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On a question of de Groot and Nishiura

by

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Abstract. Let $Z_n = [0,1]^{n+1} \setminus (0,1)^n \times \{0\}$. Then $\operatorname{cmp} Z_n < \operatorname{def} Z_n$ for $n \geq 5$. This is the answer to a question posed by de Groot and Nishiura [GN] for $n \geq 5$.

1. Introduction. A regular space X is called *rim-compact* if there exists a base $\mathcal{B}$ for the open sets of X such that the boundary $\operatorname{Bd} U$ is compact for each U in $\mathcal{B}$.

In 1942 de Groot (cf. [AN]) proved the following:

(∗) *A separable metrizable space X is rim-compact if and only if there is a metrizable compactification Y of X such that* $\operatorname{ind}(Y \setminus X) \leq 0$.

In an attempt to generalize (∗), de Groot introduced two notions, the *small inductive compactness degree* cmp and the *compactness definiency* def (we will recall the definitions in Section 2 and Section 3 respectively). It is known that $\operatorname{cmp} X \leq \operatorname{def} X$ for every separable metrizable space X. The well known conjecture of de Groot (see for example [GN]) was that the two invariants coincided in the class of separable metrizable spaces. As a way to either disprove or support the conjecture, de Groot and Nishiura [GN, p. 213] posed the following

Question 1.1. *Let*

$$Z_n = [0,1]^{n+1} \setminus (0,1)^n \times \{0\}.$$

Is it true that $\operatorname{cmp} Z_n \geq n$ *for* $n \geq 3$*?*

In the cited article, de Groot and Nishiura proved that $\operatorname{def} Z_n = n$ for every $n \geq 1$, and that $\operatorname{cmp} Z_i = i$ for $i = 1, 2$.

In [P1] R. Pol constructed a space $P \subset \mathbb{R}^4$ such that $\operatorname{cmp} P = 1 < \operatorname{def} P = 2$. The space P is a modification of an example given by Lux-

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emburg [L] of a compactum with noncoinciding transfinite inductive dimensions. After that, some other counterexamples to de Groot's conjecture were constructed by Hart (cf. [AN]), Kimura [K], Levin and Segal [LS]. However, Question 1.1 remained open (see also [P2, Question 418] and [AN, Problem 3, p. 71]).

One of our main results in this paper is the following.

THEOREM 1.1. *Let* $n \leq 2^m - 1$ *for some integer* m. *Then* $\operatorname{cmp} Z_n \leq m+1$. *In particular*, $\operatorname{cmp} Z_n < \operatorname{def} Z_n$ *for* $n \geq 5$.

This is an answer to Question 1.1 for $n \geq 5$. Our paper is based on a construction of compacta with noncoinciding transfinite inductive dimensions given in [Ch]. Our terminology follows [E] and [AN].

2. Finite sum theorem for $\mathcal{P}$-ind. In this section, all topological spaces are assumed to be regular T_1 and all classes of topological spaces considered are assumed to be nonempty and to contain any space homeomorphic to a closed subspace of one of their members. The letter $\mathcal{P}$ is used to denote such classes.

Recall the definition of the *small inductive dimension modulo* $\mathcal{P}$, $\mathcal{P}$-ind. Let X be a space.

(i) $\mathcal{P}\text{-ind}\, X = -1$ iff $X \in \mathcal{P}$.

(ii) $\mathcal{P}\text{-ind}\, X \leq n$ (≥ 0) if each point in X has arbitrarily small neighbourhoods V with $\mathcal{P}\text{-ind}\, \operatorname{Bd} V \leq n-1$.

(iii) $\mathcal{P}\text{-ind}\, X = n$ if $\mathcal{P}\text{-ind}\, X \leq n$ and $\mathcal{P}\text{-ind}\, X > n-1$.

(iv) $\mathcal{P}\text{-ind}\, X = \infty$ if $\mathcal{P}\text{-ind}\, X > n$ for $n = -1, 0, 1, \ldots$

It is clear that if $\mathcal{P} = \{\emptyset\}$ then $\mathcal{P}\text{-ind}\, X = \operatorname{ind} X$. If $\mathcal{P}$ is the class of compact spaces then $\mathcal{P}\text{-ind}\, X = \operatorname{cmp} X$.

The following properties of $\mathcal{P}$-ind will be used in the paper.

(1) If A is closed in X then $\mathcal{P}\text{-ind}\, A \leq \mathcal{P}\text{-ind}\, X$.

