FUNDAMENTA
MATHEMATICAE
172 (2002)

A method for evaluating the fractal dimension in the plane, using coverings with crosses

by

Claude Tricot (Clermont-Ferrand)

Abstract. Various methods may be used to define the Minkowski–Bouligand dimension of a compact subset E in the plane. The best known is the *box method*. After introducing the notion of ε-connected set E_ε, we consider a new method based upon coverings of E_ε with crosses of diameter 2ε. To prove that this *cross method* gives the fractal dimension for all E, the main argument consists in constructing a special pavement of the complementary set with squares. This method gives rise to a dimension formula using integrals, which generalizes the well known *variation method* for graphs of continuous functions.

1. Introduction and theorems

1.1. *Classical definitions.* In the theory of fractal dimensions, the Minkowski–Bouligand dimension is one of the best known, and the only one which can be estimated in experimental contexts. A practical evaluation is not easy to obtain, due to the very nature of the dimension, which is the order of growth of some function. But some formulations give better approximations than others. It is the purpose of this paper to propose an easy and accurate algorithm for computing the dimension, even though numerical results are not presented here. Moreover our method has some interest in itself, for a better understanding of the geometry of E. Throughout this paper, the set E is a compact subset of the plane, and $\Delta(E)$ its dimension. First let us recall the old definitions, essentially due to G. Bouligand [1].

• The *box method* uses the number ω_n of dyadic squares

$$[j\cdot 2^{-n},(j+1)2^{-n}]\times[k\cdot 2^{-n},(k+1)2^{-n}],\qquad j\in\mathbb{Z},\ k\in\mathbb{Z},$$

meeting E. Then

$$\Delta(E)=\limsup_{n\to\infty}\frac{\log\omega_n}{\log 2^n}.$$

2000 *Mathematics Subject Classification*: 28A75, 28A80.

It is usual to evaluate $\Delta(E)$ with the log-log plot $(\log 2^n, \log \omega_n)$. In the case where the lim sup is a limit, the slope gives an approximate value of the dimension for large n.

• The *Minkowski sausage method* consists in measuring the area of the *sausage*

$$E(\varepsilon) = \bigcup_{x \in E} B_\varepsilon(x). \tag{1}$$

Let $\mathcal{A}$ be the area; then

$$\Delta(E) = \limsup_{\varepsilon \to 0} \left(2 - \frac{\log \mathcal{A}(E(\varepsilon))}{\log \varepsilon} \right). \tag{2}$$

The corresponding log-log plot is $(\log(1/\varepsilon), \log(\mathcal{A}(E(\varepsilon))/\varepsilon^2))$; its slope gives an approximate value of the dimension for small values of ε.

Unfortunately these two methods give disappointing results in experimental situations. The log-log plot for the box method shows an important dispersion, partly due to the fact that ω_n takes only integer values. Moreover the slopes are very sensitive to a different scaling in the data. Log-log plots for the sausage method are systematically concave, as in the simple example of a segment of length l: if the distance used is the euclidean distance, then $\mathcal{A}(E(\varepsilon)) = l\varepsilon + \pi\varepsilon^2$ and $\log(\mathcal{A}(E(\varepsilon))/\varepsilon^2)$ is a function of $\log(1/\varepsilon)$ whose slope is less than 1 for $\varepsilon > 0$, even though it tends to 1 as ε tends to 0. It is therefore impossible to obtain a good precision for the dimension of a segment with this method.

1.2. *Variation method.* A very efficient method has been taylored exclusively for the graph Γ_z of a continuous function $z : [a, b] \to \mathbb{R}$ (see [2]): Let

$$\operatorname{osc}(z, \varepsilon, t) = \sup\{z(t') - z(t'') \mid |t - t'| \le \varepsilon,\ |t - t''| \le \varepsilon\}$$

be the ε-oscillation of z at t. If $[a, b] \cap [t - \varepsilon, t + \varepsilon] = \emptyset$, we set $\operatorname{osc}(z, \varepsilon, t) = 0$. The *variation* of z is

$$\operatorname{Var}(z, \varepsilon) = \int_{\mathbb{R}} \operatorname{osc}(z, \varepsilon, t)\, dt.$$

One can check that this value is just the area of the surface $X_{\mathrm{H}}(\Gamma_z, \varepsilon)$ defined as the union of horizontal segments centered on the graph:

$$X_{\mathrm{H}}(\Gamma_z, \varepsilon) = \bigcup_{t \in [a,b]} [t - \varepsilon, t + \varepsilon] \times \{z(t)\}.$$

If z is not constant, then the area is equivalent to $\mathcal{A}(E(\varepsilon))$, and one can write

$$\Delta(\Gamma_z) = \limsup_{\varepsilon \to 0} \left(2 - \frac{\log \int_{\mathbb{R}} \operatorname{osc}(z, \varepsilon, t)\, dt}{\log \varepsilon} \right). \tag{3}$$

Numerically, the variation is much easier to evaluate than the area of $E(\varepsilon)$. Log-log plots are very smooth in general, and exhibit a well-defined straight line part which allows one to evaluate the value of Δ with great precision [3].

In a general situation (z can be constant), one can use a cover of the graph with centered crosses:

$$X(\Gamma_z,\varepsilon) = \bigcup_{s_0\in\mathbb{R}} \{(s,t) \mid |s-s_0| \le \varepsilon,\ |t-z(s_0)| \le \varepsilon,\ (s-s_0)(t-z(s_0)) = 0\}.$$

Then [9]

$$\Delta(\Gamma_z) = \limsup_{\varepsilon\to 0} \left(2 - \frac{\log \mathcal{A}(X(\Gamma_z,\varepsilon))}{\log \varepsilon}\right). \tag{4}$$

Note that $X(\Gamma_z,\varepsilon)$ is a union of two sets: the union of horizontal segments $X_{\mathrm{H}}(\Gamma_z,\varepsilon)$ whose area is $\mathrm{Var}(z,\varepsilon)$; and the union of vertical segments $X_{\mathrm{V}}(\Gamma_z,\varepsilon)$ whose area is $2(b-a)\varepsilon$. The corresponding integral formula is

$$\Delta(\Gamma_z) = \limsup_{\varepsilon\to 0} \left(2 - \frac{\log(2(b-a)\varepsilon + \int_{\mathbb{R}} \mathrm{osc}(z,\varepsilon,t)\,dt)}{\log \varepsilon}\right). \tag{5}$$

The purpose of this paper is to obtain a similar expression for any compact set in the plane.

1.3. *The cross method.* It is not difficult to obtain the same result as (4) for an arcwise connected set. Difficulties arise when E has arbitrarily small connected components. For totally disconnected sets the union of centered crosses may have area 0. Let us first construct locally connected sets.

Throughout this paper, the plane is provided with two coordinate axes Os, Ot. Squares and rectangles are closed sets, with sides parallel to the coordinate axes. Points are denoted by x or y, their coordinates by pairs (s,t). The distance ϱ is derived from the maximum norm: If $x_i = (s_i,t_i)$, then

$$\varrho(x_1,x_2) = \max\{|s_1-s_2|, |t_1-t_2|\}.$$

Every ball $B_\varepsilon(x)$ is a square centered at x, with sides 2ε. A segment with endpoints x, y is denoted by $\overline{xy}$.

For any bounded set E, the diameter of E is

$$\mathrm{diam}(E) = \sup\{\varrho(x_1,x_2) \mid x_1,x_2\in E\}.$$

Let

$$E_\varepsilon = \bigcup\{\overline{xy} \mid x,y\in E,\ \varrho(x,y)\le\varepsilon\}. \tag{6}$$

The set E_ε is equal to E when E is a convex body, or a union of convex bodies at a distance larger than ε from each other. In general

$$E \subset E_\varepsilon \subset E(\varepsilon/2).$$

When ε tends to 0, both E_ε and $E(\varepsilon)$ tend to E.

