Op}(W_{(0,1)})$ are 1-dimensional, the linear operator defined by $\mathcal{F}(T)$ on the Hilbert space is a polynomial in $\mathrm{Op}(W_{(0,1)})$. The polynomials $S_j(x)$, $0\le j\le r-1$, form a basis for $\mathbb{C}[x]/S_{r-1}(x)$, so

$$\mathcal{F}(T) = \sum_{j=1}^{r-1} c_j S_{j-1}((0,1)), \qquad c_j\in\mathbb{C}.$$

On the other hand, the exact Egorov identity gives

$$(1,1)\mathcal{F}(T)V^k(\alpha) = \mathcal{F}(T)(1,0)V^k(\alpha). \tag{6.1}$$

Using (5.1) and the fact that $S_{j-1}((0,1))V^k(\alpha) = \frac{[jk]}{[k]}V^k(\alpha)$ for all j and k, we rewrite (6.1) as

$$\sum_j c_j \frac{[jk]}{k} t^{-1}(t^{-2k}V^{k+1}(\alpha) + t^{2k}V^{k-1}(\alpha)) \\ = \sum_j c_j \left(\frac{[j(k+1)]}{[k+1]} V^{k+1}(\alpha) + \frac{[j(k-1)]}{[k-1]} V^{k-1}(\alpha) \right).$$

Equating the coefficients of V^{k+1} on both sides yields

$$\sum_{j=1}^{r-1} c_j [j(k+1)] = \frac{[k+1]}{[k]} t^{-2k-1} \sum_{j=1}^{r-1} c_j [jk].$$

Set $\sum_j c_{j=1}^{r-1}[j] = t^{-1}u$ to get the system of equations in c_j, $j = 1, \ldots, r-1$,

$$\sum_{j=1}^{r-1} [kj] c_j = [k] t^{-k^2} u, \qquad k = 1, \ldots, r-1.$$

Recall that $[n] = \sin \frac{n\pi}{r} / \sin \frac{\pi}{r}$, so the coefficient matrix is a multiple of the matrix of the discrete sine transform. The square of the discrete sine transform is the identity map, so there is a constant C such that $c_j = C \sum_k [jk][k] t^{-k^2}$. Standard results in the theory of Gauss sums [20] show that $\sum_k [jk][k] t^{-k^2} = C'[j] t^{j^2}$ where C' is a constant independent of j. We conclude that $\mathcal{F}(T)$ is a multiple of $\sum_{j=1}^{r-1} [j] t^{j^2} S_{j-1}((0,1))$. We normalize $\mathcal{F}(T)$ to make it unitary by multiplying by

$$\eta = \sqrt{\frac{r}{2}} \sin \frac{\pi}{r} = \Big(\sum_{j=1}^{r-1} [j]^2 \Big)^{-1/2},$$

and also multiply it by t^{-1}. This is the same as the skein consisting of the standard surgery framed one-component link of the twist colored by

$$\Omega_{\mathrm{SU}(2)} = \eta \sum_{j=1}^{r-1} [j] V^j(\alpha) = \eta \sum_{j=1}^{r-1} S_{r-1}(\alpha) \in \widetilde{\mathrm{RT}}_t(S^1 \times \mathbb{D}^2).$$

And we have recovered $\Omega_{\mathrm{SU}(2)}$, which is the fundamental building block of Reshetikhin–Turaev theory. Note that the exact Egorov identity can be interpreted as a handle slide, and thus our quantum-mechanical argument hints at the handle-slide property of the element $\Omega_{\mathrm{SU}(2)}$.

Remark 6.2. The exact Egorov identity implies that γ can be slid over the one-component surgery link of the twist colored by $\Omega_{\mathrm{SU}(2)}$. Once this is noted, it is natural to try slides over knots, and to derive the Reshetikhin–Turaev formula for 3-manifold invariants [28]. So instead of stating the Reshetikhin–Turaev formula ad hoc, as always, we obtain it from quantum mechanics.

