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# Preserving P-points in definable forcing

by

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**Abstract.** I isolate a simple condition that is equivalent to preservation of P-points in definable proper forcing.

**1. Introduction.** Blass and Shelah [3], [2, Section 6.2] introduced the forcing property of preserving P-points. Here, a *P-point* is an ultrafilter $U$ on $\omega$ such that every countable subset of it has a pseudo-intersection in it: $\forall a_n \in U : n \in \omega\ \exists b \in U\ |b \setminus a_n| < \aleph_0$. While the existence of P-points is unprovable in ZFC, they are plentiful under ZFC+CH. A forcing $P$ *preserves* an ultrafilter $U$ if every set $a \subset \omega$ in the extension either contains, or is disjoint from, a ground model element of the ultrafilter $U$; otherwise, $P$ *destroys* $U$. The forcing $P$ preserves P-points if it preserves all ultrafilters that happen to be P-points.

Several circumstances make this property a natural and useful tool. Every forcing adding a real number destroys some ultrafilter [2, Theorem 6.2.2]; if the forcing adds an unbounded real, then it destroys all non-P-point ultrafilters. A P-point, if preserved by a proper forcing, will again generate a P-point in the extension. Cohen and Solovay forcings both destroy all nonprincipal ultrafilters, and so preservation of P-points excludes the introduction of Cohen or random reals into the extension. Finally, preservation of P-points is itself preserved under the countable support iteration of proper forcing [3], [2, Theorem 6.2.6].

In the context of the theory of definable proper forcing [17], the preservation of P-points has two disadvantages: it trivializes when P-points do not exist (while the important properties of a definable forcing are typically independent of circumstances of this kind), and it refers to undefinable objects such as ultrafilters. As a result, it is not clear how difficult its verification

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might be, and what tools should be used for that verification. In this paper, I will resolve this situation by isolating a simple condition that is equivalent to the preservation of P-points for definable proper forcing in the theory ZFC+LC+CH. In order to state the theorem, I will need the following definitions.

DEFINITION 1.1. A forcing $P$ *does not add splitting reals* if for every set $a \subset \omega$ in the extension there is an infinite ground model subset of $\omega$ which is either included in $a$ or disjoint from it.

This is a familiar property. Some forcings do not add splitting reals (Sacks forcing, the fat tree forcing [17, Section 4.4.3], the $E_0$ forcing [16], or Miller forcing [11], to include a diversity of examples), others do (most notably, Cohen and random forcing, as well as all the Maharam algebras [1], and with them all definable c.c.c. forcings adding a real). Clearly, a forcing adding a splitting real preserves no nonprincipal ultrafilters. I do not think that on its own not adding splitting reals is preserved under even two-step iteration. Its conjunction with the bounding property is preserved under the countable support iteration of definable forcings by [17, Corollary 6.3.8], and it is equivalent to the preservation of Ramsey ultrafilters by [17, Section 3.4].

DEFINITION 1.2. A forcing $P$ has the *weak Laver property* if for every function $g \in \omega^\omega$ in the extension dominated by some ground model function there is a ground model infinite set $a \subset \omega$ and a ground model function $h : a \to \mathcal{P}(\omega)$ such that for every number $n \in a$, both $|h(n)| < 2^n$ and $g(n) \in h(n)$ hold.

The weak Laver property is less well-known, and on the surface it appears to have nothing to do with preservation of any ultrafilters. It is a weakening of the more familiar Laver [2, Definition 6.3.27] or Sacks properties. Notably, it occurs in [2, Section 7.4.D] in parallel to the proof that the Blass–Shelah forcing preserves P-points. Some more complicated variants of it, iterable in the category of arbitrary proper forcings, appeared in [14, Section 7], to guarantee the preservation of certain more complicated properties of filters on $\omega$.

In order to precisely quantify the definability properties of the forcings involved, recall

DEFINITION 1.3. A $\sigma$-ideal $I$ on a Polish space $X$ is *universally Baire* if for every universally Baire set $A \subset 2^\omega \times X$ the set $\{y \in 2^\omega : A_x \in I\}$ is universally Baire.

