FUNDAMENTA
MATHEMATICAE
228 (2015)

Estimation of the Szlenk index of reflexive Banach spaces using generalized Baernstein spaces

by

Ryan Causey (College Station, TX)

Abstract. For each ordinal $\alpha < \omega_1$, we prove the existence of a separable, reflexive Banach space W with a basis so that $\mathrm{Sz}(W), \mathrm{Sz}(W^*) \leq \omega^{\alpha+1}$ which is universal for the class of separable, reflexive Banach spaces X satisfying $\mathrm{Sz}(X), \mathrm{Sz}(X^*) \leq \omega^\alpha$.

1. Introduction. The relatively new tool of weakly null trees has produced a number of recent results in Banach space theory. In particular, trees have facilitated the solution of questions concerning realizing a given Banach space as a subspace or quotient of a Banach space with a coordinate system (a process which we call "coordinatization") through a strong connection between trees and embedding into Banach spaces with an FDD which has prescribed properties. For example, Johnson and Zheng completely characterized when a separable reflexive space embeds into a reflexive space with unconditional basis [7] and when a separable Banach space embeds into a Banach space with shrinking, unconditional basis [8] using the UTP and w^*UTP, respectively. Odell and Schlumprecht demonstrated that for $1 < p < \infty$, a separable, reflexive space embeds into a Banach space which is the ℓ_p sum of finite-dimensional spaces if and only if every normalized, weakly null tree has a branch equivalent to the ℓ_p unit vector basis [10]. In a spirit which we continue, Odell and Schlumprecht established a strong connection between tree estimates and embeddings into Banach spaces with the corresponding block estimates (the relevant notions are defined in Section 2). These coordinatization results provide an avenue for the proof of the existence of universal Banach spaces for classes of spaces with certain tree estimates.

Our results follow the methods of Odell, Schlumprecht, and Zsák [13] and Freeman, Odell, Schlumprecht, and Zsák [5] who used Tsirelson spaces

2010 *Mathematics Subject Classification*: Primary 46B03; Secondary 46B06.
Key words and phrases: Szlenk index, universality, embedding in spaces with finite-dimensional decompositions, Baernstein spaces.

DOI: 10.4064/fm228-2-3

in their constructions. In [5], the objects of study were Banach spaces with separable dual, while in [13], the objects were separable, reflexive spaces. The former proved both a coordinatization result and a universality result concerning the classes of separable Banach spaces with Szlenk index not exceeding $\omega^{\alpha\omega}$, while the latter proved a coordinatization result and a universality result concerning the classes of separable, reflexive Banach spaces X such that the Szlenk indices of both X and X^* do not exceed $\omega^{\alpha\omega}$. In [4], results analogous to those of [5] were established using Schreier spaces. These results allowed finer gradations by working instead with the classes of separable Banach spaces with Szlenk index not exceeding ω^α for a countable ordinal α.

Two-sided estimates were not possible with Schreier spaces, which are c_0-saturated. To establish two-sided estimates, we introduce a generalization of the so-called Baernstein space, which we denote X^p_α, and which is itself a generalization of Schreier's original space. The details of the construction are given in Section 3. This allows us to improve the results of [13] by making finer gradations, as in [4].

In Section 4, we define the Szlenk index, originally used by Szlenk to prove the non-existence of a separable, reflexive Banach space which is universal for the class of separable, reflexive Banach spaces. We also recall several results concerning the Szlenk index which connect it to tree estimates.

Finally, we present the proofs of the main results in Section 5.

Our connection between the Szlenk index and tree estimates is summarized in

THEOREM 1.1. *If X is a separable, reflexive Banach space, $\alpha < \omega_1$ is such that $\mathrm{Sz}(X), \mathrm{Sz}(X^*) \le \omega^\alpha$, and $1 < p \le 2$, then X satisfies subsequential $((X^p_\alpha)^*, X^p_\alpha)$-tree estimates.*

A major idea behind Theorem 1.1 is the comparison of normalized block sequences in two Banach spaces to make a comparison of the Szlenk indices of the two spaces. For this comparison, we establish the following coordinatization result, which connects tree estimates with block estimates.

THEOREM 1.2. *Let U, V be reflexive Banach spaces with normalized, 1-unconditional bases $(u_n), (v_n)$, respectively, so that (u_n) satisfies subsequential U-upper block estimates in U, (v_n) satisfies subsequential V-lower block estimates in V, and every normalized block sequence of (v_n) is dominated by every normalized block sequence of (u_n). Then if X is a separable, reflexive Banach space which satisfies subsequential (V, U)-tree estimates, then X embeds into a reflexive Banach space Z with FDD E satisfying subsequential (V, U)-block estimates in Z.*

Finally, we employ a theorem of Johnson, Rosenthal, and Zippin from [6] to deduce the existence of a universal space with a basis. To do so, we define for each $\alpha < \omega_1$ the class

$$\mathcal{C}_\alpha = \{X : X \text{ separable, reflexive, } \mathrm{Sz}(X), \mathrm{Sz}(X^*) \leq \omega^\alpha\}.$$

THEOREM 1.3. *Let $\alpha < \omega_1$. There exists a separable, reflexive space W in $\mathcal{C}_{\alpha+1}$ with a basis which is universal for the class $\mathcal{C}_\alpha$.*

This is a strengthening of Theorem C of [13], which proved the above result in the case that $\alpha = \beta\omega$ for some $\beta < \omega_1$.

2. Definitions and notation. If Z is a Banach space and $E = (E_n)$ is a collection of finite-dimensional subspaces of Z, we say E is a *finite-dimensional decomposition*, or *FDD*, for Z if for each $z \in Z$ there exists a unique sequence (z_n) such that $z_n \in E_n$ and $z = \sum_{n=1}^\infty z_n$. If E is an FDD for a Banach space Z, for $n \in \mathbb{N}$ we denote the nth coordinate projection by P_n^E. More precisely, for $z \in Z$, if $z = \sum_{n=1}^\infty z_n$ for $z_n \in E_n$, then $P_n^E z = z_n$. For a finite $A \subset \mathbb{N}$, we let $P_A^E = \sum_{n\in A} P_n^E$. We define the *projection constant* $K = K(E, Z)$ to be

$$K = K(E, Z) = \sup_{m\leq n} \|P_{[m,n]}^E\|.$$

This is finite by the principle of uniform boundedness. We call E a *bimonotone* FDD for a Banach space Z if $K(E, Z) = 1$. If Z has an FDD, we can always endow Z with an equivalent norm which makes E a bimonotone FDD. We let $\mathrm{supp}_E\, z = \{n : P_n^E z \neq 0\}$, and call this set the *support* of z. If E is a basis, or if no confusion is possible, we write $\mathrm{supp}\, z$ in place of $\mathrm{supp}_E\, z$. We denote by $c_{00}(\bigoplus E_n)$ the collection of vectors in Z with finite support. We note that $c_{00}(\bigoplus E_n)$ is dense in any space for which E is an FDD.

We denote by $Z^{(*)}$ the closed span of $c_{00}(\bigoplus E_n^*)$ in Z^* and note that $E^* = (E_n^*)$ is an FDD for $Z^{(*)}$ with $K(E^*, Z^*) \leq K(E, Z)$. We consider E_n^* with the norm it inherits as a subspace of Z^* and not with the norm it inherits as the dual of E_n. These norms may be different if E is not bimonotone. If $Z^{(*)} = Z^*$, we say that E is a *shrinking* FDD for Z. We say that E is a *boundedly complete* FDD if for each sequence (z_n) with $z_n \in E_n$ such that $\sup_{N\in\mathbb{N}} \|\sum_{n=1}^N z_n\| < \infty$, $\sum_{n=1}^\infty z_n$ converges in Z. A Banach space Z with FDD E is reflexive if and only if the FDD is both shrinking and boundedly complete.

