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On the uniqueness of the uncentered ergodic maximal function

by

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Abstract. It is shown that if two functions share the same uncentered (two-sided) ergodic maximal function, then they are equal almost everywhere.

1. Introduction. Let (X, Σ, μ) be a measure space, and let $T : X \to X$ be a measure-preserving ergodic transformation. For an integrable function $f \in L^1(X)$, the associated ergodic one-sided maximal function is denoted by f^* and defined by

$$f^*(x) = \sup_n \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x), \qquad x \in X.$$

The uniqueness of the one-sided ergodic maximal function was recently established by Lasha Ephremidze in [2]. In particular, Ephremidze proved the following.

THEOREM 1. (a) *If $\mu(X) < \infty$, f and g are in $L^1(X)$, and $f^* = g^*$ a.e., then $f(x) = g(x)$ for a.e. $x \in X$.*

(b) *If (X, Σ, μ) is a σ-finite measure space, $\mu(X) = \infty$, f and g are in $L^1(X)$, and $f^* = g^*$ a.e., then $f = g$ a.e. on $\{x \in X : f^*(x) > 0\}$.*

In order to obtain these results, Ephremidze proved a uniqueness theorem for the one-sided discrete Hardy–Littlewood maximal operator M_1, defined on $L^1(\mathbb{Z})$ by

$$M_1 f(n) = \sup_{n \le m} \frac{1}{m-n+1} \sum_{k=n}^{m} |f(k)|.$$

In particular, he proved the following:

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Proposition 2. *If f and g are nonnegative functions in $L^1(\mathbb{Z})$ and $M_1 f = M_1 g$, then $f = g$.*

Ephremidze's proof of this proposition is quite involved and in fact entails many of the primary difficulties in his proof of Theorem 1. Also, the techniques involved in his proof take strong advantage of the fact that M_1 is a *one-sided* maximal operator and do not carry over to yield similar uniqueness results for the *two-sided* or *uncentered* discrete Hardy–Littlewood maximal operator.

The purpose of this paper is to prove the uniqueness of the uncentered discrete Hardy–Littlewood maximal function and as a corollary of the proof obtain the uniqueness of the uncentered ergodic maximal function on spaces of finite measure. Moreover we will prove that two nonnegative integrable functions on a space of infinite measure which share the same uncentered ergodic maximal function must be equal almost everywhere.

We close the introduction by remarking that Roger Jones has very recently in [3] provided an alternative proof of Theorem 1 utilizing techniques involving the Kakutani skyscraper construction. His results, however, apply only for one-sided and not two-sided ergodic maximal functions.

2. Uniqueness of the uncentered discrete maximal function

Definition 3. If f is a real-valued function on $\mathbb{Z}$, the *uncentered* (or *two-sided*) *discrete Hardy–Littlewood maximal function* of f is denoted by Mf and defined by

$$Mf(n) = \sup_{j \le n \le k} \frac{1}{k-j+1} \sum_{i=j}^{k} |f(i)|.$$

Our primary purpose in this section is to prove the following proposition.

Proposition 4. *If f and g are nonnegative functions in $L^p(\mathbb{Z})$ for some p in $[1,\infty)$ and $Mf = Mg$, then $f = g$.*

Before beginning the proof we remark that the above result does not hold for $p = \infty$. For example, if $f = \chi_{\mathbb{Z}}$ and $g = \chi_{\mathbb{N}}$, then $Mf = Mg \equiv 1$ although $f \neq g$.

Proof. It suffices to show that a nonnegative function $f \in L^p(\mathbb{Z})$ is uniquely determined by its maximal function Mf.

Note that for a nonnegative function f, Mf is identically 0 if and only if f is identically 0.