The class of universally Baire sets first appeared in [4]: these are the sets whose continuous preimages in Hausdorff spaces have the property of Baire. Suitable large cardinal assumptions imply that suitably definable subsets of Polish spaces are universally Baire [12], [8, Section 3.3], and analytic sets are universally Baire in ZFC. As [17] shows, a typical definable proper forcing adding a single real is of the form $P_I$ where $I$ is a universally Baire $\sigma$-ideal on a Polish space. The treatment of such a general class of forcings necessitates large cardinal assumptions at many ocassions. In order to prove ZFC theorems for a more restricted, but still significant, class of forcings, I will use the following definability notion considered for example by Sierpiński [7, Theorem 29.19]:

Definition 1.4. A $\sigma$-ideal $I$ on a Polish space $X$ is $\mathbf{\Pi}^1_1$ *on* $\mathbf{\Sigma}^1_1$ if for every analytic set $A \subset 2^\omega \times X$ the set $\{y \in 2^\omega : A_y \in I\}$ is coanalytic.

Most definable tree forcings are of the form $P_I$ for a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ $\sigma$-ideal $I$. Now I am ready to state the main result of the paper. On the moral level, it says that in definable proper forcing, the preservation of P-points is equivalent to the conjunction of the weak Laver property and adding no splitting reals.

Theorem 1.5. (CH) *Suppose that $P$ is a proper forcing preserving P-points. Then $P$ has the weak Laver property and adds no splitting reals.*

Theorem 1.6. *Suppose that there is a proper class of Woodin cardinals. If $I$ is a universally Baire $\sigma$-ideal on a Polish space such that the quotient forcing $P_I$ is proper*, *has the weak Laver property*, *and adds no splitting reals*, *then $P_I$ preserves P-points. If the ideal $I$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ then the large cardinal assumption is not necessary.*

The Continuum Hypothesis assumption in the former theorem is used only to ascertain the existence of many P-points. On the other hand, the definability assumption in the latter theorem is necessary:

Example 1.7. (CH) There is a proper forcing which has the Laver property, adds no splitting reals, and fails to preserve a P-point.

The theorems can be used to swiftly argue that certain forcings preserve or do not preserve P-points. For example, the paper [15] shows that countable products of forcings of the form $P_I$, where $I$ is a $\sigma$-ideal generated by a compact collection of compact sets, do not add splitting reals. These products all have the weak Laver property, their associated ideal is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ and therefore they must preserve P-points. A direct proof of this product preservation property seems to be out of reach. As another example, the forcings adding a bounded eventually different real must fail to have the weak Laver property, and so they never preserve P-points under CH. On the other hand, the Blass–Shelah forcing of [2, Section 7.4.D] adds an unbounded eventually different real and still preserves P-points.

The notation used in the paper follows the set-theoretic standard of [5]. The shorthand LC denotes the use of suitable large cardinal assumptions. If $A \subset X \times Y$ is a set and $x \in X$ is a point, then $A_x$ is the vertical section of the set $A$ corresponding to $x$.

**2. Proof of Theorem 1.5.** Suppose that the conclusion of Theorem 1.5 fails; I will argue that the assumption must fail as well. If $P$ adds a splitting real, then $P$ certainly destroys all nonprincipal ultrafilters. In the other case, the weak Laver property must fail for some function $f \in \omega^\omega$, and there is a condition $p \in P$ forcing that $\dot{g} < \check{f}$ is a counterexample. Let $U_n : n \in \omega$ be pairwise disjoint sets of the respective size $f(n)$, in some way identified with $f(n)$. Let $J$ be the ideal on the countable set $\mathrm{dom}(J) = \bigcup_n \mathcal{P}(U_n)$ generated by singletons and sets $a \subset \mathrm{dom}(J)$ such that for every number $n \in \omega$, either $a \cap \mathcal{P}(U_n) = 0$ or $|\bigcap(a \cap \mathcal{P}(U_n))| > 2^n$, or $|U_n \setminus \bigcup(a \cap \mathcal{P}(U_n))| > 2^n$.

Claim 2.1. *The ideal $J$ is an $F_\sigma$ proper ideal.*

*Proof.* The set $F$ of generators is closed, and therefore compact, in the space $\mathcal{P}(\mathrm{dom}(J))$. The ideal generated by a closed set of generators is always $F_\sigma$, since the finite union map is continuous on the compact set $F^n$ for every $n \in \omega$, its image is again a compact set, and the ideal $J$ is the union of all of these countably many compact sets.