If Z is a Banach space with FDD $E = (E_n)$ and V is a Banach space with a normalized, 1-unconditional basis (v_n), we define the space $Z^V = Z^V(E)$ to be the completion of $c_{00}(\bigoplus_{n=1}^\infty E_n)$ endowed with the norm

$$\|z\|_{Z^V} = \sup\Big\{\Big\|\sum_{k=1}^\infty \|P_{[k_{n-1},k_n)}^E z\|_Z v_{k_{n-1}}\Big\|_V : 1 \leq k_0 < k_1 < \cdots\Big\}.$$

The norm above depends upon the FDD E, but when no confusion is possible, we will write Z^V in place of $Z^V(E)$. For convenience, we will write Z^p in place of Z^{ℓ_p}.

If U is a Banach space and (u_n) is a basis for U, we say (u_n) is *R-right dominant* if for each pair of subsequences of the natural numbers $(m_n),(k_n)$ with $m_n \le k_n$ for all n, (u_{m_n}) is R-dominated by (u_{k_n}). If $B=(b_n)$ is a subsequence of the natural numbers, we let $U_B=[u_{b_n}]$. If Z is a Banach space with FDD E, and U is a Banach space with a normalized, 1-unconditional basis (u_n), we say E satisfies *subsequential C-U-upper* (respectively, *lower*) *block estimates* in Z if each normalized block sequence (z_n) is C-dominated by (respectively, C-dominates) (u_{m_n}), where $m_n=\min \operatorname{supp}_E z_n$. If U,V are Banach spaces with normalized, 1-unconditional bases $(u_n),(v_n)$, respectively, we say X satisfies *subsequential K-(V,U)-block estimates* in Z if it satisfies subsequential K-V-lower block estimates in Z and K-U-upper block estimates in Z.

We next recall a coordinate-free version of subsequential upper and lower estimates. For $\ell\in\mathbb{N}$, we define

$$T_\ell=\{(n_1,\dots,n_\ell):n_1<\dots<n_\ell,\ n_i\in\mathbb{N}\}$$

and

$$T_\infty=\bigcup_{\ell=1}^\infty T_\ell,\qquad T_\infty^{\mathrm{even}}=\bigcup_{\ell=1}^\infty T_{2\ell}.$$

An *even tree* in a Banach space X is a family $(x_t)_{t\in T_\infty^{\mathrm{even}}}$ in X. Sequences of the form $(x_{(t,k)})_{k>k_{2n-1}}$, where $n\in\mathbb{N}$ and $t=(k_1,\dots,k_{2n-1})\in T_\infty$, are called *nodes*. A sequence of the form $(k_{2n-1},x_{(k_1,\dots,k_{2n})})_{n=1}^\infty$, with $k_1<k_2<\cdots$, is called a *branch* of the tree. An even tree is called *weakly null* if every node is a weakly null sequence. If X is a dual space, an even tree is called *w^*-null* if every node is w^*-null. If X has an FDD $E=(E_n)$, a tree is called a *block even tree* of E if every node is a block sequence of E.

If $T\subset T_\infty^{\mathrm{even}}$ is closed under taking restrictions so that for each $t\in T\cup\{\emptyset\}$ and for each $m\in\mathbb{N}$ the set $\{n\in\mathbb{N}:(t,m,n)\in T\}$ is either empty or infinite, and if the latter occurs for infinitely many values of m, then we call $(x_t)_{t\in T}$ a *full subtree*. Such a tree can be relabeled to a family indexed by T_∞^{even} and such that the branches of $(x_t)_{t\in T}$ are branches of $(x_t)_{t\in T_\infty^{\mathrm{even}}}$ and that the nodes of $(x_t)_{t\in T}$ are subsequences of the nodes of $(x_t)_{t\in T_\infty^{\mathrm{even}}}$.

Let U be a Banach space with a normalized, 1-unconditional basis (u_n) and $C\ge 1$. Let X be an infinite-dimensional Banach space. We say that X satisfies *subsequential C-U-upper tree estimates* if every normalized, weakly null even tree $(x_t)_{t\in T_\infty^{\mathrm{even}}}$ in X has a branch $(k_{2n-1},x_{(k_1,\dots,k_{2n})})$ such that $(x_{(k_1,\dots,k_{2n})})_n$ is C-dominated by $(u_{k_{2n-1}})_n$. We say X satisfies *subsequential C-U-lower tree estimates* if every normalized, weakly null even tree

$(x_t)_{t \in T_\infty^{\mathrm{even}}}$ in X has a branch $(k_{2n-1}, x_{(k_1,\dots,k_{2n})})$ such that $(x_{(k_1,\dots,k_{2n})})_n$ C-dominates $(v_{k_{2n-1}})$. We say that X satisfies *subsequential U-upper* (respectively, *lower*) *tree estimates* if it satisfies C-U-upper (respectively, lower) tree estimates for some $C \geq 1$. If U, V are Banach spaces with normalized, 1-unconditional bases, we say X satisfies *subsequential C-(V,U)-tree estimates* if it satisfies subsequential C-V-lower tree estimates and C-U-upper tree estimates.

We let $\mathcal{A}_{(V,U)}$ denote the class of all separable, reflexive Banach spaces which satisfy subsequential (V,U)-tree estimates.

A simple perturbation argument yields the following.

LEMMA 2.1. *Let U be a Banach space with a normalized, 1-unconditional basis (u_n), and let Z be a Banach space with FDD $E = (E_n)$ satisfying subsequential C-U-upper (respectively, lower) block estimates in Z. Assume also that for each $n \in \mathbb{N}$, $E_n \neq \{0\}$. If (z_n) is a normalized block sequence in Z and $(k_n) \subset \mathbb{N}$ is strictly increasing such that*

$$k_n \leq \min \operatorname{supp}_E z_n \leq \max \operatorname{supp}_E z_n < k_{n+1},$$

then (z_n) is C-dominated by (respectively, C-dominates) (u_{k_n}).

Another simple but technical lemma involves the preservation of upper block estimates. We postpone the proof until Section 4.

LEMMA 2.2. *Let U, V be Banach spaces with normalized, 1-unconditional bases $(u_n), (v_n)$, respectively, so that every normalized block sequence in (u_n) dominates every normalized block sequence in (v_n). If Z is a Banach space with FDD E which satisfies subsequential U-upper block estimates in Z, then E satisfies subsequential U-upper block estimates in $Z^V(E)$.*

3. Schreier families, Schreier and Baernstein spaces. Throughout, we will assume subsets of $\mathbb{N}$ are written in increasing order. Let $[\mathbb{N}]^{<\omega}$ denote the set of all finite subsets of $\mathbb{N}$, and $[\mathbb{N}]^{\omega}$ the set of all infinite subsets of $\mathbb{N}$. We identify subsets of $\mathbb{N}$ in the natural way with strictly increasing sequences in $\mathbb{N}$. We write $E < F$ if $\max E < \min F$. By convention, $\min \emptyset = \omega$, $\max \emptyset = 0$. We consider the families $[\mathbb{N}]^{\omega}, [\mathbb{N}]^{<\omega}$ as being ordered by extension. That is, the predecessors of an element are its initial segments, and we write $E \preceq F$ if E is an initial segment of F. A family $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is called *hereditary* if, whenever $E \in \mathcal{F}$ and $F \subset E$, $F \in \mathcal{F}$. We associate a set F with the function 1_F in $\{0,1\}^{\mathbb{N}}$, topologized with the product topology. We then endow $[\mathbb{N}]^{<\omega}$ with the topology induced by this association. We note that a hereditary family is compact if and only if it contains no strictly ascending chains.

Given two (finite or infinite) subsequences $(k_n), (\ell_n) \subset \mathbb{N}$ of the same length, we say (ℓ_n) is a *spread* of (k_n) if $k_n \leq \ell_n$ for all $n \in \mathbb{N}$. We call a family $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ *spreading* if it contains all spreads of its elements.

