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Minimal number of periodic points for smooth self-maps of S^3

by

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Abstract. Let f be a continuous self-map of a smooth compact connected and simply-connected manifold of dimension $m \geq 3$ and r a fixed natural number. A topological invariant $D_r^m[f]$, introduced by the authors [Forum Math. 21 (2009)], is equal to the minimal number of r-periodic points for all smooth maps homotopic to f. In this paper we calculate $D_r^3[f]$ for all self-maps of S^3.

1. Introduction. A classical problem in periodic point theory is to determine or estimate the least number of fixed, or more generally r-periodic, points in the homotopy class of a given self-map f of a compact manifold M^m, where r is a fixed natural number (cf. [2], [11]). If M^m is simply-connected and has dimension $m \geq 3$, then one can always find a map g homotopic to f with only one point in $\mathrm{Fix}(g^r)$ (cf. [9]). This is, however, impossible if we demand additionally that g is smooth. In [6] the authors define a topological invariant $D_r^m[f]$ equal to the minimal number of elements in $\mathrm{Fix}(g^r)$ for all g which are smooth and homotopic to f. This leads to a new, smooth branch of Nielsen periodic point theory. Let us remark that $D_r^m[f]$ may be interpreted purely in terms of the smooth category, namely we may assume that f is smooth and approximate the homotopy which joins f to g by a smooth one. Then $D_r^m[f]$ gives the minimal number of periodic points in the smooth homotopy class of f.

This invariant is obtained by decomposing the Lefschetz numbers of iterations into sequences which can be realized as local fixed point indices of iterations of a C^1 map at an isolated periodic orbit. As a result, to find $D_r^m[f]$ we need two types of data: information about $\{L(f^n)\}_{n|r}$ (more precisely, about the set of so-called algebraic periods: $\{n \in \mathbb{N} : \sum_{k|n} \mu(n/k)L(f^k) \neq 0\}$,

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for $n \mid r$, where μ is the Möbius function) and the description of all possible sequences of local indices of iterations. The article joins the ideas of three papers: [6] in which $D_r^m[f]$ is defined, [12] where a description of the algebraic periods of self-maps of S^3 is given, and finally [7], giving a classification of sequences of local indices of iterations in dimension 3. Basing on these results we are able to determine $D_r^3[f]$ for all self-maps of a smooth 3-manifold which is closed, connected and simply-connected, i.e., a 3-dimensional sphere, provided that the result of G. Perelman on the Poincaré conjecture is true.

It is worth pointing out that $D_r^3[f]$, for self-maps of S^3, is almost independent of f, namely it is insensitive to the homotopy class of f, which seems rather unexpected. For example, if r is odd and f is a map with $|\deg(f)| > 1$, then $D_r^3[f] \in \{\zeta(r) - 1, \zeta(r)\}$, where $\zeta(r)$ is the number of divisors of r (cf. Theorem 4.2). This follows from the simply-connectedness of S^3 and the fact that the set of algebraic periods is equal to the set of all natural numbers. As a consequence, for S^3 the value of $D_r^3[f]$ may be perceived as an invariant of the whole space rather than of the homotopy class of f. The same remains true for self-maps of 3-dimensional manifolds (with boundary) with fast growth of the Lefschetz numbers of iterations [5].

The article is organized as follows: in Sections 2 and 3 we introduce the notation and definitions and we recall the necessary statements of [6], [7] and [12]. In Section 4 we give the main results (Theorems 4.2 and 4.7). The case of r odd is a consequence of the previously known facts (in particular Theorem 2.9 from [6]), while the case of r even needs a careful and detailed analysis.

2. Preliminary results. A sequence of indices of iterations at an isolated fixed (periodic) point plays a crucial role in minimizing the number of periodic points in a homotopy class. Let $f : U \to \mathbb{R}^m$, where U is an open subset of $\mathbb{R}^m$, be a map such that x_0 is an isolated fixed point for each iteration of f. Then the sequence $\{\mathrm{ind}(f^n, x_0)\}_{n=1}^{\infty}$ of local indices is well-defined. Below we introduce a useful notation for representing such sequences, which will be used in the next sections.

Definition 2.1. For a given $k \in \mathbb{N}$ we define

$$\mathrm{reg}_k(n) = \begin{cases} k & \text{if } k \mid n, \\ 0 & \text{if } k \nmid n. \end{cases}$$

In other words, reg_k is the periodic sequence

$$(0, \ldots, 0, k, 0, \ldots, 0, k, \ldots),$$

where the non-zero entries appear for indices divisible by k. A sequence of indices of iterations (just as any integer sequence) has the so-called *periodic* *expansion* [13],

$$\operatorname{ind}(f^n, x_0) = \sum_{k=1}^{\infty} a_k \operatorname{reg}_k(n), \tag{2.1}$$

where $a_n = n^{-1} \sum_{k|n} \mu(n/k) \operatorname{ind}(f^k, x_0)$ and μ is the classical Möbius function, i.e., $\mu : \mathbb{N} \to \mathbb{Z}$ is defined by the following three properties: $\mu(1) = 1$, $\mu(k) = (-1)^s$ if k is a product of s different primes, and $\mu(k) = 0$ otherwise.

It has turned out that the indices of iterations must satisfy some conditions, found in [4], called the Dold relations (or Dold congruences).

Theorem 2.2 (Dold relations). *All coefficients of the periodic expansion of a sequence of indices of iterations are integers*, *i.e.*, $a_k \in \mathbb{Z}$ *in* (2.1).

For $p \geq 1$ we define $P_p(f) = \operatorname{Fix}(f^p) \setminus \bigcup_{0<n<p} \operatorname{Fix}(f^n)$. If $x \in P_p(f)$, then the orbit of x will be called a *p-orbit*.

Now we introduce the notion of a *differential Dold sequence* in $\mathbb{R}^m$ for a p-orbit, briefly a *$DD^m(p)$ sequence*. This is a sequence which can be realized as a sequence of indices of iterations on an isolated p-orbit for some smooth map in m-dimensional space.

Definition 2.3. A sequence of integers $\{c_n\}_{n=1}^{\infty}$ is called a *$DD^m(p)$ sequence* if there is a C^1 map $\phi : U \to \mathbb{R}^m$ (U an open subset of $\mathbb{R}^m$) and its isolated p-orbit P such that $c_n = \operatorname{ind}(\phi^n, P)$. If this equality holds for $n \mid r$, where r is fixed, then the finite sequence $\{c_n\}_{n|r}$ will be called a $DD^m(p|r)$ sequence. The number p will be called the *multiplicity* of $\{c_n\}_n$.