Suppose Mf is not identically 0. We will inductively define a sequence $\{a_j\}$ of positive real numbers, a corresponding sequence $\{E(a_j)\}$ of sets in $\mathbb{Z}$, and a corresponding sequence $\{I_{j,k}\}$ of intervals in $\mathbb{Z}$. Let $a_1 = \sup\{Mf(n) : n \in \mathbb{Z}\}$. Note that $0 < a_1 = \|f\|_\infty \le \|f\|_p$. Let $E(a_1) = \{n \in \mathbb{Z} : Mf(n)$

$= a_1\} = \{n \in \mathbb{Z} : f(n) = a_1\}$. Let $a_2 = \sup\{Mf(n) : n \in \mathbb{Z} \text{ and } n \notin E(a_1)\}$. Let $E(a_2) = \{n \in \mathbb{Z} : Mf(n) = a_2\}$. Note that $a_2 > 0$ as Mf never takes on the value 0, and also that $a_2 < a_1$ since by Chebyshev's inequality and the weak type (p, p) boundedness of M we have $|\{n \in \mathbb{Z} : Mf(x) > a_2/2\}| < \infty$. ($|\cdot|$ here denotes the standard counting measure on $\mathbb{Z}$.) Note that since there are only finitely many integers n such that $Mf(n) > a_2/2$, Mf actually takes on the value a_2.

Suppose $a_1 > a_2 > \cdots > a_k$ have been determined, as well as the associated sets $E(a_1), \ldots, E(a_k)$. Let

$$a_{k+1} = \sup\{Mf(n) : n \in \mathbb{Z} \text{ and } Mf(n) < a_k\}.$$

As before, we have $0 < a_{k+1} < a_k$, and we define $E(a_{k+1})$ by

$$E(a_{k+1}) = \{n \in \mathbb{Z} : Mf(n) = a_{k+1}\}\,.$$

So by induction the sequences of positive real numbers a_j and sets $E(a_j)$ in $\mathbb{Z}$ are determined.

Now, if $p \in E(a_k)$, then there exists an interval I_p containing p (i.e. a set in $\mathbb{Z}$ of the form $\{p-j, p-j+1, \ldots, p-1, p, p+1, \ldots, p+k\}$) such that

$$\frac{1}{|I_p|} \sum_{j \in I_p} f(j) = a_k.$$

If several choices for I_p are available, we choose one such that I_p contains as few points as possible.

We label the points of $E(a_k)$ as $p_{k,1}, p_{k,2}, \ldots, p_{k,|E(a_k)|}$, and we let $I_{j,k} = I_{p_{j,k}}$.

We now successively find the values of f at $p_{1,1}, p_{1,2}, \ldots, p_{1,|E(a_1)|}, p_{2,1}, \ldots, p_{2,|E(a_2)|}, p_{3,1}, \ldots$.

As we have previously noted, since $f \in L^p(\mathbb{Z})$ and $a_1 = \|f\|_\infty$, if $Mf(p) = a_1$ then $f(p) = a_1$. Hence $f(p_{1,1}), f(p_{1,2}), \ldots, f(p_{1,|E(a_1)|})$ all equal a_1.

Before we find the values $f(p_{2,i})$, we prove the following useful claim regarding how the sets $I_{j,k}$ are positioned with respect to $E(a_1) \cup \cdots \cup E(a_{j-1})$ and the points in $E(a_j)$.

Claim 5.

$$I_{j,k} \subset \bigcup_{i=1}^{j-1} E(a_i) \cup \{p_{j,k}\}.$$

In particular, $I_{j,k}$ does not intersect $\bigcup_{i>j} E(a_i)$, nor does it contain any point $p_{j,l}$ where $l \neq k$.

Proof. For our notational convenience, if $\mathcal{I}$ is an interval $\{r, r+1, \ldots, r+s\}$ in $\mathbb{Z}$, we let $\ell(\mathcal{I}) = r$ and $r(\mathcal{I}) = r+s$.