We construct the *Schreier families* with more specific properties than is usually done. Let

$$\mathcal{S}_0 = \{\emptyset\} \cup \{\{n\} : n \in \mathbb{N}\}.$$

Assuming that for $\alpha < \omega_1$, $\mathcal{S}_\alpha$ has been defined, let

$$\mathcal{S}_{\alpha+1} = \Big\{\bigcup_{k=1}^{n} E_k : E_k \in \mathcal{S}_\alpha,\, E_1 < \cdots < E_n,\, n \leq \min E_1\Big\}.$$

Assume that $\alpha < \omega_1$ is a limit ordinal. Assume also that for each $0 \leq \beta < \alpha$, $\mathcal{S}_\beta$ has been defined, and for each limit ordinal $\lambda < \alpha$, there exists a sequence $\lambda_n \uparrow \lambda$ such that $\mathcal{S}_\lambda = \{E : \exists n \leq \min E,\, E \in \mathcal{S}_{\lambda_n+1}\}$. An easy induction argument shows that for any $\beta < \gamma < \alpha$ there exists a non-negative integer m such that $\mathcal{S}_\beta \subset \mathcal{S}_{\gamma+m}$. Choose some sequence $\beta_n \uparrow \alpha$. We can recursively choose non-negative integers m_n so that

$$\mathcal{S}_{\beta_n+m_n+1} \subset \mathcal{S}_{\beta_{n+1}+m_{n+1}}.$$

We let $\alpha_n = \beta_n + m_n$, so $\alpha_n \uparrow \alpha$. Therefore we have $\mathcal{S}_{\alpha_n+1} \subset \mathcal{S}_{\alpha_{n+1}}$. We let

$$\mathcal{S}_\alpha = \{E : \exists n \leq \min E,\, E \in \mathcal{S}_{\alpha_n+1}\}.$$

The families above depend on the choices we make of $\alpha_n \uparrow \alpha$ for limit ordinals α, but it is known that regardless of these choices, $\mathcal{S}_\alpha$ is spreading, hereditary, and compact.

Next, we recall the Repeated Averages Hierarchy as defined in [2]. For a partially ordered set P, we write $\mathrm{MAX}(P)$ to denote the collection of maximal elements. For each $I \in [\mathbb{N}]^\omega$, $0 \leq \alpha < \omega_1$, we define a sequence $(x_n^{\alpha,I})_n$ to be a convex blocking of the canonical c_{00} basis, denoted (e_n), which has the properties:

(i) $(x_n^{\alpha,I})_n$ is a convex blocking of (e_{i_n}),
(ii) $I = \bigcup_{n=1}^\infty \operatorname{supp} x_n^{\alpha,I}$,
(iii) $\operatorname{supp} x_n^{\alpha,I} \in \mathrm{MAX}(\mathcal{S}_\alpha)$ for each n.

For $I \in [\mathbb{N}]^\omega$, write $I = (i_n)$. We let $x_n^{0,I} = e_{i_n}$. If $(x_n^{\alpha,I})$ has been defined to have the properties above, we let

$$x_1^{\alpha+1,I} = i_1^{-1}\sum_{j=1}^{i_1} x_j^{\alpha,I}.$$

Suppose that $x_n^{\alpha+1,I}$ has been defined for $1 \leq n < N$ to be a convex blocking of (e_{i_n}), $\bigcup_{n=1}^{N-1} \operatorname{supp}(x_n^{\alpha+1,I})$ is an initial segment of I, $\operatorname{supp}(x_n^{\alpha+1,I})$ is in $\mathrm{MAX}(S_{\alpha+1})$ for each n, and

$$x_n^{\alpha+1,I} = \frac{1}{s_n}\sum_{j=p_{n-1}+1}^{p_n} x_j^{\alpha,I} \quad \text{for some } 0 = p_0 < \cdots < p_{N-1},$$

where $s_n = \min\operatorname{supp}(x_n^{\alpha+1,I})$. Then let $s_N = \min\operatorname{supp}(x_{p_{N-1}+1}^{\alpha,I})$, let $p_N = p_{N-1} + s_N$, and let

$$x_N^{\alpha+1,I} = \frac{1}{s_N}\sum_{j=p_{N-1}+1}^{p_N} x_j^{\alpha,I}.$$

Finally, assume that for a limit ordinal $\alpha < \omega_1$, $(x_n^{\beta,I'})$ has been defined for all $\beta < \alpha$ and all $I' \in [\mathbb{N}]^\omega$. Let $\alpha_n \uparrow \alpha$ be the ordinals used to define $\mathcal{S}_\alpha$. Let $m_1 = \min I$ and $x_1^{\alpha,I} = x_1^{\alpha_{m_1}+1,I}$. Given $x_n^{\alpha,I}$ for $1 \le n < N$ with the same assumptions as in the successor case, let $I_N = I \setminus \bigcup_{n=1}^{N-1}\operatorname{supp}(x_n^{\alpha,I})$, $m_N = \min I_N$, and $x_N^{\alpha,I} = x_1^{\alpha_{m_N}+1,I_N}$.

For our next lemma, we define a convenient notation. If $x \in c_{00}$ and $E \subset \mathbb{N}$, we let Ex be the sequence defined by $Ex(n) = 1_E(n)x_n$.

Lemma 3.1. *If $I = (i_n) \in [\mathbb{N}]^\omega$ is such that $3i_n \le i_{n+1}$ and $E \in S_\alpha$, then*

$$\Big\|E\Big(\sum_{n=1}^{\infty} x_n^{\alpha,I}\Big)\Big\|_1 \le 2.$$

Proof. By induction. Since S_α is hereditary for each α, it suffices to consider $E \subset I$. If $\alpha = 0$, the claim is clear, since $\emptyset \ne E \in S_0$ means E is a singleton, and $(x_n^{0,I}) = (e_{i_n})$.

Next, assume the claim holds for the ordinal α. Let $E = \bigcup_{k=1}^m E_k \in S_{\alpha+1}$, $E_k \in S_\alpha$. Let $m_n = \min\operatorname{supp}(x_n^{\alpha+1,I})$. If the set $\{n : \operatorname{supp}(x_n^{\alpha+1,I}) \cap E \ne \emptyset\}$ is empty, then the claim is trivial. Suppose this set is non-empty, and let N be its minimum. Then $m \le \min E \le m_{N+1}/3$, and inductively, $m \le m_{N+n}/3^n$ for each $n \ge 1$. Since there exists a sequence $0 = p_0 < p_1 < \cdots$ with

$$\operatorname{supp}(x_n^{\alpha+1,I}) = m_n^{-1}\sum_{j=p_{n-1}+1}^{p_n} x_j^{\alpha,I},$$

our inductive hypothesis gives, for each $j \le m$,

$$\|E_j x_n^{\alpha+1,I}\|_1 \le 2/m_n.$$

Then

$$\begin{aligned}\Big\|E\Big(\sum_{n=1}^{\infty} x_n^{\alpha,I}\Big)\Big\|_1 &\le \|x_N^{\alpha,I}\|_1 + \sum_{n=1}^{\infty}\sum_{j=1}^{m}\|E_j x_{N+n}^{\alpha,I}\|_1 \\ &\le 1 + \sum_{n=1}^{\infty}\frac{2m}{m_{N+n}} \le 1 + 2\sum_{n=1}^{\infty}3^{-n} = 2.\end{aligned}$$

Finally, let $\alpha < \omega_1$ be a limit ordinal and assume the claim holds for all $\beta < \alpha$. Let $\alpha_n \uparrow \alpha$ be the ordinals used to define $\mathcal{S}_\alpha$. If $E \in S_\alpha$, let $N = \min\{n : \operatorname{supp}(x_n^{\alpha,I}) \cap E \ne \emptyset\}$. Let $m = \min E$, $m_n = \min\operatorname{supp} x_n^{\alpha,I}$. For each $n \geq 1$, $m < m_{N+n}$. Since $E \in S_\alpha$, it follows that $E \in S_{\alpha_m+1} \subset S_{\alpha_{m_{N+n}}}$. As

$$x_{N+n}^{\alpha,I} = x_1^{\alpha_{m_{N+n}}+1,I} = m_{N+n}^{-1} \sum_{k=p_n}^{p_n+m_{N+n}} x_k^{\alpha_{m_{N+n}},I}$$

for some p_n, the inductive hypothesis implies

$$\|Ex_{N+n}^{\alpha,I}\|_1 \leq 2/m_{N+n} \leq 2/3^n.$$

As in the successor ordinal case,

$$\Big\|E\Big(\sum_{n=1}^{\infty} x_n^{\alpha,I}\Big)\Big\|_1 \leq \|x_N^{\alpha,I}\|_1 + \sum_{n=1}^{\infty} \|Ex_{N+n}^{\alpha,I}\|_1 \leq 1 + 2\sum_{n=1}^{\infty} 3^{-n} = 2. \ \blacksquare$$