(2) If $\mathcal{P}\text{-ind}\, X \leq n \geq 0$ and U is open in X then $\mathcal{P}\text{-ind}\, U \leq n$.

(3) If $X = O_1 \cup O_2$, where O_i is open in X, $i = 1, 2$, and $\max\{\mathcal{P}\text{-ind}\, O_1, \mathcal{P}\text{-ind}\, O_2\} \leq n \geq 0$, then $\mathcal{P}\text{-ind}\, X \leq n$.

(4) $\mathcal{P}\text{-ind}\, X \leq n \geq 0$ iff for each point p and each closed set G in X with $p \notin G$ there is a partition S in X between p and G such that $\mathcal{P}\text{-ind}\, S \leq n-1$.

The following statement is implicitly contained in the proofs of [Ch, Theorem 3.9] and [ChK, Theorem 2].

LEMMA 2.1. *Let* X *be a normal space such that* $X = X_1 \cup X_2$, *where each* X_i *is closed in* X, *and* A, B *be two closed disjoint subsets of* X *such that* $A \cap X_i \neq \emptyset$ *and* $B \cap X_i \neq \emptyset$, $i = 1, 2$. *Choose a partition* C_1 *in* X_1 *between* $A \cap X_1$ *and* $B \cap X_1$ *such that* $X_1 \setminus C_1 = U_1 \cup V_1$, *where* U_1, V_1 *are open in* X_1 *and disjoint*, *and* $A \cap X_1 \subset U_1$, $B \cap X_1 \subset V_1$. *Choose also* *a partition C_2 in X_2 between $A \cap X_2$ and $((C_1 \cup V_1) \cup B) \cap X_2$ such that $X_2 \setminus C_2 = U_2 \cup V_2$, where U_2, V_2 are open in X_2 and disjoint, and $A \cap X_2 \subset U_2$, $(C_1 \cup V_1) \cup B) \cap X_2 \subset V_2$. Then the set*

$$C = X \setminus (((U_1 \setminus X_2) \cup U_2) \cup (V_1 \cup (V_2 \setminus X_1)))$$

is a partition in X between A and B such that $C \subset C_1 \cup C_2 \cup (X_1 \cap X_2)$.

Moreover, if X is a regular T_1-space then the same statement is valid for a pair of closed subsets of X where one of the sets is a point.

The following theorem and corollary are generalizations of [ChK, Theorem 2] and [Ch, Corollary 3.10(a)] respectively. Although one might show them similarly to [ChK] and [Ch], we give the proofs for the convenience of the reader.

THEOREM 2.1. *Let X be a space such that $X = X_1 \cup X_2$, where X_i is closed in X and $\mathcal{P}\text{-ind}\, X_i \le n \ge 0$ for every $i = 1, 2$. Then $\mathcal{P}\text{-ind}\, X \le n+1$. Moreover, if X is normal then for any closed subsets A and B of X there exists a partition C in X between A and B such that $\mathcal{P}\text{-ind}\, C \le n$.*

Proof. Let $x \in X$. If $x \in X_1 \setminus X_2$ or $x \in X_2 \setminus X_1$ then x has arbitrarily small open neighbourhoods U with $\mathcal{P}\text{-ind}\,\mathrm{Bd}\, U \le n-1$.

Let now $x \in X_1 \cap X_2$ and B be a closed subset of X such that $x \notin B$ and $B \cap X_i \ne \emptyset$, $i = 1, 2$. Denote the point x by A. Choose partitions C_1 and C_2 as in Lemma 2.1. Then $Y = C_1 \cup C_2 \cup (X_1 \cap X_2) = Y_1 \cup Y_2$, where $Y_i = C_i \cup (X_1 \cap X_2)$. Moreover $\mathrm{Int}\, Y_1 \cup \mathrm{Int}\, Y_2 = Y$ and $\mathcal{P}\text{-ind}\, Y_i \le n$ (recall that $Y_i \subset X_i$). So by properties (1)–(3) of $\mathcal{P}\text{-ind}\, Y$ we have $\mathcal{P}\text{-ind}\, Y \le n$. By Lemma 2.1, there exists a partition C in X between A and B such that $C \subset Y$. Now just observe that $\mathcal{P}\text{-ind}\, C \le n$.