For every $x_0 = (s_0, t_0)$,

$$X_\varepsilon(x_0) = \{(s,t) \mid |s - s_0| \le \varepsilon,\ |t - t_0| \le \varepsilon,\ (s - s_0)(t - t_0) = 0\}$$

is the cross centered at x_0, of diameter 2ε. As in Section 1.2, the union of all crosses centered in E is

$$X(E,\varepsilon) = \bigcup_{x \in E} X_\varepsilon(x).$$

The inclusions $X(E,\varepsilon) \subset E(\varepsilon)$ and $X(E_\varepsilon,\varepsilon) \subset E(3\varepsilon/2)$ are always true.

If E is not finite, the area of $X(E_\varepsilon,\varepsilon)$ is not zero. The next result shows that the convergence to 0 of this area, as $\varepsilon \to 0$, is controlled by $\Delta(E)$:

THEOREM 1. *For every compact set E,*

$$\Delta(E) = \limsup_{\varepsilon \to 0} \left(2 - \frac{\log \mathcal{A}(X(E_\varepsilon,\varepsilon))}{\log \varepsilon} \right). \tag{7}$$

For every large connected component F_ε of E_ε (with diameter $\ge \varepsilon$), we will see that $X(F_\varepsilon,\varepsilon)$ has indeed an area equivalent to that of the Minkowski sausage $F_\varepsilon(\varepsilon)$. But if F_ε is a small connected component, $\mathcal{A}(X(F_\varepsilon,\varepsilon))$ may be much smaller than $\mathcal{A}(F_\varepsilon(\varepsilon))$ (see Figure 1). To prove the theorem, one has to check that the contribution of small components is negligible. The difficulty comes from the geometry of E. When $\varepsilon \to 0$, more and more components may arise, and many of them may be small.

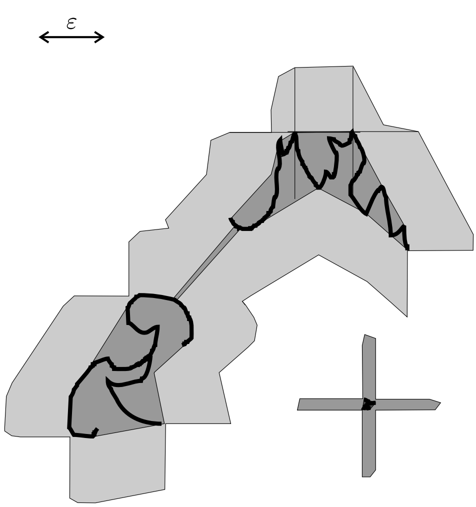

Fig. 1. The set E_ε has two connected components, F_ε (of diameter $< \varepsilon$) and G_ε (of diameter $\ge \varepsilon$). The union $X(F_\varepsilon,\varepsilon)$ of centered crosses is negligible with respect to the Minkowski sausage $F_\varepsilon(\varepsilon)$. On the contrary, $X(G_\varepsilon,\varepsilon)$ and $G_\varepsilon(\varepsilon)$ have equivalent areas.

1.4. *Integral formula.* Let

$$X_{\mathrm{H}}(E,\varepsilon)=\bigcup_{(s,t)\in E}([s-\varepsilon,s+\varepsilon]\times\{t\})$$

be the union of all horizontal segments of length 2ε centered on E, and

$$X_{\mathrm{V}}(E,\varepsilon)=\bigcup_{(s,t)\in E}(\{s\}\times[t-\varepsilon,t+\varepsilon])$$

be the union of vertical segments. Since $X(E,\varepsilon)=X_{\mathrm{H}}(E,\varepsilon)\cup X_{\mathrm{V}}(E,\varepsilon)$ for all $\varepsilon>0$, (7) implies

$$\Delta(E)=\limsup\left(2-\frac{\log(\mathcal{A}(X_{\mathrm{H}}(E_\varepsilon,\varepsilon))+\mathcal{A}(X_{\mathrm{V}}(E_\varepsilon,\varepsilon)))}{\log\varepsilon}\right). \tag{8}$$

We will replace these areas with integrals.

Assume that the cylinder $[s-\varepsilon/2,s+\varepsilon/2]\times\mathbb{R}$ contains at least one point of E. Projecting orthogonally the intersection $E\cap([s-\varepsilon/2,s+\varepsilon/2]\times\mathbb{R})$ on Ot, one gets a linear, compact set. Putting together this set and all its complementary intervals of length $\le\varepsilon$, we get

$$E_\varepsilon(s)=(\mathrm{Proj}_{Ot}(E\cap([s-\varepsilon/2,s+\varepsilon/2]\times\mathbb{R})))_\varepsilon.$$

If E is the graph of a continuous function z, then $E_\varepsilon(s)$ is the interval

$$[\min_{|s'-s|\le\varepsilon/2}z(s'),\max_{|s'-s|\le\varepsilon/2}z(s')]$$

whose length is the $\varepsilon/2$-oscillation of z at s. For a general E, we use the Lebesgue measure L and define

$$f_\varepsilon(s)=\begin{cases}L(E_\varepsilon(s)) & \text{if } E\cap([s-\varepsilon/2,s+\varepsilon/2]\times\mathbb{R})\neq\emptyset,\\ 0 & \text{otherwise.}\end{cases}$$

The area $\mathcal{A}(X_{\mathrm{H}}(E_\varepsilon,\varepsilon))$ is equivalent to the integral of f_ε (Proposition 4). Note that this area is not exactly equal to the integral: To obtain an equality, one should use the length of the set

$$\mathrm{Proj}_{Ot}(E_\varepsilon\cap([s-\varepsilon/2,s+\varepsilon/2]\times\mathbb{R}))$$

which is larger and more difficult to handle numerically.

The same approach is used for vertical segments. Let R_θ be the rotation of angle θ. Define $g_\varepsilon(s)=L((R_{\pi/2}(E))_\varepsilon(s))$. The following result is a generalization of (5):

THEOREM 2. *Let E be a compact set of the plane, and f_ε, g_ε as above. Then*

$$\Delta(E)=\limsup_{\varepsilon\to0}\left(2-\frac{\log(\int_{\mathbb{R}}f_\varepsilon(t)\,dt+\int_{\mathbb{R}}g_\varepsilon(t)\,dt)}{\log\varepsilon}\right). \tag{9}$$

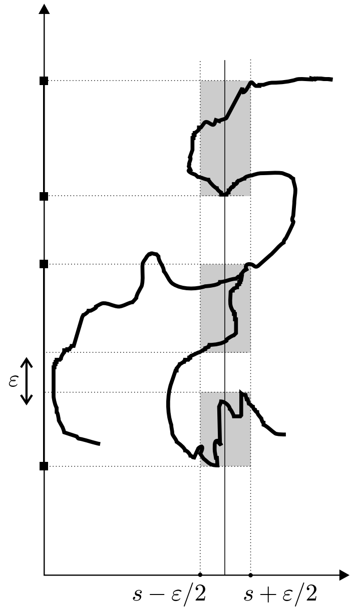

Fig. 2. Project the set $E\cap([s-\varepsilon/2,s+\varepsilon/2]\times\mathbb{R})$ on Ot, then unite all points at a distance $\le \varepsilon$. The result is the set $E_\varepsilon(s)$. The convergence to 0 of the integral of the function $L(E_\varepsilon(s))$ is controlled by the dimension of E.

Computing these integrals is numerically simple, and the corresponding log-log plot should give good estimations of the dimension for usual mathematical models.

Remark. One may use other structural elements than orthogonal crosses. It is possible to use any angle $\neq 0 \bmod(\pi)$ between the branches. Also, (9) may be changed into a formula using all possible directions: Take

$$\Delta(E)=\limsup_{\varepsilon\to 0}\left(2-\frac{\log(\int_0^\pi\int_{\mathbb{R}} f_\varepsilon^\theta(t)\,dt\,d\theta)}{\log\varepsilon}\right),$$

where $f_\varepsilon^\theta(t)=L((R_\theta(E))_\varepsilon(s))$. But in this paper, we restrict our attention to the proof of (7) and (9).