For a framed link L, let $\Omega_{\mathrm{SU}(2)}(L)$ be the skein obtained by replacing each component of L by $\Omega_{\mathrm{SU}(2)}$ so that the curve $(1,0)$ on the boundary of the solid torus is mapped to the framing. Then $\mathcal{F}(T)$ is the coloring by $\Omega_{\mathrm{SU}(2)}$ of the surgery curve of T. This is true for any twist, and since any element of the mapping class group is a product of twists and $\rho(h)$ is unique, we obtain the well-known result (see [29]), which is an analogue of Theorem 3.8:

THEOREM 6.3. *Let h be an element of the mapping class group of the torus obtained by performing surgery on a framed link L_h in $\mathbb{T}^2 \times [0,1]$. Up to multiplication by a constant, the map $\rho(h) : \widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2) \to \widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$ is given by $\rho(h)\beta = \Omega_{\mathrm{SU}(2)}(L_h)\beta$.*

One should point out that this property is obvious if L_h is decomposed as the union of the surgery curves of the Dehn twists that comprise h, but it holds in general because of the handle-slide property of $\Omega_{\mathrm{SU}(2)}$.

6.2. The structure of the reduced Reshetikhin–Turaev skein algebra of the cylinder over a surface. The next result is similar to Theorem 3.6 in [29].

LEMMA 6.4. *The quantum group quantizations of all Wilson lines on a surface generate the algebra of all linear operators on the Hilbert space of the quantization.*

THEOREM 6.5. *Given a genus g surface Σ_g, $g \geq 1$, the representation of $\widetilde{\mathrm{RT}}_t(\Sigma_g \times [0,1])$ on $\widetilde{\mathrm{RT}}_t(H_g)$ is faithful. Moreover, $\widetilde{\mathrm{RT}}_t(\Sigma_g \times [0,1])$ is the algebra of all linear operators on $\widetilde{\mathrm{RT}}_t(H_g)$.*

Proof. By Lemma 6.4, it suffices to show that the dimension of the space $\widetilde{\mathrm{RT}}_t(\Sigma_g \times [0,1])$ is the square of the dimension of $\widetilde{\mathrm{RT}}_t(H_g)$. In the case of the Kauffman bracket, part of the proof is in [30] and the case where r is an odd prime is in [8].

For compact, orientable 3-manifolds M and N, let $M \# N$ be the connected sum, obtained by removing a ball from M and N, then gluing the results along the new boundary spheres. In $M \# N$, M and N are separated by a sphere S_{sep}. By turning one H_g inside out, we get $H_g \# H_g$ as S^3 with two handlebodies removed.

LEMMA 6.6. *Given 3-dimensional manifolds M and N, the map*

$$\widetilde{\mathrm{RT}}_t(M) \otimes \widetilde{\mathrm{RT}}_t(N) \to \widetilde{\mathrm{RT}}_t(M \# N)$$

defined by $(L, L') \mapsto L \cup L'$, where L and L' are framed links in M and N respectively, is an isomorphism of vector spaces.

Proof. The proof was inspired by [27]. Any skein in $\widetilde{\mathrm{RT}}_t(M \# N)$ can be written as $\sum_{j=1}^{k} c_j \sigma_j$, where c_j's are complex coefficients and each σ_j is a skein intersecting S_{sep} along the jth Jones–Wenzl idempotent. Taking a trivial knot colored by $\Omega_{\mathrm{SU}(2)}$ and sliding it over S_{sep} we get, by Lemma 22 (the Encirclement Lemma) in [17], the equality $\eta^{-1}\sum_{j=0}^{r-2} c_j\sigma_j = \eta^{-1}c_0$.

So any skein equals one disjoint from S_{sep}. Hence the map from the statement is an epimorphism. It is a monomorphism since the skein module of a regular neighborhood of S_{sep} is trivial. So it is an isomorphism. ∎

Now we continue as in [30]. We can obtain $\Sigma_g \times [0,1]$ from $H_g \# H_g$ by surgery on a g-component framed link L_g (see Figure 12(a)). Let $N_1 \subset H_g \# H_g$ be a regular neighborhood of L_g, consisting of g solid tori. Let $N_2 \subset \Sigma_g \times [0,1]$ be the union of the g surgery tori, and L_g' the framed link in $\Sigma_g \times [0,1]$ consisting of the cores of these tori. Then $H_g \# H_g$ is obtained from $\Sigma_g \times [0,1]$ by surgery on L_g'.