COROLLARY 2.1. *Let X be a space and $q \ge 0$ be an integer. If $X = \bigcup_{k=1}^{n+1} X_k$, where each X_k is closed in X, $0 \le n \le 2^m - 1$ for some integer m and $\max\{\mathcal{P}\text{-ind}\, X_k\} \le q$, then $\mathcal{P}\text{-ind}\, X \le q+m$. In particular, if $\mathrm{cmp}\, X_k = 0$ for $k = 1, \ldots, n+1$, then $\mathrm{cmp}\, X \le m$.*

Proof. Let $n = 2^m - 1$. For every integer j such that $1 \le j \le 2^{m-1}$ put $X_j^{(1)} = X_{2j-1} \cup X_{2j}$. By Theorem 2.1, we have $\mathcal{P}\text{-ind}\, X_j^{(1)} \le q+1$. For every integer p such that $1 \le p \le 2^{m-2}$ put $X_p^{(2)} = X_{2p-1}^{(1)} \cup X_{2p}^{(1)}$. By Theorem 2.1, we have $\mathcal{P}\text{-ind}\, X_p^{(2)} \le q+2$ and so on. Observe that $X = X_1^{(m)}$. It is clear that $\mathcal{P}\text{-ind}\, X \le q+m$.

To every normal space X one assigns the *large inductive compactness degree* Cmp as follows (cf. [AN]):

(i) For $n = -1$ or 0, $\mathrm{Cmp}\, X = n$ iff $\mathrm{cmp}\, X = n$.

(ii) $\mathrm{Cmp}\, X \le n \ge 1$ if for each pair of disjoint closed subsets A and B of X there exists a partition C in X such that $\mathrm{Cmp}\, C \le n-1$.

(iii) $\operatorname{Cmp} X = n$ if $\operatorname{Cmp} X \leq n$ and $\operatorname{Cmp} X > n-1$.
(iv) $\operatorname{Cmp} X = \infty$ if $\operatorname{Cmp} X > n$ for every natural number n.

It is clear that Cmp has the following properties:

1. If A is closed in X, then $\operatorname{Cmp} A \leq \operatorname{Cmp} X$.
2. If $X = X_1 \oplus X_2$, then $\operatorname{Cmp} X = \max\{\operatorname{Cmp} X_1, \operatorname{Cmp} X_2\}$.

Corollary 2.2. *Let X be a normal space such that $X = X_1 \cup X_2$, where X_i is closed in X and $\operatorname{Cmp} X_i \leq 0$ for every i. Then $\operatorname{Cmp} X \leq 1$. Moreover, if $\operatorname{Cmp}(X_1 \cap X_2) = -1$, then $\operatorname{Cmp} X \leq 0$; if $\operatorname{Cmp} X_1 = -1$, then $\operatorname{Cmp} X = \operatorname{Cmp} X_2$.*

Proof. Observe that $\operatorname{Cmp} X_i = \operatorname{cmp} X_i \leq 0$ for every i. By Theorem 2.1, for any closed subsets A and B of X there exists a partition C in X between A and B such that $\operatorname{cmp} C = \operatorname{Cmp} C \leq 0$. So $\operatorname{Cmp} X \leq 1$. If $\operatorname{Cmp}(X_1 \cap X_2) = -1$, then (again by Theorem 2.1) there exists a base $\mathcal{B}$ for the open sets of X such that the boundary $\operatorname{Bd} U$ is compact for each $U \in \mathcal{B}$.

Now we are ready to prove the following theorem.

Theorem 2.2. *Let X be a normal space such that $X = X_1 \cup X_2$, where X_i is closed for $i = 1, 2$. Then*

$$\begin{aligned}\operatorname{Cmp} X &\leq \max\{\operatorname{Cmp} X_1, \operatorname{Cmp} X_2\} + \operatorname{Cmp}(X_1 \cap X_2) + 1 \\ &\leq \operatorname{Cmp} X_1 + \operatorname{Cmp} X_2 + 1.\end{aligned}$$