We finally define the spaces which we will use to prove our theorems, as well as deduce some of their properties. For $\alpha < \omega_1$, we define the norm $\|\cdot\|_\alpha$ on c_{00} by

$$\|x\|_\alpha = \sup_{E \in S_\alpha} \|Ex\|_1.$$

The completion of c_{00} under this norm is known as the *Schreier space of order* α, and denoted X_α. We see that the canonical basis (e_n) of c_{00} becomes a normalized, 1-unconditional basis for X_α. We note also that the canonical basis is shrinking in X_α (this follows, for example, from [9], where it was shown that X_α contains no copy of ℓ_1). We will consider spaces of the form $X_\alpha^p = (X_\alpha)^{\ell_p}$, as defined in Section 2. The space X_1^2 was introduced by Baernstein, and the generalizations X_1^p were studied by Seifert [3]. For this reason, we refer here to X_α^p as the *Baernstein space of order* α *and parameter* p.

We note that for $x \in c_{00}$,

$$\begin{aligned}\|x\|_{X_\alpha^p} &= \sup\Big\{\Big(\sum_j\Big(\sum_{i\in E_j}|x_i|\Big)^p\Big)^{1/p} : E_1 < E_2 < \cdots,\ E_j \in S_\alpha\Big\} \\ &= \sup\{\|(\|E_jx\|_1)_j\|_{\ell_p} : E_1 < E_2 < \cdots,\ E_j \in S_\alpha\},\end{aligned}$$

with the appropriate modification to the first line if $p = \infty$. The same is true if the suprema run over all finite sequences $E_1 < \cdots < E_n$, $E_j \in S_\alpha$. We collect some relevant facts about the unit vector basis (e_n) of X_α^p in the following lemma.

LEMMA 3.2. *Fix $\alpha < \omega_1$ and $1 < p < \infty$. Then the unit vector basis (e_n) of X_α^p is shrinking, boundedly complete, right dominant, and satisfies subsequential X_α^p-upper block estimates in X_α^p.*

Proof. Since the unit vector basis of X_α is shrinking, it is shrinking and boundedly complete in X_α^p by [12, Lemma 8 and Corollary 7]. Therefore X^p_α is reflexive and the coordinate functionals (e_n^*) form a normalized, 1-unconditional basis for $(X^p_\alpha)^*$.

Take $(m_n), (k_n)$ such that $m_n \le k_n$. Fix $a_n \in c_{00}$. Let $x = \sum a_n e_{m_n}$ and $y = \sum a_n e_{k_n}$. There exists a sequence $E_1 < E_2 < \cdots$ with $E_j \in S_\alpha$ for each j such that

$$\|x\|^p_{X^p_\alpha} = \sum_j \Big(\sum_{i \in I_j} |a_i| \Big)^p,$$

where $I_j = \{i : m_i \in E_j\}$. Let $M_j = \{m_i : i \in I_j\}$. Then $M_j \subset E_j$, and we can assume $M_j = E_j$. Let $K_j = \{k_i : i \in I_j\}$. Then K_j is a spread of M_j, and thus $K_j \in S_\alpha$. Clearly, we also have $K_1 < K_2 < \cdots$ and

$$\|y\|^p_{X^p_\alpha} \ge \sum_j \Big(\sum_{i \in I_j} |a_i| \Big)^p = \|x\|^p_{X^p_\alpha}.$$

Therefore (e_n) is 1-right dominant in X^p_α.

Next, take $E_1 < E_2 < \cdots$, $E_j \in S_\alpha$, and (z_n) a normalized block sequence in X^p_α with $m_n = \min \operatorname{supp} z_n$. We can write $z_n = w_n + x_n + y_n$, where $(w_n), (x_n), (y_n)$ are subnormalized and such that the support of each w_n or y_n intersects at most one E_j, and for each j there exists at most one n such that $E_j \cap \operatorname{supp} x_n \neq \emptyset$. Let

$$J = \{ j \in \mathbb{N} : E_j \cap \operatorname{supp} z_n \neq \emptyset \text{ for some } n \}.$$

By [4, Proposition 3.1], there exists a sequence $(F_j)_{j \in J}$ of successive sets such that $F_j \in S_\alpha$ for each $j \in J$ and

$$\Big\| E_j \Big(\sum a_n x_n \Big) \Big\|_1 \le 2 \Big\| F_j \Big(\sum a_n e_{m_n} \Big) \Big\|_1.$$

This means

$$\begin{aligned} \Big(\sum_j \Big\| E_j \Big(\sum_n a_n x_n \Big) \Big\|_1^p \Big)^{1/p} &\le 2 \Big(\sum_j \Big\| F_j \Big(\sum_n a_n e_{m_n} \Big) \Big\|_1^p \Big)^{1/p} \\ &\le 2 \Big\| \sum_{n=1}^\infty a_n e_{m_n} \Big\|_{X^p_\alpha}. \end{aligned}$$

Moreover, since for each n there exists at most one j_n such that $E_j \cap \operatorname{supp} w_n \neq \emptyset$, and since the unit vector basis of X^p_α clearly 1-dominates the unit vector basis of ℓ_p, we deduce (with the unindexed sums taken over all n such that there exists some j with $E_j \cap \operatorname{supp} w_n \neq \emptyset$) that

$$\begin{aligned} \sum_j \Big\| E_j \Big(\sum_k a_k w_k \Big) \Big\|_1^p &= \sum \Big\| E_{j_n} \Big(\sum_k a_k w_k \Big) \Big\|_1^p = \sum |a_n|^p \|E_{j_n} w_n\|_1^p \\ &\le \sum |a_n|^p \|w_n\|^p_{X_\alpha} \le \sum |a_n|^p \le \Big\| \sum_{n=1}^\infty a_n e_{m_n} \Big\|^p_{X^p_\alpha}. \end{aligned}$$

Similarly,

$$\sum_j \Big\| E_j\Big(\sum_n a_n y_n\Big)\Big\|_1^p \le \Big\|\sum_{n=1}^\infty a_n e_{m_n}\Big\|_{X_\alpha^p}^p.$$

Therefore

$$\begin{aligned}\Big(\sum_j \Big\| E_j\Big(\sum a_n z_n\Big)\Big\|_1^p\Big)^{1/p} & \\ \le \Big(\sum_j \Big\| E_j\Big(\sum a_n w_n\Big)\Big\|_1^p\Big)^{1/p} &+ \Big(\sum_j \Big\| E_j\Big(\sum a_n x_n\Big)\Big\|_1^p\Big)^{1/p} \\ + \Big(\sum_j \Big\| E_j\Big(\sum a_n y_n\Big)\Big\|_1^p\Big)^{1/p} & \\ \le 4\Big\|\sum a_n e_{m_n}\Big\|_{X_\alpha^p}. &\end{aligned}$$

Since $E_1 < E_2 < \cdots$ was arbitrary, we deduce that (e_n) satisfies subsequential 4-X_α^p-upper block estimates in X_α^p. ∎

We conclude this section with the following extension of Lemma 3.1.