1.5. *Outline of the paper.* Section 2 gives a few properties of the operation $E\to E_\varepsilon$ and introduces the notion of ε-connectedness. If E is arcwise connected, then the areas of $E(\varepsilon)$ and $X(E,\varepsilon)$ are equivalent as $\varepsilon\to 0$. The fractal dimension of E is related to the exponent of convergence of families of squares paving the complementary set; related references are [5], [7], [8], [4]. In Section 3 a pavement is constructed in such a way that the union of large squares (with diameter $\ge \varepsilon/2$) is a good approximation of $X(E_\varepsilon,\varepsilon)$. Technical lemmas are proved in Sections 4 and 5. In Section 6 we show that the contribution of small connected components of E_ε is negligible when calculating the dimension, and we prove Theorem 1. Finally, Theorem 2 is proved in Section 7. Theorems 1 and 2 are stated without proof in [10]. We give all details in the present paper.

2. Connected components

LEMMA 1. *Let $\varepsilon > 0$, $\eta > 0$, and E be a compact subset of the plane whose connected components have diameter $\geq \eta$. Then*

$$\mathcal{A}(E(\varepsilon)) \leq \left(1 + \frac{2\varepsilon}{\eta}\right)^2 \mathcal{A}(X(E,\eta)). \tag{10}$$

Proof. The main argument consists in using the following result [9, Annexe B]:

For any bounded set F, and $\varepsilon_2 \leq \varepsilon_1$,

$$\mathcal{A}(F(\varepsilon_1)) \leq \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2 \mathcal{A}(F(\varepsilon_2)). \tag{11}$$

Since connected components have diameter $\geq \eta$, there exists a covering $\mathcal{U}$ of E by squares of side η such that each C in $\mathcal{U}$ has the following property: one of its sides, say J, is equal to the orthogonal projection of $E \cap C$ over J. This implies that $C \subset X(E,\eta)$. Therefore $\mathcal{A}(\bigcup \mathcal{U}) \leq \mathcal{A}(X(E,\eta))$. Let F be the set of all centers of squares in $\mathcal{U}$. Since $E \subset \bigcup \mathcal{U} = F(\eta/2)$, we have $E(\varepsilon) \subset F(\varepsilon + \eta/2)$. Using (11) we get

$$\mathcal{A}(E(\varepsilon)) \leq \left(\frac{\eta + 2\varepsilon}{\eta}\right)^2 \mathcal{A}(F(\eta/2)) \leq \left(1 + \frac{2\varepsilon}{\eta}\right)^2 \mathcal{A}(X(E,\eta)). \ \blacksquare$$

COROLLARY 1. *If E has a finite number of connected components*, *none reduced to a point*, *then the conclusion of Theorem* 1 *is true.*

Proof. Each of the connected components has diameter > 0. Let $\eta > 0$ be the minimum diameter. Let $\varepsilon_0 > 0$ be the smallest distance between two components. Let $0 < \varepsilon < \min\{\eta, \varepsilon_0/2\}$. For every connected component F of E, Lemma 1 implies $\mathcal{A}(F(\varepsilon)) \leq 9\,\mathcal{A}(X(F,\varepsilon))$. Then

$$\mathcal{A}(E(\varepsilon)) \leq 9\,\mathcal{A}(X(E,\varepsilon)) \leq 9\,\mathcal{A}(X(E_\varepsilon,\varepsilon)).$$

Also $X(E_\varepsilon,\varepsilon) \subset E(2\varepsilon)$, so that $\mathcal{A}(X(E_\varepsilon,\varepsilon)) \leq \mathcal{A}(E(2\varepsilon)) \leq 4\,\mathcal{A}(E(\varepsilon))$. This shows that $\mathcal{A}(E(\varepsilon))$, $\mathcal{A}(X(E,\varepsilon))$ and $\mathcal{A}(X(E_\varepsilon,\varepsilon))$ are equivalent as ε tends to 0. $\blacksquare$

In the general case, E_ε has a number of connected components which varies with ε.

DEFINITION 1. The set E is *ε-connected* if E_ε is connected.

This property is satisfied if, and only if, for all points x, y of E, there exists a finite ε-chain $x_1 = x$, $x_2, \ldots, x_n = y$ in E. We consider the finite family $\mathcal{R}_\varepsilon$ of all subsets of E such that F_ε is a connected component of E_ε:

$$\mathcal{R}_\varepsilon = \{F \subset E \mid F \text{ is compact, } \varepsilon\text{-connected, and } F(\varepsilon) \cap E = F\}.$$

Elements of $\mathcal{R}_\varepsilon$ are the *ε-connected components* of E. Two different ε-connected components F and G are such that $\inf\{\varrho(x,y) \mid x \in F,\, y \in G\} > \varepsilon$. We will have to analyse separately the large and small components, arranged in two families:

$$\mathcal{R}'_\varepsilon = \{F \in \mathcal{R}_\varepsilon \mid \operatorname{diam}(F) \geq \varepsilon\}, \qquad \mathcal{R}''_\varepsilon = \{F \in \mathcal{R}_\varepsilon \mid \operatorname{diam}(F) < \varepsilon\}.$$

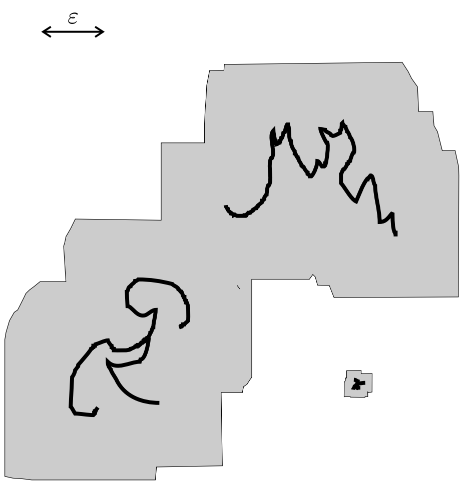

Fig. 3. The set E has two ε-connected components (see Fig. 1). If F is the component of diameter $\eta < \varepsilon$, and G that of diameter $\geq \varepsilon$, then $\widehat{R}(E,\varepsilon) = F(\eta) \cup G(\varepsilon)$.

The neighborhood $E(\varepsilon)$ has been defined for all $\varepsilon > 0$. It will be useful to write $E(0) = E$ and $E(-\infty) = \emptyset$. These conventions allow us to keep the basic relationship

$$(E(\varepsilon))(\eta) = E(\varepsilon + \eta) \tag{12}$$

for all ε, η in $\{-\infty\} \cup \mathbb{R}^+$.

Given $f : \mathbb{R}^+ \to \mathbb{R}^+$, we consider the neighborhood of ε-connected components of E, made up with balls whose size is controlled by f. Let

$$R(E,f,\varepsilon) = \bigcup \{F(\eta) \mid F \in \mathcal{R}_\varepsilon \text{ and } \eta = \varepsilon f(\operatorname{diam}(F)/\varepsilon)\}.$$

If $f = 0$, then $R(E,f,\varepsilon) = E$. If $f = 1$, $R(E,f,\varepsilon) = E(\varepsilon)$. We will then use two specific functions:

- The function $g(t) = \min\{t, 1\}$ which gives rise to the neighborhood

$$\begin{aligned} \widehat{R}(E,\varepsilon) &= \bigcup \{F(\eta) \mid F \in \mathcal{R}_\varepsilon \text{ and } \eta = \min\{\operatorname{diam}(F), \varepsilon\}\} \\ &= \bigcup \{F(\varepsilon) \mid F \in \mathcal{R}'_\varepsilon\} \cup \{F(\operatorname{diam}(F)) \mid F \in \mathcal{R}''_\varepsilon\}. \end{aligned} \tag{13}$$

• The function

$$h(t) = \begin{cases} -\infty & \text{if } 0 \le t < 1/2, \\ 1 & \text{if } t \ge 1/2, \end{cases}$$

which gives rise to the neighborhood

$$\widetilde{R}(E,\varepsilon) = \bigcup \{F(\varepsilon) \mid F \in \mathcal{R}_\varepsilon \text{ and } \operatorname{diam}(F) \ge \varepsilon/2\}. \tag{14}$$

There is no general inclusion relationship between $\widehat{R}(E,\varepsilon)$ and $\widetilde{R}(E,\varepsilon)$. Here are three simple properties:

PROPERTY 1. *If* $f_1(t) \le f_2(t)$ *for all* t, *then* $R(E, f_1, \varepsilon) \subset R(E, f_2, \varepsilon)$.