Proof. Put $\operatorname{Cmp}(X_1 \cap X_2) = k$ and $\max\{\operatorname{Cmp} X_1, \operatorname{Cmp} X_2\} = m$. Observe that $k \leq m$. Let $k = -1$. First we prove the theorem for any $m \geq -1$ ($k = -1$). By Corollary 2.2 the statement is valid for $m = -1$ and $m = 0$. Assume that it holds for $m < p \geq 1$. Put $m = p$. Consider two disjoint closed subsets A and B of X. We can suppose that $A \cap X_i \neq \emptyset$ and $B \cap X_i \neq \emptyset$, $i = 1, 2$. Choose partitions C_i, $i = 1, 2$, as in Lemma 2.1 such that $\max\{\operatorname{Cmp} C_1, \operatorname{Cmp} C_2\} \leq p-1$. Set $Y_1 = C_1 \cup C_2$ (recall that C_1 and C_2 are disjoint), $Y_2 = X_1 \cap X_2$ and $Y = Y_1 \cup Y_2$. Observe that $\operatorname{Cmp}(Y_1 \cap Y_2) = -1$, $\operatorname{Cmp} Y_1 = \max\{\operatorname{Cmp} C_1, \operatorname{Cmp} C_2\} \leq p - 1$ and $\max\{\operatorname{Cmp} Y_1, \operatorname{Cmp} Y_2\} \leq p - 1$. By the inductive assumption,

$$\begin{aligned}\operatorname{Cmp} Y &\leq \max\{\operatorname{Cmp} Y_1, \operatorname{Cmp} Y_2\} + \operatorname{Cmp}(Y_1 \cap Y_2) + 1 \\ &\leq -1 + (p-1) + 1 = p - 1.\end{aligned}$$

By Lemma 2.1, there is a partition C in X between A and B such that $C \subset Y$. Hence, $\operatorname{Cmp} X \leq p = k + m + 1$.

Assume that the assertion is valid for any pair (k, m) with $k < q \geq 0$ and $k \leq m$. Put $k = q$. Consider the case $m = k \geq 0$. If $k = m = 0$, then $\operatorname{Cmp} X_i \leq 0$ for every $i = 1, 2$, and by Corollary 2.2, $\operatorname{Cmp} X \leq 1 = k+m+1$. Let $k = m = q \geq 1$. Consider two disjoint closed subsets A and B of X. We can suppose that $A \cap X_i \neq \emptyset$ and $B \cap X_i \neq \emptyset$, $i = 1, 2$. Choose partitions C_i,

$i = 1, 2$, as in Lemma 2.1 such that $\max\{\operatorname{Cmp} C_1, \operatorname{Cmp} C_2\} \leq q-1$. Write $Y_1 = C_1 \cup C_2$ (C_1 and C_2 are disjoint), $Y_2 = X_1 \cap X_2$ and $Y = Y_1 \cup Y_2$. Observe that $\operatorname{Cmp} Y_1 = \max\{\operatorname{Cmp} C_1, \operatorname{Cmp} C_2\} \leq q-1$, $\operatorname{Cmp}(Y_1 \cap Y_2) \leq \min\{q, q-1\} = q-1 < q$ and $\max\{\operatorname{Cmp} Y_1, \operatorname{Cmp} Y_2\} \leq q$. By the inductive assumption,

$$\begin{aligned}\operatorname{Cmp} Y &\leq \max\{\operatorname{Cmp} Y_1, \operatorname{Cmp} Y_2\} + \operatorname{Cmp}(Y_1 \cap Y_2) + 1 \\ &\leq q + (q-1) + 1 = 2q.\end{aligned}$$

By Lemma 2.1, there is a partition C in X between A and B such that $C \subset Y$. Hence, $\operatorname{Cmp} X \leq 2q+1 = k+m+1$.

Assume that the assertion is valid for any m with $k \leq m < p \geq 1$ ($k = q$). Put $m = p$. Consider two disjoint closed subsets A and B of X. We can suppose that $A \cap X_i \neq \emptyset$ and $B \cap X_i \neq \emptyset$, $i = 1, 2$. Choose partitions C_i, $i = 1, 2$, as in Lemma 2.1 such that $\max\{\operatorname{Cmp} C_1, \operatorname{Cmp} C_2\} \leq p-1$. Set $Y_1 = C_1 \cup C_2$ (C_1 and C_2 are disjoint), $Y_2 = X_1 \cap X_2$ and $Y = Y_1 \cup Y_2$. Observe that $\operatorname{Cmp} Y_1 = \max\{\operatorname{Cmp} C_1, \operatorname{Cmp} C_2\} \leq p-1$, $\operatorname{Cmp}(Y_1 \cap Y_2) \leq \min\{q, p-1\} = q$ and $\max\{\operatorname{Cmp} Y_1, \operatorname{Cmp} Y_2\} \leq p-1$. By the inductive assumption,

$$\begin{aligned}\operatorname{Cmp} Y &\leq \max\{\operatorname{Cmp} Y_1, \operatorname{Cmp} Y_2\} + \operatorname{Cmp}(Y_1 \cap Y_2) + 1 \\ &\leq q + (p-1) + 1 = q+p.\end{aligned}$$

By Lemma 2.1 there is a partition C in X between A and B such that $C \subset Y$. Hence, $\operatorname{Cmp} X \leq q+p+1 = k+m+1$.