There is a close relation, established in [6], between the minimal number of r-periodic points for all smooth maps in a given homotopy class and $DD^m(p|r)$ sequences (Theorem 2.5 below).

Definition 2.4. Let $\{\xi_n\}_{n|r}$ be a sequence of integers satisfying the Dold relations, i.e., its coefficients in the periodic expansion are integers. Assume that we are able to decompose $\{\xi_n\}_{n|r}$ as

$$\xi(n) = c_1(n) + \cdots + c_s(n),$$

where c_i is a $DD^m(l_i|r)$ sequence for $i = 1, \ldots, s$. Each such decomposition determines the sum of the multiplicities, i.e., the number $l = l_1 + \cdots + l_s$. We define $D^m_r[\xi]$ to be the smallest l which can be obtained in this way.

Let $\{L(f^n)\}_{n|r}$ be the sequence of the Lefschetz numbers of iterations of f. We define $D^m_r[f] = D^m_r[\{L(f^n)\}_{n|r}]$.

Theorem 2.5 ([6]). *Let M be a smooth*, *compact*, *connected and simply-connected manifold of dimension $m \geq 3$ and $r \in \mathbb{N}$ a fixed number. For M with nonempty boundary*, *assume additionally that f has no periodic points* *on the boundary. Then*

$$\min\{\#\mathrm{Fix}(g^r) : g \textit{ is } C^1 \textit{ and is homotopic to } f\} = D_r^m[f].$$

By Definition 2.4 determining $D_r^m[f]$ requires the knowledge of all $DD^m(p)$ sequences. There are strong restrictions on such sequences, found by Chow, Mallet-Paret and Yorke [3].

It is not difficult to observe that in order to obtain any $DD^m(p)$ sequence $\{d_n\}_n$ it is enough to replace each reg_k by reg_{pk} in the periodic expansion of some $DD^m(1)$ sequence $\{c_n\}_n$ (we will say that $\{d_n\}_n$ *comes from* $\{c_n\}_n$). As a consequence, we will know all $DD^m(p)$ sequence if we know all $DD^m(1)$ sequences. This is provided in dimension 3 by the following theorem:

Theorem 2.6 ([7]). *The complete list of $DD^3(1)$ sequences is given below*:

(A) $c_A(n) = a_1\mathrm{reg}_1(n) + a_2\mathrm{reg}_2(n)$,
(B) $c_B(n) = \mathrm{reg}_1(n) + a_d\mathrm{reg}_d(n)$,
(C) $c_C(n) = -\mathrm{reg}_1(n) + a_d\mathrm{reg}_d(n)$,
(D) $c_D(n) = a_d\mathrm{reg}_d(n)$,
(E) $c_E(n) = \mathrm{reg}_1(n) - \mathrm{reg}_2(n) + a_d\mathrm{reg}_d(n)$,
(F) $c_F(n) = \mathrm{reg}_1(n) + a_d\mathrm{reg}_d(n) + a_{2d}\mathrm{reg}_{2d}(n)$, *where d is odd*,
(G) $c_G(n) = \mathrm{reg}_1(n) - \mathrm{reg}_2(n) + a_d\mathrm{reg}_d(n) + a_{2d}\mathrm{reg}_{2d}(n)$, *where d is odd.*

In all cases $d \geq 3$ and $a_i \in \mathbb{Z}$.

In fact, in dimension 3, to find $D_r^3[f]$ we only need to know some special $DD^3(2)$ sequences, in addition to $DD^3(1)$ sequences (see Lemma 2.7).

Let us list three $DD^3(2)$ sequences which come from $DD^3(1)$ sequences of the form (E), (F) and (G):

(E′) $c_{E'}(n) = \mathrm{reg}_2(n) - \mathrm{reg}_4(n) + a_{2d}\mathrm{reg}_{2d}(n)$, where $d \geq 3$,
(F′) $c_{F'}(n) = \mathrm{reg}_2(n) + a_{2d}\mathrm{reg}_{2d}(n) + a_{4d}\mathrm{reg}_{4d}(n)$, where $d \geq 3$ is odd,
(G′) $c_{G'}(n) = \mathrm{reg}_2(n) - \mathrm{reg}_4(n) + a_{2d}\mathrm{reg}_{2d}(n) + a_{4d}\mathrm{reg}_{4d}(n)$, where $d \geq 3$ is odd.

In all cases a_{2d} and a_{4d} are arbitrary integers.

Lemma 2.7 ([6]). *Let $f : M \to M$ be a C^1 map with* $\dim M = 3$. *Then in the definition of $D_r^3[f]$ it is enough to consider only $DD^3(1|r)$ sequences, i.e., sequences which for $n \,|\, r$ are of the forms* (A)–(G), *and $DD^3(2|r)$ sequences of the forms* (E′), (F′) *and* (G′).

Let r be a fixed natural number. The sequence of Lefschetz numbers $\{L(f^n)\}_{n=1}^\infty$ also satisfies the Dold relations, so we can write its periodic expansion:

$$L(f^n) = \sum_{k=1}^{\infty} b_k\mathrm{reg}_k(n). \tag{2.2}$$

DEFINITION 2.8. We define $B(f)$, the set of *algebraic periods* of f, as $B(f) = \{k \in \mathbb{N} : b_k \neq 0\}$, and $B_r(f)$, the set of *algebraic periods* of f *up to level* r, as $B_r(f) = \{k \in \mathbb{N} : k \mid r \text{ and } b_k \neq 0\}$.

Let us now rewrite the formula (2.2) for $n \mid r$ as

$$L(f^n) = b_1 \mathrm{reg}_1(n) + b_2 \mathrm{reg}_2(n) + b_4 \mathrm{reg}_4(n) + \sum_{k \in G} b_k \mathrm{reg}_k(n), \tag{2.3}$$

where b_1, b_2, b_4 are arbitrary integers, and

$$G = \{k \in \mathbb{N} : k \notin \{1,2,4\} \text{ and } b_k \neq 0\} = B_r(f) \setminus \{1,2,4\}.$$

Let

$$H = \{k \in G : k \text{ is odd and } b_k \neq 0,\ b_{2k} \neq 0\}.$$

In the next section we will use the following result proved in [6].

THEOREM 2.9. *If r is odd, then*

$$(*) \qquad D_r^3[f] = \begin{cases} \#G & \text{if } |L(f)| \leq \#G, \\ \#G + 1 & \text{otherwise.} \end{cases}$$

If r is even and $r > 4$, then

$$(**) \qquad D_r^3[f] \in [\#G - \#H, \#G - \#H + 2].$$

3. Algebraic periods. In order to calculate $D_r^3[f]$ we need to know the periodic expansion of the Lefschetz numbers. Thus, we need to know exactly the set of algebraic periods of f (cf. Definition 2.8). We will base on the description of the algebraic periods for self-maps of S^3 which is given in [12]. We use homology spaces with rational coefficients.