In particular, $R(E, f, \varepsilon) \subset E(\varepsilon)$ when f is equal to g or h.

PROPERTY 2. *If* $\varepsilon f(t/\varepsilon)$ *is an increasing function of* ε, *then*

$$\varepsilon' \le \varepsilon \;\Rightarrow\; R(E, f, \varepsilon') \subset R(E, f, \varepsilon).$$

To show this, use the fact that $\bigcup \mathcal{R}_{\varepsilon'} \subset \bigcup \mathcal{R}_\varepsilon$. Property 2 is true when $f = g$. But it is not true for h.

PROPERTY 3. $R(E, f_1, \varepsilon) \cup R(E, f_2, \varepsilon) \subset R(E, \max\{f_1, f_2\}, \varepsilon)$.

3. Paving the complementary set. We will pave the complement of E with dyadic squares in such a way that for every n, a square of side 2^{-n} meets a set $\widetilde{R}(E, 2\varepsilon)$ for some $\varepsilon \ge 2^{-n}$. Then we will use the fact that $\widetilde{R}(E,\varepsilon)$ contains only the large connected components, so that its area is equivalent to $\mathcal{A}(X(E_\varepsilon, \varepsilon))$.

For every $\varepsilon > 0$, let us define the families $\mathcal{C}'_\varepsilon$, $\mathcal{C}''_\varepsilon$, $\mathcal{G}'_\varepsilon$, $\mathcal{G}''_\varepsilon$, $\mathcal{F}'_\varepsilon$, $\mathcal{F}''_\varepsilon$, $\mathcal{F}_\varepsilon$ of squares.

• Since we look for a countable covering, we consider the family $\mathcal{C}'_\varepsilon$ of squares whose sides are parallel to the axes and whose s-projection is an ε-adic interval:

$$C \in \mathcal{C}'_\varepsilon \;\Leftrightarrow\; C = [k\varepsilon, (k+1)\varepsilon] \times [t, t+\varepsilon] \text{ for some } k \in \mathbb{Z},\ t \in \mathbb{R}.$$

• In the same way, $\mathcal{C}''_\varepsilon$ is a family of squares whose t-projection is ε-adic:

$$C \in \mathcal{C}''_\varepsilon \;\Leftrightarrow\; C = [s, s+\varepsilon] \times [k\varepsilon, (k+1)\varepsilon] \text{ for some } k \in \mathbb{Z},\ s \in \mathbb{R}.$$

• To cover the complement of a compact set E, we define

$$\mathcal{G}'_\varepsilon = \{C \in \mathcal{C}'_\varepsilon \mid C \cap \widetilde{R}(E, 2\varepsilon) \ne \emptyset \text{ and } \mathring{C} \cap E = \emptyset\}.$$

Therefore

$$C \in \mathcal{G}'_\varepsilon \;\Leftrightarrow\; C \in \mathcal{C}'_\varepsilon \text{ and there exists } F \in \mathcal{R}_{2\varepsilon} \text{ with } \operatorname{diam}(F) \ge \varepsilon \text{ such that } C \cap F(2\varepsilon) \ne \emptyset.$$

• In the same way, the family $\mathcal{G}''_\varepsilon$ corresponds to $\mathcal{C}''_\varepsilon$:

$$\mathcal{G}''_\varepsilon = \{C \in \mathcal{C}''_\varepsilon \mid C \cap \widetilde{R}(E, 2\varepsilon) \ne \emptyset \text{ and } \mathring{C} \cap E = \emptyset\}.$$

• The families $\mathcal{G}'_\varepsilon$ and $\mathcal{G}''_\varepsilon$ are not finite, nor even countable. Let the *covering index* of a set family Ω, denoted by $\mathrm{I}(\Omega)$, be the largest k such that the interiors of k distinct elements of Ω have a non-empty intersection. From $\mathcal{G}'_\varepsilon$ we can extract a subfamily of finite index as shown in Fig. 4.

Fig. 4. The gray area is $\bigcup \mathcal{G}'_\varepsilon$. It is the union of all squares C of side ε whose s-projection is ε-adic, such that $C \cap \widetilde{R}(E, 2\varepsilon) \neq \emptyset$ and the interior of C does not meet E. The boundary of $\widetilde{R}(E, 2\varepsilon)$ is the dotted line. Notice that the small connected component has disappeared (see Figs. 1 and 3).

The set $\bigcup \mathcal{G}'_\varepsilon$ is a union of rectangles of disjoint interiors, of width ε, with the length in the vertical direction (see Fig. 4). Every such rectangle R can be covered by a family $\mathcal{F}'_\varepsilon(R)$ of squares of side ε so that $\bigcup \mathcal{F}'_\varepsilon(R) = R$ and every point of R belongs to no more than two squares. The maximum number of squares in $\mathcal{F}'_\varepsilon(R)$ is $1 + [\mathrm{diam}(R)/\varepsilon]$. Let $\mathcal{F}'_\varepsilon$ be the union of all such $\mathcal{F}'_\varepsilon(R)$. It is a covering of $\bigcup \mathcal{G}'_\varepsilon$ such that $\mathrm{I}(\mathcal{F}'_\varepsilon) \leq 2$.

• Proceed in the same way with $\bigcup \mathcal{G}''_\varepsilon$, which is a union of rectangles with disjoint interiors, of width ε, with the length in the horizontal direction. There exists a family $\mathcal{F}''_\varepsilon$ of squares which covers $\bigcup \mathcal{G}''_\varepsilon$, such that $\mathrm{I}(\mathcal{F}''_\varepsilon) \leq 2$.

• Finally, the family $\mathcal{F}_\varepsilon = \mathcal{F}'_\varepsilon \cup \mathcal{F}''_\varepsilon$ is such that

$$\bigcup \mathcal{F}_\varepsilon = \bigcup \mathcal{G}'_\varepsilon \cup \bigcup \mathcal{G}''_\varepsilon \quad \text{and} \quad \mathrm{I}(\mathcal{F}_\varepsilon) \leq \mathrm{I}(\mathcal{F}'_\varepsilon) + \mathrm{I}(\mathcal{F}''_\varepsilon) \leq 4.$$

We will see that the union $\bigcup_n \mathcal{F}_{2^{-n}}$ is a pavement of the complementary set of E.

4. Technical lemmas. Let us assume that $\operatorname{diam}(E) = 1$, without loss of generality.

LEMMA 2. *For all* $\varepsilon > 0$,

$$\bigcup \mathcal{F}_\varepsilon \subset \widetilde{R}(E, 2\varepsilon)(\varepsilon) \subset E(3\varepsilon). \tag{15}$$

Proof. Every square C of $\mathcal{F}_\varepsilon$ meets $\widetilde{R}(E, 2\varepsilon)$. It is included in $\widetilde{R}(E, 2\varepsilon)(\varepsilon)$. But $\widetilde{R}(E, 2\varepsilon)$ is included in $E(2\varepsilon)$, so that $\widetilde{R}(E, 2\varepsilon)(\varepsilon) \subset E(3\varepsilon)$. ∎

LEMMA 3. *For all* ε,

$$\widehat{R}(E, 2\varepsilon) \cup \widetilde{R}(E, 2\varepsilon) \subset \widehat{R}(E, \varepsilon) \cup \bigcup \mathcal{F}_\varepsilon. \tag{16}$$