Corollary 2.3. *Let X be a normal space with $\operatorname{Cmp} X = n \geq 1$. Then*

(a) *X cannot be represented as a union of n closed subsets $P_1, \ldots, P_n$ with $\operatorname{Cmp} P_i \leq 0$ for each i.*

Suppose now that $X = \bigcup_{i=1}^{n+1} Z_i$, where each Z_i is closed and $\operatorname{Cmp} Z_i \leq 0$ for every $i = 1, \ldots, n+1$. Then:

(b) $\operatorname{Cmp}(Z_1 \cup \ldots \cup Z_{k+1}) = k$ *for any k with $0 \leq k \leq n$.*

(c) $\operatorname{Cmp}((Z_1 \cup \ldots \cup Z_{1+i}) \cap (Z_{i+2} \cup \ldots \cup Z_{i+j+2})) = \min\{i, j\}$ *for any nonnegative integers i, j such that $i+j+1 \leq n$.*

Proof. (a) Suppose that $X = \bigcup_{i=1}^{n} P_i$, where P_i is a closed subset of X with $\operatorname{Cmp} P_i \leq 0$ for each i. Applying Theorem 2.2 $n-1$ times we get $\operatorname{Cmp}(\bigcup_{i=1}^{n} P_i) \leq n-1$. This is a contradiction.

(b) By Theorem 2.2, we have $\operatorname{Cmp}(Z_1 \cup \ldots \cup Z_{k+1}) \leq k$. If $\operatorname{Cmp}(Z_1 \cup \ldots \cup Z_{k+1}) < k$ then we apply Theorem 2.2 to the union $(Z_1 \cup \ldots \cup Z_{k+1}) \cup (Z_{k+2} \cup \ldots \cup Z_{n+1})$ to get again $\operatorname{Cmp}(\bigcup_{i=1}^{n+1} Z_i) \leq n-1$.

(c) Apply (b) and Theorem 2.2.

Remark 2.1. The estimates from Corollary 2.2 and Theorem 2.2 cannot be improved (see Corollary 3.3).

3. Spaces with $\operatorname{cmp} \neq \operatorname{def}$ **(**$\operatorname{cmp} \neq \operatorname{Cmp}$**).** The deficiency def is defined in the following way: For a separable metrizable space X,

$$\operatorname{def} X = \min\{\operatorname{ind}(Y \setminus X) : Y \text{ is a metrizable compactification of } X\}.$$

In this section, the concept of B-special decomposition introduced in [Ch] is essential. A decomposition $X = F \cup \bigcup_{i=1}^{\infty} E_i$ of a metric space X into disjoint sets is called *B-special* if E_i is clopen in X and $\lim_{i\to\infty} \delta(E_i) = 0$, where $\delta(A)$ is the diameter of A.

The following proposition is easily obtained by use of [Ch, Lemma 2.3].

PROPOSITION 3.1. *Let $X = F \cup \bigcup_{i=1}^{\infty} E_i$ be a B-special decomposition of a metric space X and $n \geq 0$ be an integer. If* $\max\{\mathcal{P}\text{-}\operatorname{ind} F, \mathcal{P}\text{-}\operatorname{ind} E_i\} \leq n$, *then* $\mathcal{P}\text{-}\operatorname{ind} X \leq n$.

Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of real numbers such that $0 < x_{i+1} < x_i \leq 1$ for all i and $\lim_{i\to\infty} x_i = 0$. Put

$$C^n = (\operatorname{Bd} I^n \times \{0\}) \cup \bigcup_{i=1}^{\infty} (I^n \times [x_{2i}, x_{2i-1}]) \subset I^{n+1}.$$

THEOREM 3.1. (a) *There are closed subsets $X_1, \ldots, X_{n+1}$ of C^n such that $C^n = \bigcup_{k=1}^{n+1} X_k$ and* $\operatorname{cmp} X_k = 0$ *for each $k = 1, \ldots, n+1$.*

(b) $\operatorname{def} C^n = \operatorname{Cmp} C^n = n$ $(= \operatorname{Comp} C^n)$ (see [AN] for the definition of Comp).