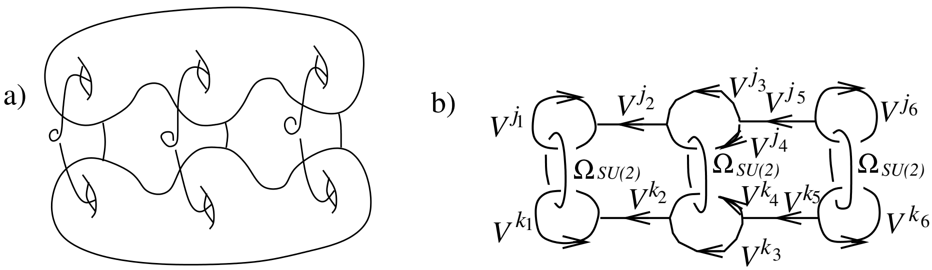

Fig. 12

Every skein in $H_g \# H_g$ respectively $\Sigma_g \times [0,1]$ can be isotoped to miss N_1 respectively N_2. The homeomorphism $\phi : (H_g \# H_g)\backslash N_1 \to (\Sigma_g \times [0,1])\backslash N_2$ induces an isomorphism

$$\phi : \widetilde{\mathrm{RT}}_t((H_g \# H_g)\backslash N_1) \to \widetilde{\mathrm{RT}}_t((\Sigma_g \times [0,1])\backslash N_2).$$

But ϕ does not induce a well-defined map between $\widetilde{\mathrm{RT}}_t(H_g \# H_g)$ and $\widetilde{\mathrm{RT}}_t(\Sigma_g \times [0,1])$. Sikora defined $F_1 : \widetilde{\mathrm{RT}}_t(H_g \# H_g) \to \widetilde{\mathrm{RT}}_t(\Sigma_g \times [0,1])$ by $F_1(\sigma) = \phi(\sigma) \cup \Omega_{\mathrm{SU}(2)}(L_g')$. The Lickorish trick (see [17]) allows us to slide $\phi(\sigma)$ along L_g'; hence this map is well defined. Its inverse is $F_2(\sigma) = \phi^{-1}(\sigma) \cup \Omega_{\mathrm{SU}(2)}(L_g)$. To see this, push L_g' off N_2 in the direction of the framing of L_g'. Then each component of $\phi^{-1}(L_g')$ is the meridian of the surgery torus, and it surrounds once the corresponding component of L_g. Again by [17, Lemma 22], $\Omega_{\mathrm{SU}(2)}(L_g) \cup \Omega_{\mathrm{SU}(2)}(\phi^{-1}(L_g')) = \emptyset \in \widetilde{\mathrm{RT}}_t(H_g \# H_g)$. Hence $F_2 \circ F_1 = \mathrm{Id}$. Similarly $F_1 \circ F_2 = \mathrm{Id}$, and we are done. ∎

Note that this result is the nonabelian analogue of Proposition 3.3.

6.3. The quantization of Wilson lines determines the Reshetikhin–Turaev representation. Just as for the torus, there is an action of the mapping class group of a surface Σ_g on the ring of regular functions on the moduli space $\mathcal{M}_g$ of flat SU(2)-connections on Σ_g given by $h \cdot f(A) = f(h_*^{-1}A)$. This induces the action on the quantum observables

$$h \cdot \mathrm{Op}(W_\gamma) = \mathrm{Op}(W_{h(\gamma)}).$$

Theorem 6.7. *There is a projective representation ρ of the mapping class group of a closed surface that satisfies the exact Egorov identity*

$$\mathrm{Op}(W_{h(\gamma)}) = \rho(h)\,\mathrm{Op}(W_\gamma)\rho(h)^{-1}$$

with the quantum group quantization of Wilson lines. Moreover, for every h, $\rho(h)$ is unique up to multiplication by a constant.

Proof. We mimic the second proof of Theorem 3.7. The bijective map $L \mapsto h(L)$ on the set of isotopy classes of framed links in the cylinder over the torus induces an automorphism of the free $\mathbb{C}[t, t^{-1}]$-module with basis these isotopy classes of links. Because the ideal defined by the skein relations is invariant under this map, the map defines an automorphism

$$\Phi : \widetilde{\mathrm{RT}}_t(\Sigma_g \times [0,1]) \to \widetilde{\mathrm{RT}}_t(\Sigma_g \times [0,1]).$$

By Theorem 6.5 the representation $\widetilde{\mathrm{RT}}_t(\Sigma_g \times [0,1]) \to L(\widetilde{\mathrm{RT}}_t(H_g))$ is faithful (this is essential) and onto. So the automorphism Φ is inner [33]. This proves the existence of $\rho(h)$. That ρ is a representation of the mapping class group and its uniqueness are consequences of Schur's lemma. ∎

Remark 6.8. Another way to prove this, as well as to prove Theorem 6.1, is to note that if h is an element of the mapping class group that extends to the handlebody H_g, then for $v \in \widetilde{\mathrm{RT}}_t(H_g)$ seen as image of a skein on the boundary the exact Egorov identity implies $\mathcal{F}(h)v\emptyset = h(v)\mathcal{F}\emptyset$, or $\mathcal{F}(h)v = h(v)$ in $\widetilde{\mathrm{RT}}_t(H_g)$. Then we use the results from §3 in [29] to derive the conclusion. We gave the above proof to exhibit similarity with the action of the mapping class group on theta functions and with the metaplectic representation, since we think that this approach can be generalized to other gauge groups.