Proof. For each point $x_0 = (s_0, t_0)$ of the plane there exists at least one pair (j, k) of integers such that $x_0 \in [j\varepsilon, (j+1)\varepsilon] \times [k\varepsilon, (k+1)\varepsilon]$. Let

$$\begin{aligned}
s_1 &= \sup\{s \mid s \le s_0 \text{ and there exists } t \in [k\varepsilon, (k+1)\varepsilon] \text{ such that } (s, t) \in E\},\\
s_2 &= \inf\{s \mid s \ge s_0 \text{ and there exists } t \in [k\varepsilon, (k+1)\varepsilon] \text{ such that } (s, t) \in E\},\\
t_3 &= \sup\{t \mid t \le t_0 \text{ and there exists } s \in [j\varepsilon, (j+1)\varepsilon] \text{ such that } (s, t) \in E\},\\
t_4 &= \inf\{t \mid t \ge t_0 \text{ and there exists } s \in [j\varepsilon, (j+1)\varepsilon] \text{ such that } (s, t) \in E\}.
\end{aligned}$$

In other words, $[s_1, s_2]$ is the largest interval containing s_0 such that $]s_1, s_2[\times [k\varepsilon, (k+1)\varepsilon]$ does not meet E, etc. The values s_1 et t_3 may be $-\infty$; the values s_2 et t_4 may be $+\infty$. There are two cases:

1) $\max\{s_2 - s_1, t_4 - t_3\} \le \varepsilon$. There exist two points $x_1 = (s_1, t_1)$ and $x_2 = (s_2, t_2)$ of E such that t_1 and t_2 belong to $[k\varepsilon, (k+1)\varepsilon]$, and two points $x_3 = (s_3, t_3)$ and $x_4 = (s_4, t_4)$ of E such that s_3 and s_4 belong to $[j\varepsilon, (j+1)\varepsilon]$. Then $\varrho(x_1, x_2) \le \varepsilon$ and $\varrho(x_3, x_4) \le \varepsilon$. Notice that the two values s_1 and s_2 cannot be on either side of $[j\varepsilon, (j+1)\varepsilon]$, so that one of them, say s_1, belongs to this interval. Similarly, one may assume that $t_4 \in [k\varepsilon, (k+1)\varepsilon]$. Therefore $\varrho(x_1, x_4) \le \varepsilon$. The set $F = \{x_1, x_2, x_3, x_4\}$ is ε-connected. Let $F_0 \in \mathcal{R}_\varepsilon$ be the ε-connected component containing F. The point x_0 lies in the smallest rectangle containing F, so that $x_0 \in F(\operatorname{diam}(F))$. But $F \subset F_0$ and $\operatorname{diam}(F) \le \eta = \min\{\operatorname{diam}(F_0), \varepsilon\}$. Therefore $x_0 \in F_0(\eta)$, and $x_0 \in \widehat{R}(E, \varepsilon)$.

2) $\max\{s_2 - s_1, t_4 - t_3\} > \varepsilon$. Note that

$$\max\{g(t), h(t)\} = \begin{cases} t & \text{if } 0 \le t < 1/2,\\ 1 & \text{if } t \ge 1/2.\end{cases}$$

Take a point x_0 in $\widehat{R}(E, 2\varepsilon) \cup \widetilde{R}(E, 2\varepsilon)$. By Property 3, x_0 belongs to some $F(\eta)$ where $F \in \mathcal{R}_{2\varepsilon}$ and

$$\eta = \begin{cases} \operatorname{diam}(F) & \text{if } \operatorname{diam}(F) < \varepsilon,\\ 2\varepsilon & \text{if } \operatorname{diam}(F) \ge \varepsilon.\end{cases}$$

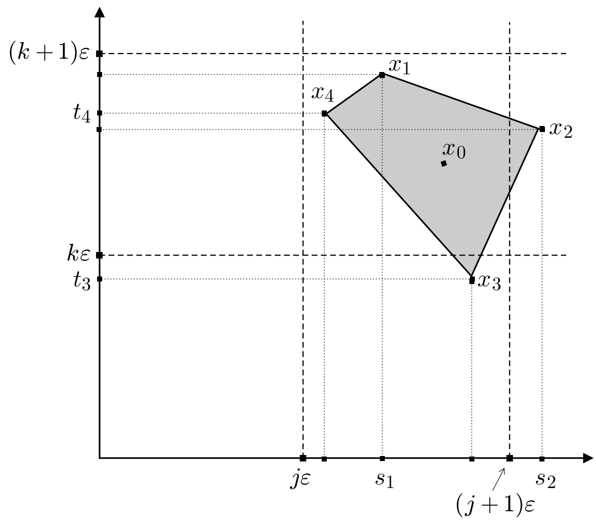

Fig. 5. The set F of the proof of Lemma 3, in the case where $\max\{s_2-s_1, t_4-t_3\} \le \varepsilon$.

(i) If $\operatorname{diam}(F) < \varepsilon$, then F is ε-connected, and $\eta = \operatorname{diam}(F)$, so that $\eta = \min\{\operatorname{diam}(F), \varepsilon\}$. We deduce that $x_0 \in \widehat{R}(E,\varepsilon)$.

(ii) If $\operatorname{diam}(F) \ge \varepsilon$, then $\eta = 2\varepsilon$ and $x_0 \in \widetilde{R}(E, 2\varepsilon)$. Assume that $s_2 - s_1 > \varepsilon$. In the rectangle $[s_1, s_2] \times [k\varepsilon, (k+1)\varepsilon]$ one can find a square C of side ε containing x_0 and such that $\mathring{C} \cap E = \emptyset$. Since $C \cap \widetilde{R}(E, 2\varepsilon) \neq \emptyset$, we have $C \in \mathcal{G}''_\varepsilon$. We deduce that $x_0 \in \bigcup \mathcal{G}''_\varepsilon$. If $t_4 - t_3 > \varepsilon$, then x_0 belongs to a square C in $[j\varepsilon, (j+1)\varepsilon] \times [t_3, t_4]$ which belongs to $\mathcal{G}''_\varepsilon$. In both cases, $x_0 \in \bigcup \mathcal{F}_\varepsilon$. ∎

Corollary 2. *For every integer n,*

$$\widehat{R}(E, 2^{-n}) \cup \widetilde{R}(E, 2^{-n}) \subset E \cup \bigcup_{k \ge n+1} \bigcup \mathcal{F}_{2^{-k}}. \tag{17}$$

Proof. One may deduce from (16) that for all n,

$$\widehat{R}(E, 2^{-n}) \subset \widehat{R}(E, 2^{-n-1}) \cup \bigcup \mathcal{F}_{2^{-n-1}}.$$

We get by induction

$$\widehat{R}(E, 2^{-n}) \subset \widehat{R}(E, 2^{-N}) \cup \bigcup_{k=n+1}^{N} \bigcup \mathcal{F}_{2^{-k}} \subset E(2^{-N}) \cup \bigcup_{k=n+1}^{\infty} \bigcup \mathcal{F}_{2^{-k}}$$

for any $N > n$. Since $\bigcap_N E(2^{-N}) = E$, we obtain

$$\widehat{R}(E, 2^{-n}) \subset E \cup \bigcup_{k=n+1}^{\infty} \bigcup \mathcal{F}_{2^{-k}}.$$

Same result for $\widetilde{R}(E, 2^{-n})$, using (16). ∎

LEMMA 4. $\mathrm{Card}(\mathcal{R}_{2^{-n}}) \le \mathrm{Card}(\bigcup_{k=1}^{n} \mathcal{F}_{2^{-k}})$.

Proof. We construct a one-to-one map $\mathcal{R}_{2^{-n}} \to \bigcup_{k=1}^{n} \mathcal{F}_{2^{-k}}$.

Let $F \in \mathcal{R}_{2^{-n}}$. Such a compact subset of E can be characterized as follows: $F_{2^{-n}}$ is connected and $F(2^{-n}) \cap E = F$. Let $k \le n$ be the integer such that

$$2^{-k} < \eta = \inf\{\varrho(x, F) \mid x \in E - F\} \le 2^{-k+1}.$$

Let $\varepsilon = 2^{-k}$.