(c) *Let m be an integer such that $0 \leq n \leq 2^m - 1$. Then* $\operatorname{cmp} C^n \leq m$. *In particular*, $\operatorname{cmp} C^n < \operatorname{Cmp} C^n = \operatorname{def} C^n$ *for* $n \geq 3$.

Proof. (a) For every i choose finite systems B_k^i, $k = 1, \ldots, n+1$, consisting of disjoint compact subsets of I^n with diameter $< 1/i$ such that $I^n = \bigcup_{k=1}^{n+1} (\bigcup B_k^i)$. We put

$$X_k = (\operatorname{Bd} I^n \times \{0\}) \cup \bigcup_{i=1}^{\infty} \left(\left(\bigcup B_k^i \right) \times [x_{2i}, x_{2i-1}] \right)$$

for every $k = 1, \ldots, n+1$. Observe that the space X_k admits a B-special decomposition into compact subsets and, by Proposition 3.1, $\operatorname{cmp} X_k = 0$ for every $k = 1, \ldots, n+1$.

(b) It is enough to prove that $\operatorname{Comp} C^n \geq n$, i.e. there exist n pairs $(F_1, G_1), \ldots, (F_n, G_n)$ of disjoint compact subsets of C^n such that for any partitions S_i in X between F_i and G_i, $i = 1, \ldots, n$, the intersection $S_1 \cap \ldots \cap S_n$ is not compact. (Recall that for every separable metrizable space W we have $\operatorname{Comp} W \leq \operatorname{Cmp} W \leq \operatorname{def} W$ (cf. [AN]) and evidently $\operatorname{def} C^n \leq n$.) For example, such pairs are $((\{0\} \times I^n) \cap C^n, (\{1\} \times I^n) \cap C^n), \ldots, ((I^{n-1} \times \{0\} \times [0,1]) \cap C^n, (I^{n-1} \times \{1\} \times [0,1]) \cap C^n)$.

Moreover, for any partition C in C^n between the sets $(\{0\} \times I^n) \cap C^n$ and $(\{1\} \times I^n) \cap C^n$, we have $\operatorname{Comp} C \geq n - 1$.

(c) Apply Corollary 2.1 (the particular case) and the statement (a).

Now we are ready to show Theorem 1.1.

Proof of Theorem 1.1. Decompose the space Z_n, $n \geq 3$, into the union of two closed subsets Z_n^1 and Z_n^2 (each homeomorphic to C^n), where

$$Z_n^1 = (\mathrm{Bd}\, I^n \times \{0\}) \cup \bigcup_{i=1}^{\infty} (I^n \times [1/(2i+1), 1/(2i)]),$$

$$Z_n^2 = (\mathrm{Bd}\, I^n \times \{0\}) \cup \bigcup_{i=1}^{\infty} (I^n \times [1/(2i), 1/(2i-1)]).$$

Let m be the integer such that $0 \leq n \leq 2^m - 1$. It follows from Theorem 3.1(c) that $\mathrm{cmp}\, Z_n^i \leq m$ for $i = 1, 2$. Thus, by Theorem 2.1, we have $\mathrm{cmp}\, Z_n \leq m + 1$.

COROLLARY 3.1. (a) $\mathrm{cmp}\, C^2 = \mathrm{cmp}(C^2 \times [0,1]) = 2$.
(b) $\mathrm{cmp}\, C^3 = 2$.

Proof. (a) Recall that for any partition C in C^2 between $(\{0\} \times I^2) \cap C^2$ and $(\{1\} \times I^2) \cap C^2$, we have $\mathrm{Comp}\, C \geq 1$, and hence $\mathrm{cmp}\, C \geq 1$. This yields $\mathrm{cmp}\, C^2 = 2$ (and even $\mathrm{cmp}\, Z_2 = 2$). Observe that the space $C^2 \times \mathbb{R}$ can be considered as an open subset of C^3. So by property (2) of $\mathcal{P}$-ind and Theorem 3.1(c), $\mathrm{cmp}(C^2 \times I) = \mathrm{cmp}(C^2 \times \mathbb{R}) \leq \mathrm{cmp}\, C^3 \leq 2$. On the other hand, $\mathrm{cmp}(C^2 \times I) \geq \mathrm{cmp}\, C^2 = 2$.