To see that $\mathrm{dom}(J) \notin J$, suppose that $a_i : i \in k$ are the generators of the ideal $J$. To show that they do not cover $\mathrm{dom}(J)$, find a number $n \in \omega$ such that $2^n > k$ and argue that there is a set $b \subset U_n$ not in any of the sets $a_i : i \in k$. First, partition $k$ into two pieces, $k = z_0 \cup z_1$, such that for $i \in z_0$, $|\bigcap(a_i \cap \mathcal{P}(U_n))| > 2^n$ holds, and for $i \in z_1$, $|U_n \setminus \bigcup(a \cap \mathcal{P}(U_n))| > 2^n$ holds. Use a counting argument to find pairwise distinct elements $u_i : i \in k$ in the set $U_n$ so that for $i \in z_0$, $u_i \in \bigcap(a_i \cap \mathcal{P}(U_n))$ holds, and for $i \in z_1$, $u_i \notin \bigcup(a \cap \mathcal{P}(U_n))$ holds. The set $b = \{u_i : i \in z_1\}$ then belongs to none of the sets $a_i : i \in k$. ■

It follows from the definition of the ideal $J$ that the forcing $P$ below the condition $p$ adds a set $b \subset \mathrm{dom}(J)$ such that no ground model $J$-positive set can be disjoint from it, or included in it. Namely, consider the set $\dot{b} = \{c \subset U_n : \dot{g}(n) \in c, n \in \omega\}$. Suppose that $q \leq p$ is a condition, and $a \subset \mathrm{dom}(J)$ is a $J$-positive set. Then there must be infinitely many numbers $n \in \omega$ such that $a \cap \mathcal{P}(U_n) \neq 0$ and $|\bigcap(a \cap \mathcal{P}(U_n))| \leq 2^n$; since $\dot{g}$ is forced by $p$ to be a counterexample to the weak Laver property, there must be a condition $r \leq q$ and a number $n \in \omega$ such that $r \Vdash \dot{g}(n) \notin \bigcap(\check{a} \cap \mathcal{P}(U_n))$ and therefore $r \Vdash \check{a} \not\subset \dot{b}$. Similarly, there must be infinitely many numbers $n \in \omega$ such that $a \cap \mathcal{P}(U_n) \neq 0$ and $|U_n \setminus \bigcup(a \cap \mathcal{P}(U_n))| \leq 2^n$, and by the failure of the weak Laver property, there must be a number $n$ and a condition $r \leq q$ forcing $\dot{g}(n) \in \bigcup(a \cap \mathcal{P}(U_n))$ and so $\check{a} \cap \dot{b} \neq 0$.

It is now enough to extend the ideal $J$ to a complement of a P-point, since then the previous paragraph shows that such a P-point cannot be preserved by the forcing $P$ below the condition $p$. Such an extension exists, since the ideal $J$ is $F_\sigma$; the construction is well-known, I am not certain to whom to attribute it, it certainly easily follows from some fairly old results.

CLAIM 2.2. (CH) *Whenever $K$ is a proper $F_\sigma$ ideal on a countable set, there is a P-point ultrafilter disjoint from $K$.*

*Proof.* By a result of [6], the quotient poset $\mathcal{P}(\omega)/I$ is countably saturated, in particular $\sigma$-closed. Any sufficiently generic filter over this poset will generate the desired P-point ultrafilter. Just build a modulo $K$ descending $\omega_1$-chain $a_\alpha : \alpha \in \omega_1$ of $K$-positive sets such that:

- $a_{\alpha+1}$ is either disjoint from or a subset of the $\alpha$th subset of $\omega$ in some fixed enumeration;
- $a_\alpha$ is modulo finite included in all sets $a_\beta : \beta \in \alpha$ for every limit ordinal $\alpha$.

The first item shows that the sets $a_\alpha : \alpha \in \omega_1$ generate an ultrafilter disjoint from $K$, the second item is to ensure that this ultrafilter will be a P-point. The induction itself is easy. At the successor step, note that if $b \subset \omega$ is the $\alpha$th subset of $\omega$ in a given enumeration, then one of the sets $a_\alpha \cap b, a_\alpha \setminus b$ will be $K$-positive, and it will serve as $a_{\alpha+1}$. At the limit stage of induction, use the result of Mazur [10] to find a lower semicontinuous submeasure $\phi$ such that $K = \{b \subset \omega : \phi(b) < \infty\}$, enumerate $\alpha = \{\beta_n : n \in \omega\}$, and choose finite sets $b_n \subset \bigcap_{m\in n} a_{\beta_m}$ of $\phi$-mass $\geq n$. The set $a_\alpha = \bigcup_n b_n$ will work. ∎