Now, we recall that by the definitions of $I_{j,k}$ and $p_{j,k}$, we have $p_{j,k} \in I_{j,k}$, $Mf(p_{j,k}) = a_j$, $|I_{j,k}|^{-1} \sum_{i \in I_{j,k}} f(i) = a_j$, and that if $\mathcal{I}$ is an interval in $\mathbb{Z}$ containing $p_{j,k}$ of size smaller than that of $I_{j,k}$ then the average of f over $\mathcal{I}$ is less than a_j.

Now, if $p \in I_{j,k}$, then $Mf(p) \geq a_j$ automatically holds, and so $p \in E(a_1) \cup \cdots \cup E(a_j)$.

So it suffices to show that if $l \neq k$, then $p_{j,l} \notin I_{j,k}$. This is seen by contradiction. Suppose $I_{j,k}$ did contain a point $p_{j,l}$ in $E(a_j)$ where $l \neq k$. We assume without loss of generality that $p_{j,k} < p_{j,l}$. We then consider the interval $\{p_{j,l}, p_{j,l}+1, \ldots, r(I_{j,k})\}$. The average of f over this interval cannot be less than or equal to a_j, since otherwise the average of f over $\{\ell(I_{j,k}), \ldots, p_{j,l}-1\}$ would be greater than or equal to a_j, contradicting either the fact that $Mf(p_{j,k}) = a_j$ or the minimality condition on the size of $I_{j,k}$. So the average of f over $\{p_{j,l}, p_{j,l}+1, \ldots, r(I_{j,k})\}$ must exceed a_j. But then $p_{j,l} \in \bigcup_{i=1}^{j-1} E(a_i)$, and so $p_{j,l} \notin E(a_j)$. ∎

We are now in a position to find the values $f(p_{2,1}), \ldots, f(p_{2,|E(a_2)|})$. We have $p_{2,j} \in I_{2,j}$ and $I_{2,j} \subset E(a_1) \cup \{p_{2,j}\}$. As

$$Mf(p_{2,j}) = \frac{1}{|I_{2,j}|}\Big[f(p_{2,j}) + \sum_{n \in I_{2,j}\setminus\{p_{2,j}\}} f(n)\Big]$$

and all the values of $f(n)$ are known for $n \in I_{2,j}\setminus\{p_{2,j}\}$, we see that $f(p_{2,j})$ is uniquely determined by $Mf(p_{2,j})$.

We proceed by induction. Suppose $f(p_{1,1}), \ldots, f(p_{1,|E(a_1)|}), f(p_{2,1}), \ldots, f(p_{2,|E(a_2)|}), \ldots, f(p_{N,1}), \ldots, f(p_{N,|E(a_N)|})$ have been determined. We now find $f(p_{N+1,1}), \ldots, f(p_{N+1,|E(a_{N+1})|})$. Well, for $j \in \{1, \ldots, |E(a_{N+1})|\}$ we have

$$I_{p_{N+1,j}} \subset E(a_1) \cup \cdots \cup E(a_N) \cup \{p_{N+1,j}\}.$$

As

$$Mf(p_{N+1,j}) = \frac{1}{|I_{N+1,j}|}\Big[f(p_{N+1,j}) + \sum_{n \in I_{N+1,j}\setminus\{p_{N+1,j}\}} f(n)\Big]$$

and all the values of $f(n)$ are (by induction) known for $n \in I_{N+1,j}\setminus\{p_{N+1,j}\}$, we see that $f(p_{N+1,j})$ is uniquely determined.

In this manner we realize that the values of $f(n)$ are known if $n \in \bigcup_{i=1}^{\infty} E(a_i)$. As $Mf(n) > 0$ for all n and $a_i \to 0$ since Mf is in weak $L^p(\mathbb{Z})$, we deduce that each integer n lies in $E(a_i)$ for some i. Hence Mf uniquely determines f. ∎

One technical point we must deal with is that the uncentered ergodic maximal operator allows for cancellation, whereas the uncentered discrete Hardy–Littlewood maximal operator does not. Accordingly we define an associated discrete maximal operator that does allow for cancellation and for this operator provide a suitable uniqueness result which will in turn readily yield the desired uniqueness result for the uncentered ergodic maximal operator.