Let us recall that

$$H_i(S^3) = \begin{cases} \mathbb{Q} & \text{for } i = 0, 3, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f : S^3 \to S^3$. The homomorphism $f_{*3} : H_3(S^3;\mathbb{Q}) \to H_3(S^3;\mathbb{Q})$ is multiplication by a number $\beta \in \mathbb{Z}$, called the degree of f (denoted $\deg(f)$). Fix $r \in \mathbb{N}$. We consider the periodic expansion of $\{L(f^n)\}_{n=1}^{\infty}$ given by (2.2). Recall that by Definition 2.8, $b_n \neq 0$ is equivalent to $n \in B(f)$.

LEMMA 3.1 ([12, Theorem 1.2]). *Let b_n denote the nth coefficient of the periodic expansion of $\{L(f^n)\}_{n=1}^{\infty}$. Then*:

(a) $b_1 = 1 - \beta$.
(b) $b_2 = 0$ *if and only if* $\beta \in \{0, 1\}$.
(c) *If* $n > 2$, *then* $b_n = 0$ *if and only if* $\beta \in \{-1, 0, 1\}$.

Let $\zeta(r)$ denote the number of divisors of r. Assume that $\beta \notin \{-1,0,1\}$. Then, by Lemma 3.1, $B(f) = \mathbb{N}$ and the following proposition holds:

PROPOSITION 3.2. *$B_r(f)=\{n \in \mathbb{N} : n \mid r\}$, or equivalently $\#B_r(f)=\zeta(r)$.*

By Proposition 3.2 and the definition of G we obtain:

PROPOSITION 3.3.

(1) *For r odd, $G = B_r(f) \setminus \{1\}$, thus $\#G = \zeta(r) - 1$.*
(2) *For r even:*

- *if $4 \mid r$, then $G = B_r(f) \setminus \{1, 2, 4\}$, thus $\#G = \zeta(r) - 3$,*
- *if $4 \nmid r$, then $G = B_r(f) \setminus \{1, 2\}$, thus $\#G = \zeta(r) - 2$.*

LEMMA 3.4. *If r is even, then $\#H = \eta(r) - 1$, where $\eta(r)$ denotes the number of odd divisors of r.*

Proof. Observe that $\#H$ is the number of pairs $k, 2k$, where $k \mid r$, $k > 1$ is odd, in G. As each natural $2k$ is also an algebraic period, every odd $k > 1$ determines such a pair and thus an element in H. ∎

4. Minimal number of periodic points for self-maps of S^3. The exact determination of the minimal number of r-periodic points for all smooth maps homotopic to a map $f : S^3 \to S^3$ of degree β will be given in Proposition 4.1 and Theorems 4.2 and 4.7 below.

PROPOSITION 4.1. *For each self-map f of S^3 we have $L(f^n) = 1 - \beta^n$. This implies the following statements.*

- *If $\beta = 1$, then $L(f^n) = 0$ for all n, hence $D^3_r[f] = 0$.*
- *If $\beta = -1$, then $L(f^n) = 2\mathrm{reg}_1(n) - \mathrm{reg}_2(n)$, hence $D^3_r[f] = 1$ for all r, because the Lefschetz numbers of iterations form a sequence of the type* (A).
- *If $\beta = 0$, then $L(f^n) = \mathrm{reg}_1(n)$, and analogously to the previous case, $D^3_r[f] = 1$.*

In order to find $D^3_r[f]$ for f such that $|\beta| = |\deg(f)| > 1$, we will use Theorem 2.9. For r odd, Theorem 2.9 together with Proposition 3.3(1) and the fact that $L(f) = 1 - \beta$ immediately give the value of $D^3_r[f]$:

THEOREM 4.2. *For r odd and $|\beta| > 1$,*

$$D^3_r[f] = \begin{cases} \zeta(r) - 1 & \text{if } \zeta(r) \geq \beta \geq 2 - \zeta(r), \\ \zeta(r) & \text{otherwise.} \end{cases}$$

4.1. *The case of r even.* Let

$$L(f^n) = b_1\mathrm{reg}_1(n) + b_2\mathrm{reg}_2(n) + b_4\mathrm{reg}_4(n) + \sum_{k \in G} b_k \mathrm{reg}_k(n) = \sum_i c_i(n) \tag{4.1}$$

be a minimal decomposition of Lefschetz numbers, where each c_i is a $DD^3(p_i|r)$ sequence and $\sum_i p_i = D^3_r[f]$ (cf. Definition 2.4). Let $\mathcal{A}$ be the set consisting of the sequences c_i.

Note that, by Lemma 2.7, each c_i has one of the forms (A)–(G), (E′)–(G′), which implies that $p_i \le 2$. Recall that by Theorem 2.9,

$$D_r^3[f] \in [\#G - \#H, \#G - \#H + 2]. \tag{4.2}$$

DEFINITION 4.3. We will say that a sequence ψ of one of the types (A)–(G), (E′)–(G′) *reduces* a sequence $b_k \mathrm{reg}_k$ in the periodic expansion of Lefschetz numbers (4.1) if $k \in B_r(f)$ and $a_k = b_k$, i.e. $b_k \mathrm{reg}_k$ appears in the periodic expansion of ψ.

For a self-map f of S^3, the following lemma holds.

LEMMA 4.4. *If $|\beta| = |\deg(f)| > 1$, $4 \,|\, r$ and $r > 4$ then each set $\mathcal{A}$ of sequences realizing $D_r^3[f]$ contains a sequence of one of the types* (B)–(E) *with the term $a_4 \mathrm{reg}_4$ ($a_4 \ne 0$).*

Proof. Suppose that in $\mathcal{A}$ there is no sequence of any of the types (B)–(E) with $a_4 \mathrm{reg}_4$, where $a_4 \ne 0$. Then $b_4 = (\beta^2 - \beta^4)/4 \le -3$ (as $|\beta| > 1$) implies the existence in $\mathcal{A}$ of at least three sequences:

$$\gamma_1, \gamma_2, \gamma_3 \tag{4.3}$$

of the types (G′) or (E′), since only these give a negative contribution to b_4. We will show that this leads to a contradiction with the minimality of $\mathcal{A}$.