LEMMA 3.3. *Fix $1 \le p < \infty$. If $I = (i_n) \in [\mathbb{N}]^\omega$ is such that $i_{n+1} \ge 3i_n$, $\alpha < \omega_1$, and $(x_n^{\alpha,I})$ is the sequence of repeated averages, then $(x_n^{\alpha,I})$ as a sequence in X_α^p is 5-equivalent to the unit vector basis of ℓ_p.*

Proof. Since $\|x_n^{\alpha,I}\|_1 = 1$ and $\operatorname{supp} x_n^{\alpha,I} \in S_\alpha$, the sequence of repeated averages is a normalized block sequence in X_α^p. Consequently, it 1-dominates the unit vector basis of ℓ_p. Fix $E_1 < E_2 < \cdots$, so that $E_j \in S_\alpha$ for each j. Fix $(a_n) \in c_{00}$. Let $z = \sum a_n x_n^{\alpha,I}$. We can assume $E_j \subset I$ for each j by replacing E_j with $E_j \cap I$ without changing the value of $\sum \|E_j z\|_1^p$. As before, we can decompose $x_n^{\alpha,I} = w_n + x_n + y_n$ so that $(w_n), (x_n), (y_n)$ are subnormalized block sequences, for each n, $\operatorname{supp} w_n$ meets E_j for at most one j, $\operatorname{supp} y_n$ meets E_j for at most one j, and for each j, E_j meets $\operatorname{supp} x_n$ for at most one n. Let $J_n = \{j : E_j \cap \operatorname{supp} x_n \ne \emptyset\}$, and note that $J_1 < J_2 < \cdots$. Let $x = \sum a_n x_n$. Then

$$\sum_j \|E_j x\|_1^p = \sum_{n=1}^\infty |a_n|^p \sum_{j\in J_n} \|E_j x_n\|_1^p \le \sum_{n=1}^\infty |a_n|^p \|x_n\|_{X_\alpha^p}^p \le \sum_{n=1}^\infty |a_n|^p.$$

Next, let $N_j = \{n : E_j \cap \operatorname{supp} w_n \ne \emptyset\}$, $w = \sum a_n w_n$. Note that $N_1 < N_2 < \cdots$. By Lemma 3.1,

$$\|E_j w\|_1 \le 2 \max_{n\in N_j} |a_n| \le 2\Big(\sum_{n\in N_j} |a_n|^p\Big)^{1/p}.$$

Therefore

$$\sum_j \|E_j w\|_1^p \le 2^p \sum_j \sum_{n\in N_j} |a_n|^p \le 2^p \sum_n |a_n|^p.$$

Similarly, if $y = \sum a_n y_n$, then

$$\sum_j \|E_j y\|_1^p \le 2^p \sum |a_n|^p.$$

Therefore

$$\|z\|_{X_\alpha^p} \le \|w\|_{X_\alpha^p} + \|x\|_{X_\alpha^p} + \|y\|_{X_\alpha^p} \le 5\Big(\sum |a_n|^p\Big)^{1/p}. \ \blacksquare$$

4. Ordinal indices. First, we recall the Szlenk index of a separable Banach space. Let X be a Banach space, and K be a weak* compact subset of X^*. For $\varepsilon > 0$, we define

$$(K)'_\varepsilon = \{z \in K : \text{for all } w^*\text{-neighborhoods } U \text{ of } z, \operatorname{diam}(U \cap K) > \varepsilon\}.$$

It is easily verified that $(K)'_\varepsilon$ is also weak* compact. We let

$$P_0(K,\varepsilon) = K, \quad P_{\alpha+1}(K,\varepsilon) = (P_\alpha(K,\varepsilon))'_\varepsilon, \quad \alpha < \omega_1,$$

and

$$P_\alpha(K,\varepsilon) = \bigcap_{\beta<\alpha} P_\beta(K,\varepsilon), \quad \alpha < \omega_1,\ \alpha \text{ a limit ordinal}.$$

If there exists some $\alpha < \omega_1$ so that $P_\alpha(K,\varepsilon) = \emptyset$, we define

$$\eta(K,\varepsilon) = \min\{\alpha : P_\alpha(K) = \emptyset\}.$$

Otherwise, we set $\eta(K,\varepsilon) = \omega_1$. Then we define the *Szlenk index* of a Banach space X, denoted $\mathrm{Sz}(X)$, to be

$$\mathrm{Sz}(X) = \sup_{\varepsilon>0} \eta(B_{X^*},\varepsilon).$$

The Szlenk index is one of several slicing indices. The following two facts come from [15]:

(1) For a Banach space X, $\mathrm{Sz}(X) < \omega_1$ if and only if X^* is separable.
(2) If X embeds isomorphically into Y, then $\mathrm{Sz}(X) \le \mathrm{Sz}(Y)$.

The above definition of the index is, in some cases, intractable. A connection between weak indices and the Szlenk index has been very useful in computations. For this, we will be concerned with a specific type of tree.

For a Banach space X and $\rho \in (0,1]$, we let

$$\mathcal{H}_\rho^X = \Big\{(x_n) \in S_X^{<\omega} : \Big\|\sum a_n x_n\Big\| \ge \rho \sum a_n,\ \forall (a_n) \subset \mathbb{R}^+\Big\}.$$

We will compute the Szlenk index of Baernstein spaces by combining several facts about the Szlenk index.

Theorem 4.1 ([1, Theorems 3.22, 4.2], [13, Proposition 5]). *If X is a Banach space such that X^* is separable, there exists some ordinal $\alpha < \omega_1$ so that $\mathrm{Sz}(X) = \omega^\alpha$. Moreover for any $\alpha < \omega_1$, $\mathrm{Sz}(X) > \omega^\alpha$ if and only if there exist $\rho \in (0,1]$ and $(x_E)_{E\in\mathcal{S}_\alpha\setminus\{\emptyset\}} \subset S_X$ such that for each $E \in \mathcal{S}_\alpha \setminus \mathrm{MAX}(\mathcal{S}_\alpha)$, $(x_{E\cup\{n\}})_{n>E}$ is weakly null and for each branch $E_1 \prec \cdots \prec E_n$ of $\mathcal{S}_\alpha \setminus \{\emptyset\}$, $(x_{E_i})_{i=1}^n \in \mathcal{H}_\rho^X$.*

With this, we can prove the following.

Proposition 4.2. *For $\alpha < \omega_1$ and $p \in (1,\infty)$, $\mathrm{Sz}(X_\alpha^p) = \omega^{\alpha+1}$.*

Proof. Let (e_n) denote the unit vector basis of X_α^p. For $E \in S_\alpha \setminus \{\emptyset\}$, let $x_E = e_{\max E}$. If $E_1, \dots, E_n$ is a branch of S_α, then $(x_{E_i})_{i=1}^n = (e_i)_{i\in E_n}$. Clearly

$$\Big\|\sum_{i=1}^n a_i x_{E_i}\Big\|_{X_\alpha^p} = \sum_{i\in E} a_i \quad \text{for } a_i \geq 0.$$

Since the basis is normalized and shrinking, we deduce that for $E \in S_\alpha \setminus \mathrm{MAX}(S_\alpha)$, $(x_{E\cup\{n\}})_{n>E} = (e_n)_{n>E}$ is weakly null. Then Theorem 4.1 guarantees $\mathrm{Sz}(X_\alpha^p) > \omega^\alpha$. We must therefore only show that $\mathrm{Sz}(X_\alpha^p) \leq \omega^{\alpha+1}$.

Suppose not. By Theorem 4.1, there must exist some normalized tree $(x_E)_{E\in S_{\alpha+1}\setminus\{\emptyset\}} \subset \mathcal{H}_\rho^{X_\alpha^p}$ with $x_{E\cup\{n\}} \underset{w}{\to} 0$. By standard perturbation and pruning arguments, we can assume this tree is a block tree. For $E \in S_{\alpha+1}\setminus\{\emptyset\}$, let $m(E) = \min\operatorname{supp} x_E$. Because the basis is normalized, shrinking, and satisfies subsequential 4-X_α^p-upper block estimates in X_α^p, we can replace ρ with $\rho/4$ and replace the tree $(x_E)_{E\in S_{\alpha+1}\setminus\{\emptyset\}}$ with $(e_{m(E)})_{E\in S_{\alpha+1}\setminus\{\emptyset\}}$ while maintaining the two properties mentioned above. Choose i_1 so large that $5i_1^{1/p} < (\rho/16)i_1$. Next, choose $i_2 < \cdots < i_N$ such that $i_n > 3i_{n-1}$ and $m(\{i_1,\dots,i_{n-1}\}) < i_n$ for each $n = 2,\dots,N$ and $E = \{i_1,\dots,i_N\} \in \mathrm{MAX}(\mathcal{S}_{\alpha+1})$. Since $\mathcal{S}_{\alpha+1}$ is compact, it can contain no strictly increasing infinite chain, so such a set must exist.