Definition 6. If f is a function on $\mathbb{Z}$, the associated discrete uncentered (two-sided) maximal function of f is denoted by $\widetilde{M}f$ and defined by

$$\widetilde{M}f(n) = \sup_{j \le n \le k} \frac{1}{k-j+1} \sum_{i=j}^{k} f(i).$$

We note that if f and g are in $L^1(\mathbb{Z})$ and $\widetilde{M}(f) = \widetilde{M}(g)$, it does *not* necessarily follow that $f = g$. For example, letting $f = -\chi_{\{0\}}$ and $g \equiv 0$, we have $\widetilde{M}f = \widetilde{M}g \equiv 0$, although $f \neq g$. We do have, however, the following uniqueness result.

Proposition 7. *Let $f : \mathbb{Z} \to \mathbb{R}$ be such that for each $x \in \mathbb{Z}$,*

$$\widetilde{M}f(x) = \frac{1}{k-j+1} \sum_{n=j}^{k} f(n)$$

for some $j \le x \le k$. Then $\widetilde{M}f$ uniquely determines f.

Proof. Let $p \in \mathbb{Z}$. Let I_p be an interval containing p such that

$$\widetilde{M}f(p) = \frac{1}{|I_p|} \sum_{n \in I_p} f(n).$$

Now, if $q \in I_p$ then there exists an interval $I_{p,q}$ containing q and contained in I_p such that

$$\widetilde{M}f(q) = \frac{1}{|I_{p,q}|} \sum_{n \in I_{p,q}} f(n).$$

This is easily seen by contradiction. Suppose such an interval $I_{p,q}$ did not exist. Let J be an interval containing q such that

$$\widetilde{M}f(q) = \frac{1}{|J|} \sum_{n \in J} f(n).$$

Now, $\widetilde{M}f(q) > \widetilde{M}f(p)$ as J could not be I_p itself. Assume without loss of generality that $r(J) > r(I_p)$ and $\ell(J) \in I_p$. Note that the average of f over $\{r(I_p)+1, \dots, r(J)\}$ exceeds $\widetilde{M}f(q)$ and hence $\widetilde{M}f(p)$. But then the average of f over $\{\ell(I_p), \dots, r(J)\}$ exceeds $\widetilde{M}f(p)$, contradicting the definition of $\widetilde{M}f(p)$.

Note that the proof of Proposition 4 now readily carries over to this situation to uniquely determine f on I_p from $\widetilde{M}f$. The only modification necessary is that the use of the Chebyshev inequality and the weak type boundedness of M is replaced by the knowledge that, given a point $p \in \mathbb{Z}$ and associated interval I_p as above, there are only a finite number of averages of f over intervals contained in I_p. Hence a finite sequence $a_1 > a_2 > \cdots > a_{k_p}$ of all the values of $\widetilde{M}f$ over I_p may be obtained, as well as the associated sets $E(a_k)$ in I_p. The rest of the proof follows as before, taking advantage of the fact that for all $x \in I_p$ the intervals I_x may be chosen to be contained in I_p.

So, if $x \in \mathbb{Z}$, let I_x be the interval in $\mathbb{Z}$ containing x such that

$$\widetilde{M}f(x) = \frac{1}{|I_x|} \sum_{n \in I_x} f(n).$$

As we can reconstruct f from $\widetilde{M}f$ on I_x, we can in particular find $f(x)$. Hence f is uniquely determined from $\widetilde{M}f$. ∎