There exists a point $x_0 = (s_0, t_0)$ of F and a point $y_0 = (u_0, v_0)$ of $E - F$ such that $\varrho(x_0, y_0) = \eta$. The point x_0 belongs to the boundary of the square $B_\eta(y_0)$. Among all possible cases, let us assume, for example, that $s_0 = u_0 - \eta$ and $t_0 \le v_0$ (see Figure 6).

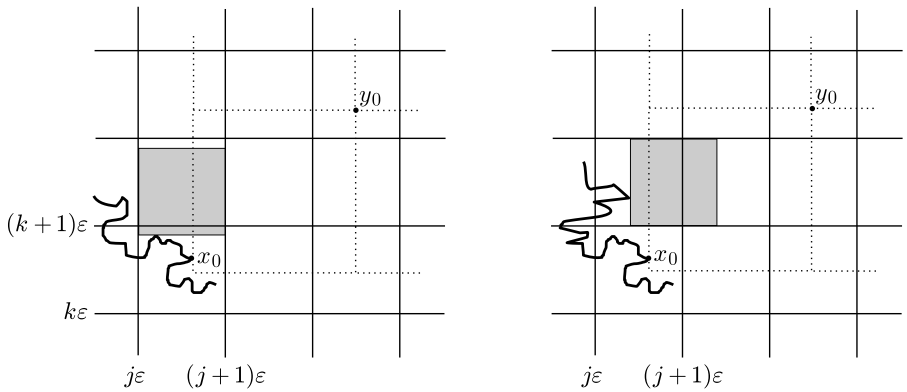

Fig. 6. For the proof of Lemma 4, a construction of the square $C(F)$ in two cases: (i) left, $(]j\varepsilon, (j+1)\varepsilon[\times](k+1)\varepsilon, (k+2)\varepsilon[) \cap E = \emptyset$, and (ii) right, $(]j\varepsilon, (j+1)\varepsilon[\times](k+1)\varepsilon, (k+2)\varepsilon[) \cap E \ne \emptyset$.

Let j and k be two integers such that $x_0 \in [j\varepsilon, (j+1)\varepsilon] \times [k\varepsilon, (k+1)\varepsilon]$. We consider two cases.

(i) $(]j\varepsilon, (j+1)\varepsilon[\times](k+1)\varepsilon, (k+2)\varepsilon[) \cap E = \emptyset$. Then there exists a $t_1 \in [t_0, (k+1)\varepsilon]$ such that

$$t_1 = \inf\{t \ge t_0 \mid (]j\varepsilon, (j+1)\varepsilon[\times]t, t+\varepsilon[) \cap E = \emptyset\}.$$

Let $C = [j\varepsilon, (j+1)\varepsilon] \times [t_1, t_1+\varepsilon]$. The intersection $C \cap F$ contains a point x_1 of y-coordinate t_1 on the lower side of C, and $\mathring{C} \cap E = \emptyset$. Let $F' = F \cup \{y_0\}$. Since $\varepsilon < \varrho(x_0, y_0) \le 2\varepsilon$, F' is 2ε-connected and $\mathrm{diam}(F') \ge \varepsilon$. Therefore $F'(2\varepsilon) \subset \widetilde{R}(E, 2\varepsilon)$. Since C meets F', $C \subset \widetilde{R}(E, 2\varepsilon)$, so that $C \in \mathcal{G}'_\varepsilon$. Using the notations introduced in Section 3, C is in a rectangle $R' \subset \bigcup \mathcal{G}'_\varepsilon$ of the type $[j\varepsilon, (j+1)\varepsilon] \times [t_1, t_1+\varepsilon']$ where $\varepsilon' \ge \varepsilon$. One of the squares of the family $\mathcal{F}_\varepsilon(R')$ has x_1 on its boundary. This square must be C itself. We deduce that $C \in \mathcal{F}_\varepsilon$.

(ii) $(]j\varepsilon,(j+1)\varepsilon[\times](k+1)\varepsilon,(k+2)\varepsilon[)\cap E\neq\emptyset$. Then there exists an $s_1\in[j\varepsilon,s_0]$ such that

$$s_1=\inf\{s\geq j\varepsilon \mid (]s,s+\varepsilon[\times](k+1)\varepsilon,(k+2)\varepsilon[)\cap E=\emptyset\}.$$

Let $C=[s_1,s_1+\varepsilon]\times[(k+1)\varepsilon,(k+2)\varepsilon]$. This square has the same properties as above: $C\subset F(\varepsilon)$ and $C\in\mathcal{F}_\varepsilon$.

A map $C:\mathcal{R}_{2^{-n}}\to\bigcup_{k=1}^{n}\mathcal{F}_{2^{-k}}$ is defined by letting $C(F)=C$.

We still have to verify that C is one-to-one. Let F and G be in $\mathcal{R}_{2^{-n}}$. Assume that $C(F)=C(G)\in\mathcal{F}_{2^{-k}}$. From the previous construction, $C(F)\subset F(2^{-k})$ and $C(G)\cap G\neq\emptyset$. We deduce that $F(2^{-k})\cap G\neq\emptyset$. Moreover $F(2^{-k})\cap E=F$ so that $F(2^{-k})\cap G\subset F$. Therefore $F\cap G\neq\emptyset$. Since these sets are components, $F=G$. ∎

5. Exponents of convergence. We use the notation $a_n\preceq b_n$ for two positive sequences such that the ratio a_n/b_n is bounded as n tends to ∞, and $a_n\simeq b_n$ if both $a_n\preceq b_n$ and $b_n\preceq a_n$.

Let ω_n be the number of elements of $\mathcal{F}_{2^{-n}}$. The order of growth of the sequence ω_n can be defined as

$$e=\limsup_n\frac{\log\omega_n}{n\log 2},$$

which can be expressed as

$$e=\inf\Big\{\alpha\,\Big|\,\sum_{n=1}^{\infty}\omega_n2^{-n\alpha}<\infty\Big\}=\inf\{\alpha\mid\omega_n\preceq 2^{n\alpha}\}. \tag{18}$$

We will use the following indices:

$$e_1=\inf\Big\{\alpha\,\Big|\,2^{-n\alpha}\sum_{k=1}^{n}\omega_k\to 0\Big\},\qquad e_2=\inf\Big\{\alpha\,\Big|\,2^{n(2-\alpha)}\sum_{k=n}^{\infty}\omega_k2^{-2k}\to 0\Big\}.$$

LEMMA 5. *The equality $e=e_1$ is always true. If $e<2$, then $\sum\omega_k2^{-2k}$ converges and $e=e_2$.*

Proof. a) $e\leq e_1$: For all $\alpha>e_1$, $2^{-n\alpha}\sum_{k=1}^{n}\omega_k\to 0$, so that $\sum_{k=1}^{n}\omega_k\preceq 2^{n\alpha}$. We deduce that $\omega_n\preceq 2^{n\alpha}$, and $\alpha\geq e$.

b) $e_1\leq e$: For all $\alpha>e$, we have $\omega_n\preceq 2^{n\alpha}$, and $\sum_{k=1}^{n}\omega_k\preceq 2^{n\alpha}$. We deduce that $2^{-n\alpha}\sum_{k=1}^{n}\omega_k$ is bounded, and that $e_1\leq\alpha$.

c) If $e<2$, for all $\alpha\in\,]e,2[$ we have $\omega_n\preceq 2^{n\alpha}$, so that $\sum_{k=1}^{n}\omega_k2^{-2k}\preceq\sum_{k=1}^{n}2^{-n(2-\alpha)}$ which converges. Let us show that $e\leq e_2$.