Since $i_n \leq m(\{i_1,\dots,i_n\}) < i_{n+1}$, the sequence $(e_{i_n})_{n=1}^N$ 4-dominates $(e_{m(\{i_1,\dots,i_n\})})_{i=1}^N$. This follows from an application of Lemma 2.1 after we recall that (e_n) satisfies subsequential 4-X_α^p-upper block estimates in X_α^p. Therefore for any $a_n \geq 0$,

$$\Big\|\sum_{n=1}^N a_n e_{i_n}\Big\|_{X_\alpha^p} \geq \frac{1}{4}\Big\|\sum_{n=1}^N a_n e_{m(\{i_1,\dots,i_n\})}\Big\|_{X_\alpha^p} \geq \frac{\rho}{16}\sum a_n. \tag{4.1}$$

Since $E \in S_{\alpha+1}$, $\min E = i_1$, and $E \in \mathrm{MAX}(S_{\alpha+1})$, there exist unique $E_n \in S_\alpha$ with $E_1 < \cdots < E_{i_1}$ and $E = \bigcup_{n=1}^{i_1} E_n$. Let $I = E \cup \{3^k i_N : k \in \mathbb{N}\}$. Then $i_{n+1} \geq 3i_n$ for each n. If $(x_n^{\alpha,I})$ is the sequence of repeated averages, then $\operatorname{supp} x_n^{\alpha,I} = E_n$ for $1 \leq n \leq i_1$. Let a_j be such that $x_n^{\alpha,I} = \sum_{j\in E_n} a_j e_j$.

Then $\sum_{j\in E_n} a_j = 1$, so

$$\sum_{n=1}^{i_1}\sum_{j\in E_n} a_j = i_1. \tag{4.2}$$

But by Lemma 3.3,

$$\Big\|\sum_{n=1}^{i_1}\sum_{j\in E_n} a_j e_j\Big\|_{X^p_\alpha} = \Big\|\sum_{n=1}^{i_1} x_n^{\alpha,I}\Big\|_{X^p_\alpha} \le 5\Big\|\sum_{n=1}^{i_1} e_n\Big\|_{\ell_p} = 5i_1^{1/p}. \tag{4.3}$$

Combining (4.1)–(4.3), we deduce that

$$5i^{1/p} \ge \Big\|\sum_{n=1}^{i_1}\sum_{j\in E_n} a_j e_{i_j}\Big\|_{X^p_\alpha} \ge \frac{\rho}{16}\sum_{n=1}^{i_1}\sum_{j\in E_n} a_j = \frac{\rho}{16} i_1.$$

But this contradicts our choice of i_1, and completes the proof. ■

5. Main theorems. Throughout this section, $Z^{V^{(*)}}$ will denote $Z^{(V^{(*)})}$.

Proof of Lemma 2.2. First, we observe that if every normalized block of (v_n) is dominated by every normalized block of (u_n), then there exists C such that every normalized block of (v_n) is C-dominated by every normalized block of (u_n). By replacing C with a larger constant if necessary, we may also assume that E satisfies subsequential C-U-upper block estimates in Z. We may assume that E is bimonotone in Z, since renorming Z to make E bimonotone will have the consequence of equivalently renorming $Z^V(E)$.

Fix $(a_n)\in c_{00}$ and let $u=\sum_n a_n u_n$. Fix $1\le k_0<k_1<\cdots$. Let $N=\{n\in\mathbb{N} : P_{[k_{n-1},k_n)}u\ne 0\}$. For $n\in N$, let $x_n=P_{[k_{n-1},k_n)}u$, $y_n=x_n/\|x_n\|$, and $c_n=\|x_n\|$. Then $u=\sum_{n\in N} c_n y_n$. Moreover,

$$\begin{aligned}\Big\|\sum_n \|P_{[k_{n-1},k_n)}u\|_U v_{k_{n-1}}\Big\|_V &= \Big\|\sum_{n\in N} \|P_{[k_{n-1},k_n)}u\|_U v_{k_{n-1}}\Big\|_V\\ &= \Big\|\sum_{n\in N} c_n v_{k_{n-1}}\Big\|_V \le C\Big\|\sum_{n\in N} c_n y_n\Big\|_U = C\|u\|.\end{aligned}$$

This means the U- and $U^V(u_n)$-norms are C-equivalent on c_{00}.

Fix a normalized block sequence (z_n) in $Z^V(E)$. Let $m_n=\min\operatorname{ran}_E(z_n)$. Fix $(a_n)\in c_{00}$ and let $z=\sum a_n z_n$. Choose $1\le k_1<\cdots<k_N$ so that

$$\|z\|_{Z^V(E)} = \Big\|\sum_{i=1}^N \|P^E_{[k_{i-1},k_i)}z\|_Z v_{k_{i-1}}\Big\|_V.$$

For $n\in\mathbb{N}$, let

$$I_n=\{i\le N : [k_{i-1},k_i)\subset[\min\operatorname{ran}_E(z_n),\min\operatorname{ran}_E(z_{n+1}))\}.$$

Let $I=\{1,\dots,N\}\setminus\bigcup_n I_n$. For each $i\in I$, let
$$J_i=\{n\in\mathbb{N}:[k_{i-1},k_i)\cap\operatorname{ran}_E(z_n)\neq\emptyset\}.$$
Note that the $(I_n)_{n\in\mathbb{N}}$ are pairwise disjoint. The $(J_i)_{i\in I}$ need not be pairwise disjoint, but if $I=I'\cup I''$ is a partition of I so that neither I' nor I'' contains consecutive elements of I, then $(J_i)_{i\in I'}$ are pairwise disjoint, and so are $(J_i)_{i\in I''}$. We have
$$\begin{aligned}\|z\|_{Z^V(E)}&\le\Big\|\sum_{i\notin I}\|P^E_{[k_{i-1},k_i)}z\|_Z v_{k_{i-1}}\Big\|_V+\Big\|\sum_{i\in I}\|P^E_{[k_{i-1},k_i)}z\|_Z v_{k_{i-1}}\Big\|_V\\&\le\Big\|\sum_n\sum_{i\in I_n}a_n\|P^E_{[k_{i-1},k_i)}z_n\|_Z v_{k_{i-1}}\Big\|_V\\&\quad+\Big\|\sum_{i\in I'}\Big\|P^E_{[k_{i-1},k_i)}\Big(\sum_{n\in J_i}a_nz_n\Big)\Big\|_Z v_{k_{i-1}}\Big\|_V\\&\quad+\Big\|\sum_{i\in I''}\Big\|P^E_{[k_{i-1},k_i)}\Big(\sum_{n\in J_i}a_nz_n\Big)\Big\|_Z v_{k_{i-1}}\Big\|_V.\end{aligned}$$
We will bound each term by a multiple of $\|\sum a_nu_{m_n}\|_U$. Let
$$y_n=\sum_{i\in I_n}\|P^E_{[k_{i-1},k_i)}z_n\|_Z v_{k_{i-1}}.$$
Then $\|y_n\|_V\le\|z_n\|_{Z^V(E)}\le 1$. Hence
$$\Big\|\sum_n\sum_{i\in I_n}a_n\|P^E_{[k_{i-1},k_i)}z_n\|_Z v_{k_{i-1}}\Big\|_V=\Big\|\sum a_ny_n\Big\|_V\le C\Big\|\sum a_nu_{m_n}\Big\|_U.$$

Moreover, by bimonotonicity and the fact that E satisfies subsequential C-U-upper block estimates in U, we can use Proposition 2.1 to deduce that for each $i\in I$,
$$\Big\|P^E_{[k_{i-1},k_i)}\Big(\sum_{n\in J_i}a_nz_n\Big)\Big\|_Z\le\Big\|\sum_{n\in J_i}a_nz_n\Big\|_Z\le\Big\|\sum_{n\in J_i}a_nu_{m_n}\Big\|_U.$$
Then
$$\begin{aligned}\Big\|\sum_{i\in I'}\Big\|P^E_{[k_{i-1},k_i)}\Big(\sum_{n\in J_i}a_nz_n\Big)\Big\|_Z v_{k_{i-1}}\Big\|_V&\le C\Big\|\sum_{i\in I'}\Big\|\sum_{n\in J_i}a_nu_{m_n}\Big\|_U v_{k_{i-1}}\Big\|_V\\&\le C\Big\|\sum_{i\in I'}\sum_{n\in J_i}a_nu_{m_n}\Big\|_{U^V(u_n)}\\&\le C^2\Big\|\sum a_nu_{m_n}\Big\|_U.\end{aligned}$$
A similar estimate holds for the sum over I''. ∎

Our first major theorem generalizes [12, Theorem 15].