For odd d ($d > 1, 4d \,|\, r$) let us consider the following triple which appears in the formula (4.1):

$$b_d \mathrm{reg}_d(n) + b_{2d} \mathrm{reg}_{2d}(n) + b_{4d} \mathrm{reg}_{4d}(n). \tag{4.4}$$

By Lemma 3.1 each coefficient in (4.4) is non-zero. A (G′) sequence may be used in $\mathcal{A}$ only if it reduces part of such a triple, namely its last two terms. Then we have to use one (B)–(E) sequence to reduce $b_d \mathrm{reg}_d$, which makes the contribution of the triple (4.4) to $D_r^3[f]$ equal to $2+1=3$. On the other hand, we may use (F) or (G) to reduce the first two terms and one (B)–(E) sequence to reduce the last. As a result we get the smaller contribution $1+1=2$.

Suppose that among γ_1, γ_2, γ_3 in (4.3) there are three (G′)'s. Their contribution to $D_r^3[f]$ is 9. On the other hand, independently of the values of b_1, b_2, b_4, there is a smaller realization: we may reduce these three triples by six other sequences in the way indicated above and $b_1 \mathrm{reg}_1(n) + b_2 \mathrm{reg}_2(n) + b_4 \mathrm{reg}_4(n)$ by one (A) sequence and one (B)–(E) sequence. This gives eight sequences in total, contradicting the minimality.

Similarly, we get a contradiction when we assume that at least one of the three sequences γ_1, γ_2, γ_3 is of the type (E′) (we replace the expression $c_{E'} = \mathrm{reg}_2(n) - \mathrm{reg}_4(n) + a_{2d} \mathrm{reg}_{2d}(n)$ which counts with multiplicity 2 with $c_D = a_{2d} \mathrm{reg}_{2d}(n)$ of multiplicity 1 and repeat the same reasoning as in the case of three (G) sequences). ∎

COROLLARY 4.5. *Under the assumptions of Lemma* 4.4,

$$D_r^3[f] \in \{\#G - \#H + 1, \#G - \#H + 2\}.$$

Proof. By Theorem 2.9 it is enough to exclude the possibility $D_r^3[f] = \#G - \#H$. Assume that a set $\mathcal{A}$ of sequences realizes $D_r^3[f] = \#G - \#H$. By Lemma 4.4 there is a sequence of one of the types (B)–(E) in $\mathcal{A}$ with the term $a_4\mathrm{reg}_4$. The remaining $\#G - \#H - 1$ sequences (counting multiplicity) must reduce $b_k\mathrm{reg}_k$ for $k \in G$. Only (F) and (G) sequences reduce two $b_k\mathrm{reg}_k$, $k \in G$, and we may use them in such a way $\#H$ times ((F′), (G′) also reduce two $b_k\mathrm{reg}_k$'s but they are counted twice). Now the remaining $\#G - 2\#H$ $b_k\mathrm{reg}_k$'s must be reduced by $\#G - 2\#H - 1$ sequences of types (A)–(E), (E′)–(G′) (counting multiplicity). This is impossible, since each of (A)–(E) reduces at most one $b_k\mathrm{reg}_k$ with $k \in G$, and each (E′)–(G′) is counted twice and reduces at most two $b_k\mathrm{reg}_k$ with $k \in G$. ∎

LEMMA 4.6. *Suppose that* $|\deg(f)| > 1$ *and* $D_r^3[f] = \#G - \#H + 1$. *Then there is a minimal set realizing* $D_r^3[f]$ *containing only* (B)–(G) *sequences.*

Moreover, we may assume that this set contains one sequence of the type (B)–(E) *with* $a_d\mathrm{reg}_d$ *such that* $d = 4$ *and* $\#G - 2\#H$ *sequences of the types* (B)–(E) *with* $d \neq 4$ *and* $\#H$ *sequences of the types* (F)–(G).

Proof. Let us fix a minimal set $\mathcal{A}$. By Lemma 4.4 there is a sequence $c_{(4)}$ in $\mathcal{A}$ of one of the types (B)–(E) with $a_4\mathrm{reg}_4$. We then take $a_4 = b_4$. The remaining sequences realize $b_k\mathrm{reg}_k$ for $k \in G$, hence each of them must realize at least one $b_k\mathrm{reg}_k$ so none of them is of the type (A).

Let us assume that $\mathcal{A}$ contains a sequence of the type (G′),

$$c_{G'} = \mathrm{reg}_2 - \mathrm{reg}_4 + a_{2d}\mathrm{reg}_{2d} + a_{4d}\mathrm{reg}_{4d}.$$

We will show how to change $\mathcal{A}$ into another minimal system $\mathcal{A}'$ in which $c_{G'}$ does not appear. Namely, we change three sequences in $\mathcal{A}$:

- Instead of $c_{G'}$ we take two sequences of the type (D): $a_{2d}\mathrm{reg}_{2d}$, $a_{4d}\mathrm{reg}_{4d}$.
- Instead of $c_{(4)}$ with $a_4\mathrm{reg}_4$ we take $c'_{(4)}$ with $(a_4 - 1)\mathrm{reg}_4$.
- Let us notice that $b_2 = (\beta - \beta^2)/2 < 0$ implies the existence of a sequence ψ of the type (E) or (G), since only these have a negative contribution to b_2. Instead of such a sequence we take $\psi'(n) = \mathrm{reg}_2(n) + \psi(n)$, which is an expression of the type (B) or (F) respectively.

This gives a system $\mathcal{A}'$ with the same sum of multiplicities as $\mathcal{A}$ (so also minimal) with one fewer expression of the type (G′). Similarly we may remove expressions of the type (E′) and (F′). ∎

THEOREM 4.7. *Let* r *be even and* $|\beta| > 1$. *Then* $D_r^3[f] \in \{\zeta(r) - \eta(r) - 1, \zeta(r) - \eta(r)\}$. *What is more,* $D_r^3[f] = \zeta(r) - \eta(r) - 1$ *if and only if one of* *the following two conjunctions holds*:

$$(4.5)\qquad \begin{array}{lll} \text{(i)} & \zeta(r)-\eta(r)\geq\beta^2 & \textit{and}\quad (\beta^2-\beta)/2-\eta(r)\geq 0,\\ \text{(ii)} & \zeta(r)-3\eta(r)\geq\beta-2 & \textit{and}\quad (\beta^2-\beta)/2-\eta(r)\leq -1. \end{array}$$

Before giving the proof of the above theorem, we illustrate it by the following example:

EXAMPLE 4.8. Let $f: S^3\to S^3$ have degree $\beta=2$ and let $r=12$. The Lefschetz numbers $L(f^n)$ are equal to $1-2^n$. We represent this sequence (for $n\,|\,12$) in the form of a periodic expansion. We get $b_1=1-\beta=-1$, $b_2=(\beta-\beta^2)/2=-1$, $b_4=(\beta^2-\beta^4)/4=-3$, thus

$$(4.6)\qquad \begin{aligned} L(f^n) = &-\mathrm{reg}_1(n)-\mathrm{reg}_2(n)-3\mathrm{reg}_4(n)\\ &+b_3\mathrm{reg}_3(n)+b_6\mathrm{reg}_6(n)+b_{12}\mathrm{reg}_{12}(n). \end{aligned}$$

We have $\zeta(12)=6$ and $\eta(12)=2$. Because $\zeta(r)-3\eta(r)=\beta-2=0$ and $(\beta^2-\beta)/2-\eta(r)=-1$, we see that the condition (ii) of (4.5) is satisfied and thus $D^3_r[f]=\zeta(r)-\eta(r)-1=3$. Indeed, we may take the following three sequences, which together realize $\{L(f^n)\}_{n|12}$:

$$\begin{aligned} c_G(n)&=\mathrm{reg}_1(n)-\mathrm{reg}_2(n)+b_3\mathrm{reg}_3(n)+b_6\mathrm{reg}_6(n),\\ c_{C_1}(n)&=-\mathrm{reg}_1(n)-3\mathrm{reg}_4(n),\\ c_{C_2}(n)&=-\mathrm{reg}_1(n)+b_{12}\mathrm{reg}_{12}(n). \end{aligned}$$

Proof of Theorem 4.7. First notice that if $|\beta|>1$, then $D^3_2[f]=1$ and $D^3_4[f]=2$, thus we verify directly that for $r=2$ and $r=4$ the assertion holds. For $r>4$ we divide the proof into two cases.

PART (I): $4\,|\,r$. By Corollary 4.5, $D^3_r[f]\in\{\#G-\#H+1,\#G-\#H+2\}$, which by Proposition 3.3(2) for $4\,|\,r$ and Lemma 3.4 is equal to $\{\zeta(r)-\eta(r)-1, \zeta(r)-\eta(r)\}$. We will now find conditions equivalent to the statement that $D^3_r[f]=\#G-\#H+1=\zeta(r)-\eta(r)-1$.

By Lemma 4.6 we may assume that there is a minimal set $\mathcal{A}$ of sequences realizing $D^3_r[f]$ with $\#H$ sequences of the types (F) or (G) and $\#G-2\#H$ sequences of the types (B)–(F), which reduce $b_k\mathrm{reg}_k$ for $k\neq 4$, and one extra expression $c_{(4)}$ of the type (B), (C), (D) or (E), which reduces $b_4\mathrm{reg}_4$. Thus there are $\#G-\#H$ sequences plus one extra in $\mathcal{A}$.

Let us denote the contribution of the single sequence $c_{(4)}$ to the first two terms of the formula (4.1) by $\epsilon_1\mathrm{reg}_1(n)+\epsilon_2\mathrm{reg}_2(n)$, where $(\epsilon_1,\epsilon_2)\in\{(1,0),(-1,0),(0,0),(1,-1)\}$.

Let m_X, where $X\in\{B,C,D,E,F,G\}$, denote the number of sequences (not counting $c_{(4)}$) of the given type in the minimal realization $\mathcal{A}$. Then $D^3_r[f]=\#G-\#H+1$ if and only if there are integers

$$(\epsilon_1,\epsilon_2)\in\{(1,0),(-1,0),(0,0),(1,-1)\}$$

such that the following system of equations has an integer solution in nonnegative unknowns m_X:

$$(4.7)\qquad \begin{aligned} m_B + m_C + m_D + m_E + m_F + m_G &= \#G - \#H,\\ m_B - m_C + m_E + m_F + m_G &= b_1 - \epsilon_1,\\ -m_E - m_G &= b_2 - \epsilon_2,\\ m_F + m_G &= \#H, \end{aligned}$$

where the first and last equations describe the number of sequences (not counting $c_{(4)}$), and the second and third give their contribution to the first two terms of the periodic expansion.

The above system is equivalent to

$$(4.8)\qquad \begin{aligned} m_D + m_B + m_F + m_G &= \#G - \#H - m_C - m_E,\\ m_B + m_F + m_G &= b_1 + m_C - m_E - \epsilon_1,\\ m_F + m_G &= \#H,\\ m_G &= -b_2 - m_E + \epsilon_2. \end{aligned}$$

Notice that:

- For any fixed m_C, m_E, (4.8) is a Cramer system with determinant $+1$, thus m_D, m_B, m_F, m_G are uniquely determined.
- If m_C, m_E are integers, then the other unknowns must be integers.
- For any fixed values of $m_C, m_E \geq 0$ the solutions of (4.8) are nonnegative if and only if the following system of inequalities holds:

$$(4.9)\qquad \begin{aligned} 0 \leq -b_2 - m_E + \epsilon_2 \leq \#H \leq b_1 + m_C - m_E - \epsilon_1\\ \leq \#G - \#H - m_C - m_E. \end{aligned}$$

As a consequence, to find the solution of the system (4.7), it is enough to solve (4.9) with integer m_C, m_E such that $m_C, m_E \geq 0$. We rewrite (4.9) as a system of four inequalities:

$$(4.10)\qquad m_E \leq -b_2 + \epsilon_2,$$

$$(4.11)\qquad m_E \geq -b_2 + \epsilon_2 - \#H,$$

$$(4.12)\qquad m_C - m_E \geq \#H - b_1 + \epsilon_1,$$

$$(4.13)\qquad m_C \leq \frac{1}{2}(-b_1 + \#G - \#H + \epsilon_1).$$

The problem reduces to finding a point $(m_C, m_E) \in \mathbb{Z}^2$ with $m_C, m_E \geq 0$ for which the inequalities (4.10)–(4.13) are satisfied.