3. Uniqueness of the uncentered ergodic maximal function. Let (X, Σ, μ) be a measure space, and let $T : X \to X$ be a measure-preserving ergodic transformation. For an integrable function $f \in L^1(X)$, the associated uncentered (or two-sided) ergodic maximal function is denoted by $\widetilde{f}$ and defined by

$$\widetilde{f}(x) = \sup_{j \le 0 \le k} \frac{1}{k-j+1} \sum_{i=j}^{k} f(T^i x), \qquad x \in X.$$

As in the one-sided case, the uniqueness of the uncentered ergodic maximal function depends on the measure of X itself. If X has finite μ-measure, we obtain a positive result:

THEOREM 8. *If $\mu(X) < \infty$, f and g are in $L^1(X)$, and $\widetilde{f} = \widetilde{g}$ a.e., then $f(x) = g(x)$ for a.e. $x \in X$.*

Proof. We will use the following result of Ephremidze:

LEMMA 9 ([2]). *Let T be a measure preserving ergodic transformation of a finite measure space (X, Σ, μ) and let $\int_X h \, d\mu = 0$. Then $\mu(E) = 0$, where*

$$E = \Big\{ x \in X : \sum_{k=0}^{n} h(T^k x) < 0 \text{ for all } n \ge 0 \Big\}.$$

We will also need the following lemma:

LEMMA 10. *Let $h \in L^1(X)$ and let*

$$(1) \qquad F_h = \Big\{ x \in X : \widetilde{h}(x) = \frac{1}{k-j+1} \sum_{i=j}^{k} h(T^i x) \text{ for some } j \le 0 \le k \Big\}.$$

Then $\mu(F_h) = \mu(X)$, *and consequently*

(2) $$\mu(\{x \in X : T^n x \in F_h \textit{ for all } n \in \mathbb{Z}\}) = \mu(X).$$

Proof. Note that the pointwise ergodic theorem (see, e.g., [5]) tells us that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} h(T^k x) = \frac{1}{\mu(X)} \int_X h\, d\mu \equiv \lambda_0$$

for a.e. $x \in X$. So $\mu(\{x \in X : \widetilde{h}(x) \geq \lambda_0\}) = \mu(X)$. Now, $(h-\lambda_0)\widetilde{\ } = \widetilde{h} - \lambda_0$. If

$$\sum_{i=j}^{k} (h-\lambda_0)(T^i x) < 0$$

for all $j \leq 0 \leq k$, then $\sum_{i=0}^{n} (h-\lambda_0)(T^i x) < 0$ for all $n \geq 0$, which only occurs on a set of measure zero by Lemma 9 and the fact that $\int_X (h-\lambda_0)\, d\mu = 0$.

So then a.e. $x \in \{z \in X : \widetilde{h}(z) \geq \lambda_0\}$ belongs to F_h. Hence (1) holds.

As

$$\{x \in X : T^n x \in F_h \text{ for all } n \in \mathbb{Z}\} = \bigcap_{n=-\infty}^{\infty} T^{-n}(F_j),$$

(2) holds as well. ■

Note that by Lemma 10 the set $G \subset X$ defined by

$$G = \bigcup_{n\in\mathbb{Z}} T^{-n}(\{z \in X : \widetilde{f}(z) \neq \widetilde{g}(z)\} \cup F_f^{\mathrm{c}} \cup F_g^{\mathrm{c}})$$

is of measure zero. So if $x \notin G$, setting

$$\alpha_x(n) = f(T^n x), \qquad \beta_x(n) = g(T^n x)$$

by Proposition 7 we have

$$f(x) = \alpha_x(0) = \beta_x(0) = g(x).$$

So $f(x) = g(x)$ almost everywhere on X. ■

We now consider the case where $\mu(X) = \infty$. Uniqueness in general does not hold in this case. For example, if $X = \mathbb{Z}$ and $T(x) = x+1$, by letting $f(n) = -\chi_{\{0\}}(n)$ and $g(n) \equiv 0$, we have $\widetilde{f} = \widetilde{g} = 0$ although $f \neq g$. We do have the following uniqueness result, however:

THEOREM 11. *Let T be a measure preserving ergodic transformation of a σ-finite measure space (X, Σ, μ), where $\mu(X) = \infty$. If $f, g \in L^1(X)$ and $\widetilde{f} = \widetilde{g}$ a.e. on X, then $f = g$ a.e. on $\{x \in X : \widetilde{f}(x) > 0\}$.*

Proof. We will need the following result of Ephremidze:

Lemma 12 ([2]). *Let T be a measure preserving ergodic transformation of a σ-finite measure space (X,Σ,μ). If $\mu(Q_0)<\infty$ and $Q_0^{\mathrm{c}}=X\backslash Q_0$, then*

$$\mu\Big(Q_0\setminus\bigcup_{m=-\infty}^{\infty}T^{-m}(Q_0^{\mathrm{c}})\Big)=0.$$

It suffices to show that f is uniquely determined almost everywhere on $\{x\in X:\widetilde{f}(x)>0\}$ from $\widetilde{f}$. Now, by the maximal ergodic theorem (see, e.g., [4]), for each $\lambda>0$ we have

$$\mu(\{x\in X:\widetilde{f}(x)>\lambda\})<\infty.$$

Hence by Lemma 12 we have

$$\mu\Big(\{x\in X:\widetilde{f}(x)>\lambda\}\setminus\bigcup_{m=-\infty}^{\infty}T^{-m}\{x\in X:\widetilde{f}(x)\le\lambda\}\Big)=0.$$

So for a.e. x in $\{x\in X:\widetilde{f}(x)>\lambda\}$ there exists $m=m(x)$ such that $\widetilde{f}(T^m x)\le\lambda<\widetilde{f}(x)$. As $\lambda>0$ is arbitrary, it follows that for a.e. x in $\{x\in X:\widetilde{f}(x)>0\}$ there exists $m=m(x)$ such that $\widetilde{f}(T^m(x))<\widetilde{f}(x)$.

It suffices then to show that if $\widetilde{f}(x)>0$ and $\widetilde{f}(T^m x)<\widetilde{f}(x)$ for some $m=m(x)$, then $f(x)$ may be determined uniquely from $\widetilde{M}f$. For notational convenience, let

$$\alpha(n)=f(T^n x),\qquad \widetilde{\alpha}(n)=\widetilde{f}(T^n x).$$

It is enough to show that $f(x)=\alpha(0)$ may be uniquely determined from $\widetilde{\alpha}(n)$, $n\in\mathbb{Z}$. To do this, we first show that there exists a finite interval $\mathcal{I}$ in $\mathbb{Z}$ containing 0 such that

$$\widetilde{M}\alpha(0)=\frac{1}{|\mathcal{I}|}\sum_{i\in\mathcal{I}}f(i).$$

This is seen by contradiction. Suppose such an interval $\mathcal{I}$ did not exist. Then there would exist a sequence $\{\mathcal{I}_j\}$ of intervals containing 0 such that $\lim_{j\to\infty}|\mathcal{I}_j|=\infty$ and

$$\lim_{j\to\infty}\frac{1}{|\mathcal{I}_j|}\sum_{i\in\mathcal{I}_j}f(i)=\widetilde{M}\alpha(0).$$

Note that these intervals may be chosen such that none contain the point $m\in\mathbb{Z}$. Without loss of generality we assume $m>0$. But then

$$\lim_{j\to\infty}\frac{1}{m-\ell(\mathcal{I}_j)+1}\sum_{i=\ell(\mathcal{I}_j)}^{m}f(i)=\widetilde{M}\alpha(0),$$

implying that $\widetilde{\alpha}(m)\ge\widetilde{\alpha}(0)$, contradicting the fact that $\widetilde{f}(T^m x)<\widetilde{f}(x)$.