THEOREM 5.1. *Let U, V be reflexive Banach spaces with normalized, 1-unconditional bases $(u_n), (v_n)$, respectively, so that (u_n) is right dominant and satisfies subsequential U-upper block estimates in U, (v_n) is left dominant and satisfies subsequential V-lower block estimates in V, and so that every normalized block sequence of (v_n) is dominated by every normalized block sequence of (u_n). If X is a separable, reflexive Banach space which satisfies subsequential (V, U)-tree estimates, then X embeds into a reflexive Banach space $\tilde{X}$ with bimonotone FDD E satisfying subsequential (V, U)-block estimates.*

Proof. Since X satisfies subsequential U-upper tree estimates, [12, Proposition 4] implies that X^* satisfies subsequential U^*-lower tree estimates. By [12, Theorem 12(b)], there exists a Banach space Y with bimonotone shrinking FDD F and $M \in [\mathbb{N}]^\omega$ such that X^* is a quotient of $Z = Y^{U^*_M}(F)$. By [4, Lemma 2.11], F satisfies subsequential U^*_M-lower block estimates in Z. By [4, Lemma 2.13], the space $W = Z \oplus U^*_{\mathbb{N}\setminus M}$ has a bimontone FDD G satisfying subsequential U^*-lower block estimates. Then X^* is a quotient of W, and W is reflexive by [12, Corollaries 7, 9]. By duality, X is a subspace of W^*, which is reflexive with bimonotone FDD $G^* = (G^*_n)$ satisfying subsequential U-upper block estimates in W^*.

By [12, Theorem 12(a)], there exists a blocking H of G^* defined by $H_k = \bigoplus_{i=b_k}^{b_{k+1}-1} G^*_i$ for some $1 = b_1 < b_2 < \cdots$ and $C \in [\mathbb{N}]^\omega$ so that $X \hookrightarrow (W^*)^{V_C}(H)$. We deduce from the fact that G^* satisfies subsequential U-upper block estimates in W^* that H satisfies subsequential U_B-upper block estimates in W^*. Let $k_i = \max\{b_i, c_i\}$. Since (u_n) is right dominant, H satisfies subsequential U_K-upper block estimates in W^*. Lemma 2.2 implies that H satisfies subsequential U_K-upper block estimates in $(W^*)^{V_C}(H)$. By [4, Lemma 2.11], H satisfies subsequential V_C-lower block estimates in $(W^*)^{V_C}(H)$, and since (v_n) is left dominant, H satisfies subsequential V_K-lower block estimates in $(W^*)^{V_C}(H)$. By the proof of [12, Lemma 2], we deduce that $\tilde{X} = (W^*)^{V_C}(H) \oplus V_{\mathbb{N}\setminus K}$ has bimonotone FDD satisfying subsequential (V, U)-block estimates in $\tilde{X}$. Again, [12, Corollaries 7, 9] guarantee that $\tilde{X}$ is reflexive, and this completes the proof. ∎

The next theorem is a generalization of [12, Theorem 21], and an adaptation of [4, Theorem 5.4] to the present situation. Let us recall that if U, V are Banach spaces with normalized, 1-unconditional bases, then $\mathcal{A}_{(V,U)}$ denotes the class of separable, reflexive Banach spaces satisfying subsequential (V, U)-tree estimates.

THEOREM 5.2. *Let U, V be Banach spaces with basis satisfying the hypotheses of Theorem 5.1. Then the class $\mathcal{A}_{(V,U)}$ contains a reflexive universal element with bimonotone FDD.*

Proof. Fix constants R, L, K so that (u_n) is R-right dominant and satisfies subsequential K-U-upper block estimates in U, and so that (v_n) is L-left dominant and satisfies subsequential K-V-lower block estimates in V.

By a result of Schechtman [14], there exists a Banach space W with bimontone FDD $E = (E_n)$ with the property that any Banach space with bimonotone FDD embeds almost isometrically into $\overline{\bigoplus_{n=1}^\infty E_{k_n}}$ for some subsequence (k_n) of the natural numbers, and this subspace is 1-complemented in W. More precisely, given a Banach space X with bimonotone FDD (F_i) and $\varepsilon > 0$, there is a subsequence (E_{k_n}) of (E_n) and a $(1+\varepsilon)$-embedding $T : X \to W$ such that $T(F_n) = E_{k_n}$ for all $n \in \mathbb{N}$, and $\sum_{n=1}^\infty P_{k_n}^E$ is a norm-1 projection of W onto $\overline{\bigoplus_{n=1}^\infty E_{k_n}}$.

We next consider the space $W_0 = (W^{(*)})^{U^*}(E^*)$. By [12, Corollary 7], the sequence (E_n^*) is a boundedly complete and bimonotone FDD for this space. This means that $W_0 = (W_0^{(*)})^*$ and $(E_n^{**}) = (E_n)$ is a shrinking, bimonotone FDD for $W_0^{(*)}$. Therefore W_0 is naturally the dual of the space $Y = W_0^{(*)}$ with bimonotone shrinking FDD E. By duality and [4, Lemma 2.11], we deduce that E satisfies subsequential $2K$-U-upper block estimates in Y.

Let $Z = Y^V(E)$. By Lemma 2.2, E satisfies subsequential U-upper block estimates in Z. By [4, Lemma 2.11], E satisfies subsequential V-lower block estimates in Z. By [12, Corollary 7 and Lemma 8], E is a shrinking, boundedly complete FDD for Z. Therefore $Z \in \mathcal{A}_{(V,U)}$. We see also that E is bimonotone in Z. It remains to show the universality of Z for $\mathcal{A}_{(V,U)}$.

Let $D \geq 1$ and assume X satisfies subsequential D-(V,U)-tree estimates. By Theorem 5.1, there exists a reflexive Banach space $\tilde{X}$ with bimonotone FDD D satisfying subsequential (V,U)-block estimates in $\tilde{X}$ so that X embeds isomorphically into $\tilde{X}$. Thus it suffices to assume that X itself has a bimonotone FDD F satisfying subsequential D_1-(V,U)-block estimates and show that X embeds into Z. We can find a subsequence (k_n) of $\mathbb{N}$ and a 2-embedding $T : X \to W$ so that $T(F_n) = E_{k_n}$ for all $n \in \mathbb{N}$ and $\sum_n P_{k_n}^E$ is a norm-1 projection of W onto $\overline{\bigoplus_n E_{k_n}}$. It follows that (E_{k_n}) satisfies subsequential $2D_1$-(V,U)-estimates in W. By duality, $(E_{k_n}^*)$ satisfies subsequential (U^*, V^*)-estimates in $W^{(*)}$. We will finally prove that the norms $\|\cdot\|_W, \|\cdot\|_Y, \|\cdot\|_Z$ are equivalent when restricted to $c_{00}(\bigoplus_n E_{k_n})$.