We substitute the values of $\#G$ and $\#H$ using Lemma 3.3(2) and Lemma 3.4 and the values of b_1 and b_2 calculated directly:

$$\#G = \zeta(r) - 3, \quad \#H = \eta(r) - 1, \quad b_1 = 1 - \beta, \quad b_2 = \frac{1}{2}(\beta - \beta^2).$$

Then the inequalities (4.10)–(4.13) can be rewritten (in a different order) as

$$m_C \le \frac{1}{2}(\zeta(r) - \eta(r) + \beta - 3 + \epsilon_1), \tag{4.14}$$

$$m_E \ge \frac{1}{2}(\beta^2 - \beta) - \eta(r) + 1 + \epsilon_2, \tag{4.15}$$

$$m_E \le \frac{1}{2}(\beta^2 - \beta) + \epsilon_2, \tag{4.16}$$

$$m_E \le m_C - \eta(r) - \beta + 2 - \epsilon_1. \tag{4.17}$$

Now the problem transforms into the following: for which $r \in \mathbb{N}$ and $\beta \in \mathbb{Z}$ ($4 \mid r$, $|\beta| \ge 2$) can one choose $(\epsilon_1, \epsilon_2) \in \{(0,0), (1,0), (-1,0), (1,-1)\}$ so that the inequalities (4.14)–(4.17) have a nonnegative integer solution (m_C, m_E)? To simplify the notations we write

$$\begin{aligned} b &= \frac{1}{2}(\zeta(r) - \eta(r) + \beta - 3 + \epsilon_1), \\ c &= \frac{1}{2}(\beta^2 - \beta) - \eta(r) + 1 + \epsilon_2, \\ d &= \frac{1}{2}(\beta^2 - \beta) + \epsilon_2, \\ e &= -\eta(r) - \beta + 2 - \epsilon_1. \end{aligned}$$

Now the system of inequalities (4.14)–(4.17) takes the form

$$m_C \le b, \quad c \le m_E, \quad m_E \le d, \quad m_E \le m_C + e. \tag{4.18}$$

LEMMA 4.9. *Let $b, c, d, e \in \mathbb{R}$ satisfy $c, e \in \mathbb{Z}$ and $c \le d$. Then the inequalities* (4.18) *have a nonnegative integer solution (m_C, m_E) if and only if $b \ge 0$, $d \ge 0$, and $\max\{c, 0\} \le [b] + e$, where $[b]$ denotes the integer part of b.*

To prove the above lemma it is enough to notice that the first three inequalities describe $(-\infty, b] \times [c, d]$ while the last defines the closed half-plane under the line $m_E = m_C + e$ in (m_C, m_E) coordinates.

Finally, by Lemma 4.9 the problem becomes: for which β and r can one choose (ϵ_1, ϵ_2) such that the following inequalities hold:

$$\zeta(r) - \eta(r) + \beta - 3 + \epsilon_1 \ge 0, \tag{4.19}$$

$$\frac{1}{2}(\beta^2 - \beta) + \epsilon_2 \ge 0, \tag{4.20}$$

$$\max\left\{\frac{1}{2}(\beta^2 - \beta) - \eta(r) + 1 + \epsilon_2, 0\right\} \le \left[\frac{1}{2}(\zeta(r) - \eta(r) + \beta - 3 + \epsilon_1)\right] - \eta(r) - \beta + 2 - \epsilon_1. \tag{4.21}$$

We notice that the inequality (4.20) always holds. In fact, $|\beta| \geq 2$ implies $\beta^2 - \beta \geq 2$ and $\frac{1}{2}(\beta^2 - \beta) + \epsilon_2 \geq 1 - 1 = 0$.

Now we study (4.19). We will consider four cases.

Case (1): $\zeta(r) - \eta(r) + \beta \leq 1$. Then the inequality (4.19) never holds, hence the system has no solution.

Case (2): $\zeta(r) - \eta(r) + \beta = 2$ and Case (3): $\zeta(r) - \eta(r) + \beta = 3$ will be discussed separately below.

Case (4): $\zeta(r) - \eta(r) + \beta \geq 4$. Then (4.19) holds for each ϵ_1.

We will consider these cases (in reverse order: starting from Case 4 to Case 1) as assumptions in the next subcases. We will look for solutions of the inequality (4.21).

Case (4). We assume that $\zeta(r) - \eta(r) + \beta \geq 4$ (the inequality (4.19) holds for each ϵ_1). To get rid of the maximum and the integer part in (4.21) we consider several subcases.

Subcase (4.$\geq$): $\frac{1}{2}(\beta^2 - \beta) - \eta(r) \geq 0$. Now $c = \frac{1}{2}(\beta^2 - \beta) - \eta(r) + 1 + \epsilon_2 \geq 0$, since $\epsilon_2 \in \{-1, 0\}$. The inequality (4.21) takes the form

$$\frac{1}{2}(\beta^2 - \beta) - \eta(r) + 1 + \epsilon_2 \leq \left[\frac{1}{2}(\zeta(r) - \eta(r) + \beta - 3 + \epsilon_1)\right] - \eta(r) - \beta + 2 - \epsilon_1,$$

or

$$\frac{1}{2}(\beta^2 + \beta) - 1 + \epsilon_1 + \epsilon_2 \leq \left[\frac{1}{2}(\zeta(r) - \eta(r) + \beta - 3 + \epsilon_1)\right].$$

If the above inequality holds for some $(\epsilon_1, \epsilon_2) \in \{(0,0), (1,0), (-1,0), (1,-1)\}$ then it also holds for $(-1, 0)$, thus it is enough to solve

$$\frac{1}{2}(\beta^2 + \beta) - 2 \leq \left[\frac{1}{2}(\zeta(r) - \eta(r) + \beta - 4)\right],$$

hence

$$\frac{1}{2}(\beta^2 + \beta) \leq \left[\frac{1}{2}(\zeta(r) - \eta(r) + \beta)\right].$$

Subsubcase (4.$\geq$.even): *$\zeta(r) - \eta(r) + \beta$ is even.* Now we may omit the integer part:

$$\frac{1}{2}(\beta^2 + \beta) \leq \frac{1}{2}(\zeta(r) - \eta(r) + \beta),$$

which implies

$$\beta^2 \leq \zeta(r) - \eta(r).$$

Subsubcase (4.$\geq$.odd): *$\zeta(r) - \eta(r) + \beta$ is odd.* Now we get

$$\frac{1}{2}(\beta^2 + \beta) \leq \frac{1}{2}(\zeta(r) - \eta(r) + \beta) - \frac{1}{2},$$

which implies

$$\beta^2 \leq \zeta(r) - \eta(r) - 1.$$

Moreover, we notice that in this subsubcase the above inequality is equivalent to

$$\beta^2 \le \zeta(r) - \eta(r).$$

In fact, by the parity assumption in this subsubcase we have $\zeta(r) - \eta(r) + \beta \equiv 1 \pmod 2$, and thus $\zeta(r) - \eta(r) - \beta^2 \equiv 1 \pmod 2$. As a consequence, the equality $\beta^2 = \zeta(r) - \eta(r)$ cannot hold.