Each element of the mapping class group preserves the Atiyah–Bott symplectic form, so it induces a symplectomorphism of $\mathcal{M}_g$. Theorem 6.7 proves that the symplectomorphisms of $\mathcal{M}_g$ arising from elements of the mapping class group can be quantized. Their quantization plays the role of the Fourier transform for nonabelian theta functions.

Recall also that the projective representation of the mapping class group can be made into a true representation by passing to a $\mathbb{Z}$-extension of the mapping class group [32], [34].

Appendix. For completeness, we conclude with the proof of the following result mentioned in §5.2 and whose importance was explained in §5.3.

PROPOSITION 6.9. *The bilinear pairing used in the definition of the quantum group quantization from §5.2 is nondegenerate.*

Proof. We first give a description of the inner product on the Hilbert space $\widetilde{\mathrm{RT}}_t(H_g)$ by diagrams, following [8]. The handlebody H_g has a natural orientation reversing symmetry s that leaves its core invariant. Glue two copies of H_g along their boundaries by the restriction of s to the boundary to obtain a connected sum of g copies of $S^1 \times S^2$, $\#_g S^1 \times S^2$. This induces a pairing

$$\langle \cdot, \cdot \rangle_0 : \widetilde{\mathrm{RT}}_t(H_g) \times \widetilde{\mathrm{RT}}_t(H_g) \to \widetilde{\mathrm{RT}}_t(\#_g S^1 \times S^2).$$

The manifold $\#_g S^2 \times S^2$ is obtained from S^3 by performing surgery on the trivial link with g components. Identifying $\widetilde{\mathrm{RT}}_t(\#_g S^1 \times S^2)$ with $\widetilde{\mathrm{RT}}_t(S^3)$ via Sikora's isomorphism as in Theorem 6.5, we find that the pairing takes values in $\mathbb{C}$.

The pairing by $\langle \cdot, \cdot \rangle_0$ of two basis elements is given by the Reshetikhin–Turaev invariant of a graph like the one in Figure 12(b). We argue on this figure, but one should keep in mind that there are many different graphs that can be the cores of the same handlebody. By [17, Lemma 22], in order for this Reshetikhin–Turaev invariant to be nonzero, in each pair of edges linked by a circle colored by $\Omega_{\mathrm{SU}(2)}$ the colors must be equal. This is because in order for the tensor product $V^{j_i} \otimes V^{k_i}$ to contain a copy of V^1, the dimensions of the two irreducible representations must be equal. Note also that because we work in S^3, the pairs of edges like V^{j_3} and V^{k_3} are also linked by a circle colored by $\Omega_{\mathrm{SU}(2)}$, namely the circle that links V^{j_4} and V^{k_4}. In general, the edges corresponding to decomposition circles that do not disconnect the surface fall in this category.

Let us next examine the pairs of edges that are not linked by surgery circles, such as those colored by V^{j_2}, V^{k_2} in the figure. In general, the edges that come from decomposition circles that disconnect the surface fall in this category. Rotating the graph by 90° and evaluating the Reshetikhin–Turaev invariant by the rules we obtain a homomorphism $\mathbb{C} = V^1 \to V^{j_2} \otimes V^{k_2}$. This homomorphism is nonzero if and only if $j_2 = k_2$. We conclude that the pairing of two distinct basis elements is zero. On the other hand, computing the pairing of a basis element with itself we can trace a V^1 from the bottom to the top, and the value of the pairing is $\Omega_{\mathrm{SU}(2)}(O) = \eta^{-g}$. Hence $\langle \cdot, \cdot \rangle_0 = \eta^{-g} \langle \cdot, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product.

The bilinear pairing $[\cdot, \cdot]$ from §5.2 is defined by gluing two copies of H_g along an orientation reversing homeomorphism so as to obtain S^3. The homeomorphism is of the form $s \circ h$, so $[e_i, e_j] = \langle e_i, \rho(h) e_j \rangle$. Because $\rho(h)$ is an automorphism of the Hilbert space of the quantization, the pairing is nondegenerate. ∎

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