Let $\alpha>e_2$. Since $2^{n(2-\alpha)}\sum_{k=n}^{\infty}\omega_k2^{-2k}$ tends to 0, we have $\sum_{k=n}^{\infty}\omega_k2^{-2k}\preceq 2^{n(\alpha-2)}$. Using Dini's theorem, we deduce that for all $a\in\,]0,1[$, the series

$$\sum\omega_n2^{-2n}\Big(\sum_{k=n}^{\infty}\omega_k2^{-2k}\Big)^{-a}$$

converges. Therefore

$$\sum_{n=1}^{\infty} \omega_n 2^{-2n}(2^{n(\alpha-2)})^{-a} = \sum_{n=1}^{\infty} \omega_n 2^{-n(a\alpha+2(1-a))}$$

converges as well, which gives $e \le a\alpha + 2(1-a)$. Then let a tend to 1 and α tend to e_2.

d) $e_2 \le e$: Take α and β such that $e < \alpha < \beta < 2$. Since $\omega_n \preceq 2^{n\alpha}$, we have $\sum_{k=n}^{\infty} \omega_k 2^{-2k} \preceq 2^{n(\alpha-2)}$. Therefore $2^{n(2-\beta)} \sum_{k=n}^{\infty} \omega_k 2^{-2k} \preceq 2^{n(\alpha-\beta)}$. Since the right member tends to 0, $e_2 \le \beta$. Then let α and β tend to e. ∎

Similar arguments can be found in [6, 7, 8].

6. Results on the dimensions

PROPOSITION 1. *Let e be the exponent of convergence of the sequence (ω_n), where ω_n is the number of squares in $\mathcal{F}_{2^{-n}}$. Let*

$$\Delta_{\text{ext}}(E) = \limsup \left(2 - \frac{\log \mathcal{A}(E(\varepsilon) - E)}{\log \varepsilon}\right).$$

Then

$$e = \Delta_{\text{ext}}(E). \tag{19}$$

This dimension $\Delta_{\text{ext}}(E)$ is sometimes called the *exterior dimension* and used in the study of *fat fractal sets* such that $\mathcal{A}(E) > 0$, and in the lateral properties of a curve [9].

Proof. It suffices to use the discrete values $\varepsilon = 2^{-n}$. We can write

$$\Delta_{\text{ext}}(E) = \inf\{\alpha \mid 2^{n(\alpha-2)} \mathcal{A}(E(2^{-n}) - E) \to 0\}.$$

Since E is bounded, $\mathcal{A}(E(2^{-n}) - E)$ is bounded above and $\Delta_{\text{ext}}(E) \le 2$.

a) Let us show that $e \le \Delta_{\text{ext}}(E)$. By (15),

$$\bigcup \mathcal{F}_{2^{-n}} \subset E(2^{-n+2}).$$

For every square $C \in \mathcal{F}_{2^{-n}}$, we have $\mathring{C} \cap E = \emptyset$, so that

$$\mathcal{A}(\mathcal{F}_{2^{-n}}) \le \mathcal{A}(E(2^{-n+2}) - E).$$

Since $\mathrm{I}(\mathcal{F}_{2^{-n}}) \le 4$,

$$\omega_n \, 2^{-2n} \le 4\, \mathcal{A}(E(2^{-n+2}) - E).$$

Therefore

$$2^{-n\alpha} \omega_n \preceq 2^{n(2-\alpha)} \mathcal{A}(E(2^{-n+2}) - E) \preceq 2^{(n-2)(2-\alpha)} \mathcal{A}(E(2^{-n+2}) - E).$$

For every $\alpha > \Delta_{\text{ext}}(E)$, the right member tends to 0, so that the left member tends to 0; this shows that $e \le \alpha$. Hence $e \le \Delta_{\text{ext}}(E)$. If $e = 2$, then $e = \Delta_{\text{ext}}(E)$.

b) Let us show that $e \leq \Delta_{\mathrm{ext}}(E)$. We may assume that $e < 2$. Let $\varepsilon = 2^{-n}$. Consider an ε-connected component F of E.

If $F \in \mathcal{R}'_{2^{-n}}$, then $F(2^{-n}) \subset \widetilde{R}(E, 2^{-n})$. Therefore

$$\Big(\bigcup \mathcal{R}'_{2^{-n}}\Big)(2^{-n}) - E \subset \widetilde{R}(E, 2^{-n}) - E.$$

From (17), this set is included in $\bigcup_{k\geq n}\bigcup \mathcal{F}_{2^{-k}}$, so that

$$\mathcal{A}\Big(\Big(\bigcup \mathcal{R}'_{2^{-n}}\Big)(2^{-n}) - E\Big) \leq \sum_{k\geq n} \omega_k 2^{-2k}. \tag{20}$$

If $F \in \mathcal{R}''_{2^{-n}}$, then $\operatorname{diam}(F) < 2^{-n}$, so that $F(2^{-n})$ is included in a square of side $3\cdot 2^{-n}$. We deduce that $\mathcal{A}(F(\varepsilon)) < 9\cdot 2^{-2n}$. By Lemma 4, $\operatorname{Card}\mathcal{R}''_{2^{-n}} \leq \operatorname{Card}(\bigcup_{k=1}^{n} \mathcal{F}_{2^{-k}})$. Then

$$\mathcal{A}\Big(\Big(\bigcup \mathcal{R}''_{2^{-n}}\Big)(2^{-n})\Big) \preceq 2^{-2n}\sum_{k\leq n}\omega_k. \tag{21}$$

For every ε, $E \subset \bigcup \mathcal{R}'_\varepsilon \cup \bigcup \mathcal{R}''_\varepsilon$, hence $E(\varepsilon) \subset (\bigcup \mathcal{R}'_\varepsilon)(\varepsilon) \cup (\bigcup \mathcal{R}''_\varepsilon)(\varepsilon)$. From (20) and (21) we deduce that

$$\mathcal{A}(E(2^{-n}) - E) \preceq 2^{-2n}\sum_{k\leq n}\omega_k + \sum_{k\geq n}\omega_k 2^{-2k},$$

which implies that

$$2^{n(2-\alpha)}\mathcal{A}(E(2^{-n}) - E) \preceq 2^{-n\alpha}\sum_{k=1}^{n}\omega_k + 2^{n(2-\alpha)}\sum_{k=n}^{\infty}\omega_k 2^{-2k}.$$

Therefore $\Delta_{\mathrm{ext}}(E) \leq \max\{e_1, e_2\}$, equal to e (Lemma 5). ∎

Proposition 2. *Let*

$$\Delta_X(E) = \limsup_{\varepsilon\to 0}\left(2 - \frac{\log \mathcal{A}(X(E_\varepsilon, \varepsilon))}{\log \varepsilon}\right)$$

and e as before. Then

$$\Delta_X(E) \geq e. \tag{22}$$

Proof. Let $\varepsilon = 2^{-n}$, and $U_\varepsilon = \bigcup\{F_\varepsilon \mid F \in \mathcal{R}_\varepsilon \text{ and } \operatorname{diam}(F) \geq \varepsilon/2\}$. The set $U_{2\varepsilon}$ is included in $E_{2\varepsilon}$, and its connected components have a diameter $\geq \varepsilon$. From (11), $\mathcal{A}(U_{2\varepsilon}(3\varepsilon)) \leq 49\mathcal{A}(X(U_{2\varepsilon}, \varepsilon)) \leq 49\mathcal{A}(X(E_{2\varepsilon}, 2\varepsilon))$. By (15), $\bigcup \mathcal{F}_\varepsilon \subset \widetilde{R}(E, 2\varepsilon)(\varepsilon)$, which is included in $U_{2\varepsilon}(3\varepsilon)$. This gives $2^{-2n}\omega_n \preceq \mathcal{A}(X(E_{2^{-n+1}}, 2^{-n+1}))$. If $\alpha > \Delta_X(E)$, then $\varepsilon^{\alpha-2}\mathcal{A}(X(E_\varepsilon, \varepsilon)) \to 0$, so that $2^{-n\alpha}\omega_n \to 0$. This proves that $\alpha \geq e$. ∎

Corollary 3. *For every compact set E, $\Delta_X(E) = \Delta(E)$.*

Proof. If E has area 0, then $\Delta_{\mathrm{ext}}(E) = \Delta(E)$. Propositions 1 and 2 give $\Delta_{\mathrm{ext}}(E) \leq \Delta_X(E)$. On the other hand, the inequality $\Delta_X(E) \leq \Delta(E)$ is trivial. Therefore $\Delta_X(E) = \Delta(E)$. If E has area > 0, then $\Delta_X(E)$ and $\Delta(E)$ are both equal to 2. ∎

This completes the proof of Theorem 1.