Thus the assumptions of Case 4, Subcase (4.$\ge$) and the above inequality give the following system of conditions:

$$\zeta(r) - \eta(r) + \beta \ge 4, \quad \frac{\beta^2 - \beta}{2} - \eta(r) \ge 0, \quad \zeta(r) - \eta(r) \ge \beta^2. \tag{4.22}$$

Subcase (4.<): $\frac{1}{2}(\beta^2 - \beta) - \eta(r) < 0$. Now $c = \frac{1}{2}(\beta^2 - \beta) - \eta(r) + 1 + \epsilon_2 \le 0$ (for any ϵ_2), hence the inequality (4.21) becomes

$$0 \le \left[\frac{1}{2}\left(\zeta(r) - \eta(r) + \beta - 3 + \epsilon_1\right)\right] - \eta(r) - \beta + 2 - \epsilon_1.$$

Let us notice that if the above inequality holds for some $\epsilon_1 \in \{-1, 0, 1\}$ then it also holds for $\epsilon_1 = -1$, hence we get

$$\eta(r) + \beta - 3 \le \left[\frac{1}{2}\left(\zeta(r) - \eta(r) + \beta\right) - 2\right].$$

Subsubcase (4.<.even): $\zeta(r) - \eta(r) + \beta$ is even. Now $\eta(r) + \beta - 3 \le \frac{1}{2}\left(\zeta(r) - \eta(r) + \beta\right) - 2$ or

$$\zeta(r) - 3\eta(r) \ge \beta - 2.$$

Subsubcase (4.<.odd): $\zeta(r) - \eta(r) + \beta$ *is odd.* Now $\eta(r) + \beta - 3 \le \frac{1}{2}\left(\zeta(r) - \eta(r) + \beta\right) - 2 - \frac{1}{2}$, hence we get

$$\zeta(r) - 3\eta(r) \ge \beta - 1.$$

Moreover, in this subsubcase the above inequality is equivalent to

$$\zeta(r) - 3\eta(r) \ge \beta - 2.$$

In fact, the equality $\zeta(r) - 3\eta(r) = \beta - 1$ cannot hold because of the parity assumptions in (4.<.odd).

Thus, Case 4, Subcase (4.<) and the above inequality give the following system of conditions:

$$\zeta(r) - \eta(r) + \beta \ge 4, \quad \frac{\beta^2 - \beta}{2} - \eta(r) \le -1, \quad \zeta(r) - 3\eta(r) \ge \beta - 2. \tag{4.23}$$

Case (3): $\zeta(r) - \eta(r) + \beta = 3$. The assumption of Case 3 implies that

$$b = \frac{1}{2}\left(\zeta(r) - \eta(r) + \beta - 3 + \epsilon_1\right) = \frac{1}{2}\epsilon_1.$$

Now $b \geq 0$ for $\epsilon_1 = 0$ or $\epsilon_1 = 1$.

Subcase (3.0): $\epsilon_1 = 0$. Since $\epsilon_1 = 0$ implies $\epsilon_2 = 0$, the inequality (4.21) takes the form

$$\max\left\{\frac{1}{2}(\beta^2 - \beta) - \eta(r) + 1, 0\right\} \leq -\eta(r) - \beta + 2.$$

Subsubcase (3.0.≥): $\frac{1}{2}(\beta^2 - \beta) - \eta(r) + 1 \geq 0$. Now we get

$$\frac{1}{2}(\beta^2 - \beta) - \eta(r) + 1 \leq -\eta(r) - \beta + 2,$$

which is equivalent to

$$\beta^2 + \beta \leq 2$$

and the last holds only for $\beta = -2$. Then (3.0.≥) and the assumptions of Case (3) take the form $4 \geq \eta(r)$ and $\zeta(r) = \eta(r) + 5$ respectively. Since $\eta(r) \mid \zeta(r)$, we obtain $\eta(r) = 1$ and $\zeta(r) = 6$. This implies $r = 2^5$.

Subsubcase (3.0.<): $\frac{1}{2}(\beta^2 - \beta) - \eta(r) + 1 < 0$. We get

$$0 \leq -\eta(r) - \beta + 2.$$

In other words, $\eta(r) \leq -\beta + 2$. On the other hand, the assumption (3.0.<) gives $\beta^2 - \beta < 2\eta(r) - 2$. The above inequalities imply $\beta^2 + \beta - 2 < 0$, which is never true for $|\beta| > 1$.

Subcase (3.1): $\epsilon_1 = 1$. The inequality (4.21) takes the form

$$\max\left\{\frac{1}{2}(\beta^2 - \beta) - \eta(r) + 1 + \epsilon_2, 0\right\} \leq -\eta(r) - \beta + 1.$$

Since ϵ_2 may be 0 or -1, here we may put $\epsilon_2 = -1$, which implies

$$\max\left\{\frac{1}{2}(\beta^2 - \beta) - \eta(r), 0\right\} \leq -\eta(r) - \beta + 1.$$

Subsubcase (3.1.≥): $\frac{1}{2}(\beta^2 - \beta) - \eta(r) \geq 0$. Now

$$\frac{1}{2}(\beta^2 - \beta) - \eta(r) \leq -\eta(r) - \beta + 1$$

implies $\beta^2 + \beta - 2 \leq 0$, hence $\beta = -2$. Then the assumption of Case 3, $\zeta(r) - \eta(r) + \beta = 3$, gives $\zeta(r) = \eta(r) + 5$. On the other hand, the condition (3.1.≥) takes the form $3 \geq \eta(r)$. Again $\eta(r) \mid \zeta(r)$ implies $\eta(r) = 1$, $\zeta(r) = 6$ and thus $r = 2^5$.

Subsubcase (3.1.<): $\frac{1}{2}(\beta^2 - \beta) - \eta(r) < 0$. The inequality (4.21) takes the form

$$0 \leq -\eta(r) - \beta + 1,$$

hence implies $\eta(r) \leq -\beta+1$. We combine this inequality with the condition (3.1.<) to get

$$\frac{1}{2}(\beta^2-\beta) < \eta(r) \leq -\beta+1.$$

This implies the inequality

$$\frac{1}{2}(\beta^2-\beta) < -\beta+1,$$

which is not valid for any $|\beta|>1$.