Fix $w^* \in c_{00}(\bigoplus_n E_{k_n})$. We know that $\|w^*\|_{W^{(*)}} \leq \|w^*\|_{Y^*}$. Choose $1 \leq m_0 < m_1 < \cdots < m_N$ in $\mathbb{N}$ such that

$$\|w^*\|_{Y^*} = \Big\|\sum_{n=1}^N \|P_{[m_{n-1},m_n)}^{E^*} w^*\|_{W^{(*)}} u_{m_{n-1}}^*\Big\|_V.$$

By discarding any m_n such that $P_{[m_{n-1},m_n)}^{E^*} w^* = 0$, we assume $P_{[m_{n-1},m_n)}^{E^*} w^*$

$\neq 0$ for each $1 \leq n \leq N$ without changing the sum. There exist $j_1 < \cdots < j_N$ such that $m_n > k_{j_n} = \min \operatorname{supp}_{E^*} P^{E^*}_{[m_{n-1},m_n)} w^* \geq m_{n-1}$ for each $1 \leq n \leq N$. Since (u_n^*) satisfies subsequential K-lower block estimates in U^* and is R-left dominant, and since $(E^*_{k_n})$ satisfies subsequential $2D_1$-U^*-lower block estimates in $W^{(*)}$, we see that

$$\begin{aligned}\|w^*\|_{Y^*} &\leq K \Big\| \sum_{n=1}^{N} \|P^{E^*}_{[m_{n-1},m_n)} w^*\|_{W^{(*)}} u^*_{k_{j_{n-1}}} \Big\|_{(U^p)^*} \\ &\leq KR \Big\| \sum_{n=1}^{N} \|P^{E^*}_{[m_{n-1},m_n)} w^*\|_{W^{(*)}} u^*_{j_n} \Big\|_{(U^p)^*} \leq 2KRD_1 \|w^*\|_{W^{(*)}}.\end{aligned}$$

This shows $\|\cdot\|_{W^{(*)}}$ and $\|\cdot\|_{Y^*}$ are equivalent on $c_{00}(\sum_n E^*_{k_n})$. One easily sees that $\sum_n P^{E^*}_{k_n}$, which defines a norm-1 projection of $W^{(*)}$ onto $\overline{\bigoplus_n E^*_{k_n}}$, is also a norm-1 projection of Y^* onto $\overline{\bigoplus_n E^*_{k_n}}$. It follows that

$$\frac{1}{2KRD_1} \|w\|_W \leq \|w\|_Y \leq \|w\|_W$$

for all $w \in c_{00}(\sum_n E_{k_n})$.

A very similar argument shows that $\|y\|_Y \leq \|y\|_Z \leq 2KLD_1 \|y\|_Y$ for each $y \in c_{00}(\sum_n E_{k_n})$. Therefore the map $T : X \to W$ becomes an $8K^2RLD_1^2$-embedding of X into Z. ∎

For our next theorem, we define, for an ordinal $\alpha < \omega_1$,

$$\mathcal{C}_\alpha = \{X : X \text{ separable, reflexive, } \operatorname{Sz}(X), \operatorname{Sz}(X^*) \leq \omega^\alpha\}.$$

Theorem 5.3. *For any $\alpha < \omega_1$ and $p \in (1,2]$, there exists a Banach space $Z \in \mathcal{C}_{\alpha+1}$ with bimonotone FDD satisfying subsequential $((X^p_\alpha)^*, X^p_\alpha)$-block estimates such that if $X \in \mathcal{C}_\alpha$, then X is isomorphic to a subspace of Z. Moreover, there exists $W \in \mathcal{C}_{\alpha+1}$ with a basis such that if $X \in \mathcal{C}_\alpha$, then X is isomorphic to a subspace of W.*

Proof. Let Z be the universal element of $\mathcal{A}_{((X^p_\alpha)^*, X^p_\alpha)}$ guaranteed by Theorem 5.2. Then Z, Z^* satisfy subsequential X^p_α-upper block estimates, and $\operatorname{Sz}(Z), \operatorname{Sz}(Z^*) \leq \operatorname{Sz}(X^p_\alpha) = \omega^{\alpha+1}$ by [4, Corollary 4.5]. Therefore $Z \in \mathcal{C}_{\alpha+1}$. By [6, Corollary 4.12], there exists a sequence of finite-dimensional spaces (H_n) such that if $D = (\bigoplus_{n=1}^\infty H_n)_2$, then $W = Z \oplus_2 D$ is reflexive and has a basis. Since the FDD (H_n) satisfies ℓ_2-upper block estimates in D, $\operatorname{Sz}(D) \leq \omega$ [11, Theorem 3]. From [13, Proposition 14],

$$\operatorname{Sz}(W) = \max\{\operatorname{Sz}(Z), \operatorname{Sz}(D)\} \leq \omega^{\alpha+1}.$$

By the same reasononing, $\operatorname{Sz}(W^*) = \operatorname{Sz}(Z^* \oplus_2 D^*) \leq \omega^{\alpha+1}$. Therefore $W \in \mathcal{C}_{\alpha+1}$.

If $X \in \mathcal{C}_\alpha$, then [4, Theorem 1.1] implies that X, X^* both satisfy subsequential X_α-upper tree estimates, and therefore also X_α^p-upper tree estimates. Then by [5, Lemma 2.7], X satisfies subsequential $((X_\alpha^p)^*, X_\alpha^p)$-tree estimates. As Z is universal, X embeds isomorphically in Z, and therefore X embeds isomorphically into W. ∎

Acknowledgments. This paper was completed at Texas A&M under the direction of Thomas Schlumprecht as part of the author's doctoral dissertation. The author thanks Dr. Schlumprecht for his insights and direction during its completion.

The author's research was partially supported by the National Science Foundation grant DMS1160633.

References

[1] D. Alspach, R. Judd, and E. Odell, *The Szlenk index and local ℓ_1-indices*, Positivity 9 (2005), 1–44.

[2] S. A. Argyros, S. Mercourakis, and A. Tsarpalias, *Convex unconditionality and summability of weakly null sequences*, Israel J. Math. 107 (1998), 157–193.

[3] P. Casazza, and T. Shura, *Tsirelson's Space*, Lecture Notes in Math. 1363, Springer, Berlin, 1989.

[4] R. Causey, *Estimation of the Szlenk index of Banach spaces via Schreier spaces*, Studia Math. 216 (2013), 149–178.

[5] D. Freeman, E. Odell, Th. Schlumprecht and A. Zsák, *Banach spaces of bounded Szlenk index II*, Fund. Math. 205 (2009), 161–177.

[6] W. B. Johnson, H. Rosenthal and M. Zippin, *On bases, finite-dimensional decompositions, and weaker structures in Banach spaces*, Israel J. Math. 9 (1971), 488–506.

[7] W. B. Johnson and B. Zheng, *A characterization of subspaces and quotients of reflexive Banach spaces with unconditional bases*, Duke Math J. 141 (2008), 505–518.

[8] W. B. Johnson and B. Zheng, *Subspaces and quotients of Banach spaces with shrinking unconditional bases*, Israel J. Math. 185 (2011), 375–388.

[9] R. Judd and E. Odell, *Concerning the Bourgain ℓ_1 index of a Banach space*, Israel J. Math. 108 (1998), 145–171.

[10] E. Odell and Th. Schlumprecht, *Trees and branches in Banach spaces*, Trans. Amer. Math. Soc. 354 (2002), 4085–4108.

[11] E. Odell and Th. Schlumprecht, *Embedding into Banach spaces with finite-dimensional decompositions*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. 100 (2006), 295–323.

[12] E. Odell, Th. Schlumprecht and A. Zsák, *A new infinite game in Banach spaces with applications*, in: Banach Spaces and Their Applications in Analysis, de Gruyter, Berlin, 2007, 147–182.

[13] E. Odell, Th. Schlumprecht and A. Zsák, *Banach spaces of bounded Szlenk index*, Studia Math. 183 (2007), 63–97.

[14] G. Schechtman, *On Pełczyński's paper "Universal bases"*, Israel J. Math. 22 (1975), 181–184.

[15] W. Szlenk, *The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces*, Studia Math. 30 (1968), 53–61.

Ryan Causey
Department of Mathematics
Texas A&M University
College Station, TX 77845, U.S.A.
E-mail: rcausey@math.tamu.edu

Received 25 August 2013;
in revised form 26 August 2014