(b) Just observe that C^2 can be considered as a closed subspace of C^3.

The following question is discussed in [AN, Problem 6, p. 71].

QUESTION 3.1. *For any k and m with $0 < k < m$, does there exist a separable metrizable space X such that* $\mathrm{cmp}\, X = k$ *and* $\mathrm{def}\, X = m$*?*

We partially answer the question as follows:

COROLLARY 3.2. *Let m be an integer and $l(m) = [\log_2(m)] + 1$. Then for every k with $m \geq k \geq l(m)$ there exists a separable metrizable space X such that* $\mathrm{cmp}\, X = k$ *and* $\mathrm{def}\, X = m$.

Proof. Observe that $l(m) = \min\{p : m \leq 2^p - 1\}$. Consider the space $Y = \mathbb{Q}' \times I^k$, where $\mathbb{Q}' = \mathbb{Q} \times I$, $\mathbb{Q}$ is the space of rational numbers and k is as in the theorem. Recall from [AN] that $\mathrm{cmp}\, Y = \mathrm{def}\, Y = k$. The required space X is the sum $Y \oplus C^m$.

REMARK 3.1. Observe that $\lim_{m\to\infty}(m - l(m)) = \infty$ (see also [K]).

Let C^n be the space defined above and $X_1, \ldots, X_{n+1}$ be the closed subsets of C^n described in Theorem 3.1. It follows from Theorem 3.1(a) and Corollary 2.3 that $\mathrm{Cmp}(X_1 \cup \ldots \cup X_{k+1}) = k$ for each k with $0 \leq k \leq n$.

However, we do not know the value of the deficiency of $X_1 \cup \ldots \cup X_{k+1}$. So we can ask the following.

QUESTION 3.2. *Is it true that* $\operatorname{def}(X_1 \cup \ldots \cup X_{k+1}) = k$ *for* $1 \le k < n$?

The question might be interesting when we consider a problem posed by Aarts and Nishiura [AN, Problem 6, p. 71]: Exhibit a separable metrizable space X such that $\operatorname{cmp} X < \operatorname{Cmp} X < \operatorname{def} X$. If Question 3.2 had a negative answer for example for the case of $n = 4$ and $k = 3$, then we would have $\operatorname{def}(X_1 \cup X_2 \cup X_3 \cup X_4) = 4$. We put $Y = X_1 \cup X_2 \cup X_3 \cup X_4$. Then, by the argument above, $\operatorname{Cmp} Y = 3$. On the other hand, by Theorem 3.1(a) and Corollary 2.1, $\operatorname{cmp} Y \le 2$. Hence $\operatorname{cmp} Y < \operatorname{Cmp} Y < \operatorname{def} Y$. Even if Question 3.2 had an affirmative answer, then one gets an interesting counterpart of Corollary 3.3 (see below) for def.

Now we will obtain a complement to Theorem 2.2 showing the exactness of the theorem's estimates.

COROLLARY 3.3. *For any integer* $n \ge 1$ *there exists a compact space* X_n $(= C^n)$ *with* $\operatorname{Cmp} X_n = n$ *such that for any nonnegative integers* p, q *with* $p + q = n - 1$ *there exist closed subsets* $X_n^{(p)}$ *and* $X_n^{(q)}$ *of* X_n *such that* $X_n = X_n^{(p)} \cup X_n^{(q)}$, $\operatorname{Cmp} X_n^{(p)} = p$, $\operatorname{Cmp} X_n^{(q)} = q$ *and* $\operatorname{Cmp}(X_n^{(p)} \cap X_n^{(q)}) = \min\{p, q\}$.

Proof. Let $n \ge 1$, C^n be the space defined at the beginning of this section, and $X_1, \ldots, X_{n+1}$ be the closed subsets of C^n described in Theorem 3.1. We put $X^{(p)} = X_1 \cup \ldots \cup X_{p+1}$ and $X^{(q)} = X_{p+2} \cup \ldots \cup X_{n+1}$. By Theorem 3.1(b), it follows that $\operatorname{Cmp} C^n = n$. By Corollary 2.3(b), we have $\operatorname{Cmp} X^{(p)} = p$ and $\operatorname{Cmp} X^{(q)} = q$. Furthermore, it follows from Corollary 2.3(c) that $\operatorname{Cmp}(X^{(p)} \cap X^{(q)}) = \min\{p, q\}$.

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