Case (2): $\zeta(r)-\eta(r)+\beta=2$. Here we get

$$b = \frac{1}{2}(\zeta(r)-\eta(r)+\beta-3+\epsilon_1) = \frac{1}{2}(\epsilon_1-1).$$

Notice that $b \geq 0$ only for $\epsilon_1=+1$, and then $b=0$. This shows that (4.21) has the form

$$\max\left\{\frac{1}{2}(\beta^2-\beta)-\eta(r)+1+\epsilon_2, 0\right\} \leq -\eta(r)-\beta+1.$$

Since ϵ_2 may be 0 or -1, it is enough to consider $\epsilon_2=-1$. We then get

$$\max\left\{\frac{1}{2}(\beta^2-\beta)-\eta(r), 0\right\} \leq -\eta(r)-\beta+1.$$

Subcase (2.≥): $\frac{1}{2}(\beta^2-\beta)-\eta(r) \geq 0$. The inequality (4.21) takes the form

$$\frac{1}{2}(\beta^2-\beta)-\eta(r) \leq -\eta(r)-\beta+1,$$

which implies $\beta^2+\beta \leq 2$, hence $\beta=-2$. Moreover, (2.≥) and the condition of Case (2) become $3 \geq \eta(r)$ and $\zeta(r)=\eta(r)+4$ respectively. Since $\eta(r) \mid \zeta(r)$, there are two possibilities: $\eta(r)=1$ or $\eta(r)=2$. In the first case $\zeta(r)=5$ and $r=2^4$. In the second case $\zeta(r)=6$ and $r=2^2p$ where p is a prime number greater than 2. Now we get either

- $\beta=-2$ and $r=2^4$, or
- $\beta=-2$ and $r=4p$ for an odd prime p.

Subcase (2.<): $\frac{1}{2}(\beta^2-\beta)-\eta(r) < 0$. Now the inequality (4.21) takes the form $0 \leq -\eta(r)-\beta+1$, which implies $\eta(r) \leq -\beta+1$. On the other hand, the condition (2.<) gives $\beta^2-\beta < 2\eta(r)$. This implies $\beta^2+\beta<2$, which has no solution for $|\beta|>1$.

Summing up the previous considerations, we conclude that $D_r^3[f] = \zeta(r)-\eta(r)-1$ if and only if one of the following five conditions holds (cf.

(4.22), (4.23), Cases (3) and (2)):

(1) $\zeta(r)-\eta(r)+\beta\geq 4,\quad \zeta(r)-\eta(r)\geq\beta^2,\qquad \dfrac{\beta^2-\beta}{2}-\eta(r)\geq 0,$

(2) $\zeta(r)-\eta(r)+\beta\geq 4,\quad \zeta(r)-3\eta(r)\geq\beta-2,\quad \dfrac{\beta^2-\beta}{2}-\eta(r)\leq -1,$

(3) $\beta=-2$ and $r=16$,

(4) $\beta=-2$ and $r=32$,

(5) $\beta=-2$ and $r=4p$ for an odd prime p.

We now show that

$$[(\mathrm{i})\text{ or }(\mathrm{ii})]\ \Leftrightarrow\ [(1)\text{ or }(2)\text{ or }(3)\text{ or }(4)\text{ or }(5)],$$

where (i) and (ii) are the conditions of Theorem 4.5, which we recall below:

$$\begin{array}{rlll}(\mathrm{i}) & \zeta(r)-\eta(r)\geq\beta^2 & \text{and} & (\beta^2-\beta)/2-\eta(r)\geq 0,\\ (\mathrm{ii}) & \zeta(r)-3\eta(r)\geq\beta-2 & \text{and} & (\beta^2-\beta)/2-\eta(r)\leq -1.\end{array}\tag{4.24}$$

First we prove that (ii)⇔(2).

⇐ is trivial. To prove ⇒ it suffice to show that the second and third inequalities in (2) imply the first. The second and the third inequality give respectively $\zeta(r)-\eta(r)\geq 2\eta(r)+\beta-2$ and $\eta(r)\geq(\beta^2-\beta)/2+1$. This implies that $\zeta(r)-\eta(r)\geq\beta^2$, and as $\beta^2\geq 4-\beta$ for $\beta\neq-2$, we see that the first inequality results from the second and the third for $\beta\neq-2$. If $\beta=-2$, we get $\eta(r)\geq 4$ and we should check whether $\zeta(r)-\eta(r)\geq 2\eta(r)-4$ implies $\zeta(r)-\eta(r)\geq 6$. This is obviously satisfied if $\eta(r)>4$, and for $\eta(r)=4$ we find (because $4\mid r$) that $\zeta(r)$ must be greater than 10, so $\zeta(r)-\eta(r)\geq 6$ is also satisfied in this case.

It remains to show that

$$(\mathrm{i})\ \Leftrightarrow\ [(1)\text{ or }(3)\text{ or }(4)\text{ or }(5)].$$

⇐ (1) implies (i) in a trivial way. Then we check case by case that each of (3), (4), (5) implies (i).

⇒ We show that (i) and the negation of (1) imply the alternative [(3) or (4) or (5)].

Recall that (i) means

$$\zeta(r)-\eta(r)\geq\beta^2\quad\text{and}\quad\frac{\beta^2-\beta}{2}-\eta(r)\geq 0.$$

Now the negation of (1) means in particular that

$$\zeta(r)-\eta(r)+\beta<4.$$

The above two inequalities imply $\beta^2\leq\zeta(r)-\eta(r)<4-\beta$, hence $\beta^2+\beta-4<0$, which holds only for $\beta=-2$. This in turn implies that the above inequalities take the forms

$$\zeta(r) - \eta(r) \geq 4, \quad \eta(r) \leq 3, \quad \zeta(r) - \eta(r) < 6.$$

Now we get ($\zeta(r) - \eta(r) = 4$ or 5) and $\eta(r) \leq 3$.

If $\zeta(r) - \eta(r) = 5$ then $\eta(r) = 1$, hence $r = 2^5 = 32$, so we get (4).

If $\zeta(r) - \eta(r) = 4$ then $\eta(r) = 1$ or 2. For $\eta(r) = 1$ we get $r = 2^4 = 16$, so we obtain (3). Finally, for $\eta(r) = 2$, we obtain $r = 4p$, where p is an odd prime, which gives (5). This ends the proof of Part I.

PART (II): $2 \mid r$ but $4 \nmid r$. Since $b_4 = 0$, we see that $D^3_r[f] \in \{\#G - \#H, \#G - \#H + 1\} = \{\zeta(r) - \eta(r) - 1, \zeta(r) - \eta(r)\}$, where the last equality results from Lemma 3.4 and Proposition 3.3(2) for $4 \nmid r$.

Searching for conditions equivalent to $D^3_r[f] = \#G - \#H$, we repeat the same reasoning as in the case $4 \mid r$, with the only difference being that we do not need an additional sequence to reduce $b_4 \mathrm{reg}_4$, so we have to find nonnegative integer solutions of (4.7) without parameters ϵ_1, ϵ_2.

Then similar computations give conditions (i) and (ii) of (4.5), which completes the proof of the theorem.

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