The next result shows that the value of the dimension depends only on the large connected components. For the definition of $\widetilde{R}(E,\varepsilon)$, see (14).

PROPOSITION 3. *For any compact set E,*

$$\Delta(E) = \limsup \left(2 - \frac{\log \mathcal{A}(\widetilde{R}(E,\varepsilon))}{\log \varepsilon}\right). \tag{23}$$

Proof. Let $\Delta'(E)$ be the right side. From (15) we get $\mathcal{A}(\bigcup \mathcal{F}_\varepsilon) \le \mathcal{A}(\widetilde{R}(E,2\varepsilon)(\varepsilon))$. On the other hand, $\widetilde{R}(E,2\varepsilon)(\varepsilon) \subset \bigcup\{F(3\varepsilon) \mid F \in \mathcal{R}_{2\varepsilon}, \operatorname{diam}(F) \ge \varepsilon\}$. By (11),

$$\begin{aligned}\mathcal{A}\Big(\bigcup\{F(3\varepsilon) \mid F \in \mathcal{R}_{2\varepsilon}, \operatorname{diam}(F) \ge \varepsilon\}\Big)& \\ &\le \frac{9}{4}\mathcal{A}(\{F(2\varepsilon) \mid F \in \mathcal{R}_{2\varepsilon}, \operatorname{diam}(F) \ge \varepsilon\}).\end{aligned}$$

We deduce that $\mathcal{A}(\bigcup \mathcal{F}_\varepsilon) \preceq \mathcal{A}(\widetilde{R}(E,2\varepsilon))$. Hence $\omega_n 2^{-2n} \preceq \mathcal{A}(\widetilde{R}(E,2^{-n+1}))$. This gives $e \le \Delta'(E)$, that is, $\Delta(E) \le \Delta'(E)$. The converse inequality comes directly from the inclusion $\widetilde{R}(E,\varepsilon) \subset E(\varepsilon)$. ∎

7. Integral formulas.

Let us consider the functions g_ε and f_ε defined in Section 1.4.

PROPOSITION 4. *For any compact set E,*

$$\mathcal{A}(X_{\mathrm{H}}(E_{\varepsilon/4}, \varepsilon/4)) \le \int_{\mathbb{R}} f_\varepsilon(s)\, ds \le \mathcal{A}(X_{\mathrm{H}}(E_\varepsilon, \varepsilon)). \tag{24}$$

Proof. The vertical sections of $X_{\mathrm{H}}(E_\varepsilon, \varepsilon)$ can be expressed as

$$F_\varepsilon(s) = \mathrm{Proj}_{Ot}(X_{\mathrm{H}}(E_\varepsilon, \varepsilon) \cap \mathrm{D}(s))$$

where $\mathrm{D}(s)$ is the vertical line of abscissa s. The area of $X_{\mathrm{H}}(E_\varepsilon, \varepsilon)$ is the s-integral of the length $L(F_\varepsilon(s))$.

a) Fix s_0 such that $E_\varepsilon(s_0) \ne \emptyset$. We want to show that $E_\varepsilon(s_0) \subset F_\varepsilon(s_0)$. Let $t_0 \in E_\varepsilon(s_0)$.

There exist t_1 and t_2 in $\mathrm{Proj}_{Ot}(E \cap [s_0 - \varepsilon/2, s_0 + \varepsilon/2] \times \mathbb{R})$ such that

$$t_1 \le t_0 \le t_2 \le t_1 + \varepsilon.$$

We deduce that there exist s_1, s_2 such that $|s_0 - s_i| \le \varepsilon/2$ and the points $x_i = (s_i, t_i)$ are in E. Since $\varrho(x_1, x_2) \le \varepsilon$, the segment $\overline{x_1 x_2}$ is in E_ε. On this segment there exists a point with second coordinate t_0. Let s' be its first coordinate. Since $|s_0 - s'| \le \varepsilon/2$, we deduce that (s_0, t_0) belongs to the horizontal segment of length ε centered at (s', t_0). Therefore $(s_0, t_0) \in$

$X_{\mathrm{H}}(E_\varepsilon,\varepsilon)$. Since (s_0,t_0) also belongs to the line $\mathrm{D}(s_0)$, we deduce that $t_0 \in \mathrm{Proj}_{Ot}(X_{\mathrm{H}}(E_\varepsilon,\varepsilon)\cap \mathrm{D}(s_0)) = F_\varepsilon(s_0)$.

By integration, we conclude that $\int f_\varepsilon(s)\,ds \le \mathcal{A}(X_{\mathrm{H}}(E_\varepsilon,\varepsilon))$.

b) Fix s_0 such that $F_\varepsilon(s_0)\neq\emptyset$. We will show that $F_\varepsilon(s_0)\subset E_{4\varepsilon}(s_0)$. Let $t_0\in F_\varepsilon(s_0)$.

There exists a point $x=(s',t_0)$ in E_ε such that $|s_0-s'|\le\varepsilon$. It belongs to a segment $\overline{x_1x_2}$ such that $x_i\in E$ and $\varrho(x_1,x_2)\le\varepsilon$. Let $x_i=(s_i,t_i)$. Since $|s'-s_i|\le\varepsilon$, we have $|s_0-s_i|\le 2\varepsilon$. Therefore $x_i\in E\cap([s_0-2\varepsilon,s_0+2\varepsilon]\times\mathbb{R})$. Since t_0 lies between t_1 and t_2, and $|t_1-t_2|\le\varepsilon$, we deduce that $t\in E_{4\varepsilon}(s_0)$.

By integration this gives $\mathcal{A}(X_{\mathrm{H}}(E_\varepsilon,\varepsilon))\le\int f_{4\varepsilon}(s)\,ds$. ∎

In a similar way, we obtain the following:

$$\mathcal{A}(X_{\mathrm{V}}(E_{\varepsilon/4},\varepsilon/4)) \le \int_{\mathbb{R}} g_\varepsilon(s)\,ds \le \mathcal{A}(X_{\mathrm{V}}(E_\varepsilon,\varepsilon)). \tag{25}$$

Estimates (24) and (25) imply

$$\frac{1}{2}\Big(\int_{\mathbb{R}} f_\varepsilon(s)\,ds + \int_{\mathbb{R}} g_\varepsilon(s)\,ds\Big) \le \mathcal{A}(X(E,\varepsilon))$$

and

$$\mathcal{A}(X(E_{\varepsilon/4},\varepsilon/4)) \le \int_{\mathbb{R}} f_\varepsilon(s)\,ds + \int_{\mathbb{R}} g_\varepsilon(s)\,ds.$$

These inequalities, together with Theorem 1, prove (9). This completes the proof of Theorem 2.

Remark. (9) may also be written as

$$\varDelta_X(E) = \max\Big\{\limsup_{\varepsilon\to 0}\Big(2-\frac{\log\int_{\mathbb{R}} f_\varepsilon(s)\,ds}{\log\varepsilon}\Big), \limsup_{\varepsilon\to 0}\Big(2-\frac{\log\int_{\mathbb{R}} g_\varepsilon(s)\,ds}{\log\varepsilon}\Big)\Big\}.$$

In the case of a continuous, non-constant function, $\int_{\mathbb{R}} f_\varepsilon(s)\,ds$ tends to 0 more slowly than $\int_{\mathbb{R}} g_\varepsilon(s)\,ds$, so that $\varDelta_X(E)$ is equal to the first term: This gives the variation method (Section 1.2).

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Laboratoire de Mathématiques Pures
Université Blaise Pascal
63177 Aubière Cedex, France
E-mail: ctricot@wanadoo.fr

Received 5 July 2001