So there does exist a finite interval $\mathcal{I} \subset \mathbb{Z}$ containing 0 such that

$$\widetilde{M}\alpha(0) = \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} f(i).$$

Note that if $j \in \mathcal{I}$, then there exists an interval $\mathcal{I}_j \subset \mathcal{I}$ containing j such that

$$\widetilde{M}\alpha(j) = \frac{1}{|\mathcal{I}_j|} \sum_{i \in \mathcal{I}_j} f(i).$$

This is seen by contradiction. Clearly $\widetilde{M}\alpha(j) \geq \widetilde{M}\alpha(0)$ as $j \in \mathcal{I}$. So if the desired interval $\mathcal{I}_j \subset \mathcal{I}$ could not be found, there would exist an interval $\mathcal{I}'_j$ containing j such that

$$\frac{1}{|\mathcal{I}'_j|} \sum_{i \in \mathcal{I}'_j} f(i) > \widetilde{M}\alpha(0).$$

We assume without loss of generality that $\ell(\mathcal{I}'_j) < \ell(\mathcal{I})$. But then

$$\frac{1}{\ell(\mathcal{I}) - \ell(\mathcal{I}'_j)} \sum_{i=\ell(\mathcal{I}'_j)}^{\ell(\mathcal{I})-1} f(i) \geq \widetilde{M}\alpha(j) > \widetilde{M}\alpha(0),$$

and hence

$$\frac{1}{r(\mathcal{I}) - \ell(\mathcal{I}'_j) + 1} \sum_{i=\ell(\mathcal{I}'_j)}^{r(\mathcal{I})} f(i) > \widetilde{M}\alpha(0),$$

contradicting the definition of $\widetilde{M}\alpha(0)$.

So if $j \in \mathcal{I}$ there exists an interval $\mathcal{I}_j \subset \mathcal{I}$ containing j such that

$$\widetilde{M}\alpha(j) = \frac{1}{|\mathcal{I}_j|} \sum_{i \in \mathcal{I}_j} f(i).$$

As this implies that

$$(\widetilde{M}\alpha)\chi_{\mathcal{I}} = (\widetilde{M}(\alpha\chi_{\mathcal{I}}))\chi_{\mathcal{I}},$$

we then conclude by Proposition 7 that for $n \in \mathcal{I}$, $\alpha(n)$ is uniquely determined by $\widetilde{M}\alpha$. As of course $0 \in \mathcal{I}$, the desired result follows. ∎

Corollary 13. *Let T be a measure preserving ergodic transformation of a σ-finite measure space (X, Σ, μ) with $\mu(X) = \infty$. If f and g are nonnegative functions in $L^1(X)$ and $\widetilde{f} = \widetilde{g}$ almost everywhere, then $f = g$ almost everywhere.*

Proof. $f = g$ a.e. on $\{x \in X : \widetilde{f}(x) > 0\} = \{x \in X : \widetilde{g}(x) > 0\}$ by the above theorem. Since f and g are nonnegative, f and g are identically zero on $\{x \in X : \widetilde{f}(x) = 0\} = \{x \in X : \widetilde{g}(x) = 0\}$. As $X = \{x \in X : \widetilde{f}(x) > 0\} \cup \{x \in X : \widetilde{f}(x) = 0\}$ because f is nonnegative, the result follows. ∎

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References

[1] L. Ephremidze, *On the uniqueness of maximal functions*, Georgian Math. J. 3 (1996), 49–52.
[2] —, *On the uniqueness of the ergodic maximal function*, Fund. Math. 174 (2002), 217–228.
[3] R. L. Jones, *On the uniqueness of the ergodic maximal function*, Proc. Amer. Math. Soc. 132 (2003), 1087–1090.
[4] U. Krengel, *Ergodic Theorems*, de Gruyter Stud. Math. 6, de Gruyter, Berlin, 1985.
[5] K. E. Petersen, *Ergodic Theory*, Cambridge Univ. Press, Cambridge, 1983.
[6] A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press, 1959.

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