**3. Proof of Theorem 1.6.** This is more exciting. Assume that the assumptions hold. There are two auxiliary claims.

CLAIM 3.1. *If $K$ is an $F_\sigma$ ideal on $\omega$, $p \in P$ is a condition, and $p \Vdash \dot{b} \subset \omega$, then there are a ground model $K$-positive set and a condition $r \leq p$ forcing it to be either disjoint from, or a subset of, the set $\dot{b}$.*

*Proof.* Use the result of Mazur [10] to find a lower semicontinuous submeasure $\phi$ on $\omega$ such that $J = \{c \subset \omega : \phi(c) < \infty\}$. Find pairwise disjoint sets $c_n \subset \omega$ such that $\phi(c_n) > n \cdot 2^{2^n}$, this for every $n \in \omega$. Use the weak Laver property to find an infinite set $a \subset \omega$, sets $d_n \subset \mathcal{P}(c_n)$ of the respective size $\leq 2^n$, and a condition $q \leq p$ such that $q \Vdash \forall n \in \check{a}\ \dot{b} \cap \check{c}_n \in \check{d}_n$. Use the subadditivity of the submeasure $\phi$ to find sets $e_n \subset c_n$ of submeasure $\geq n$ such that $\forall f \in d_n\ f \cap e_n = 0 \vee e_n \subset f$, this for every $n \in a$. Thus $q \Vdash \forall n \in a\ \check{e}_n \subset \dot{b} \vee \check{e}_n \cap \dot{b} = 0$. Since $P$ adds no splitting reals, there is a condition $r \leq q$ and an infinite subset $a' \subset a$ such that $r \Vdash \forall n \in a'\ \check{e}_n \subset \dot{b} \vee \forall n \in a'\ \check{e}_n \cap \dot{b} = 0$. In the first case, the ground model

$J$-positive set $\bigcup_{n\in a'} e_n$ is forced to be a subset of $\dot{b}$, in the other case, this set is forced to be disjoint from $\dot{b}$ as desired. ■

Claim 3.2. (ZFC + LC) *If $U$ is a P-point and $J$ is a universally Baire ideal disjoint from $U$, then there is an $F_\sigma$ ideal $K \supset J$ disjoint from $U$. If $J$ is analytic then no large cardinals are needed.*

Note that Claims 2.2 and 3.2 together yield a complete characterization of analytic ideals on $\omega$ that are disjoint from a P-point under CH: these are exactly those ideals that can be extended to nontrivial $F_\sigma$ ideals.

*Proof.* I will prove the large cardinal version with a direct determinacy argument and then use the Kechris–Louveau–Woodin dichotomy to argue for the analytic case in ZFC.

Recall the Galvin–Shelah game theoretic characterization of P-points: the ultrafilter $U$ is a P-point if and only if Player I has no winning strategy in the P-point game where he chooses sets $a_n \in U$, Player II chooses their finite subsets $b_n \subset a_n$, and Player II wins if $\bigcup_n b_n \in U$ [2, Theorem 4.4.4]. Now consider the same game, except the winning condition for Player II is replaced with $\bigcup_n b_n \notin J$. This is certainly easier to win for Player II, and so Player I still does not have a winning strategy. Now, however, the payoff set is universally Baire and one can use the large cardinal assumptions and determinacy results [9] to argue that the game is determined and Player II must have a winning strategy $\sigma$.

Let $M$ be a countable elementary submodel of a large enough structure containing the strategy $\sigma$. For every position $p \in M$ of the game that respects the strategy $\sigma$ and ends with a move of Player II, let $u_p = \{b \in [\omega]^{<\aleph_0} : \exists a \in U\ p^\frown a^\frown b$ is a position respecting the strategy $\sigma\}$ and let $F_p = \{c \subset \omega : c$ has no subset in $u_p\}$. The sets $F_p \subset \mathcal{P}(\omega)$ are closed and disjoint from the ultrafilter $U$, since for every set $a \in U$ the strategy $\sigma$ must answer $a$ with its subset. Thus, the sets $F_p : p \in M$ generate an $F_\sigma$ ideal $K$ on $\omega$ disjoint from the ultrafilter $U$. I must show that $J \subset K$ holds.

Suppose $c \subset \omega$ is not in the ideal $K$. By induction on $n \in \omega$ find sets $a_n \in U \cap M$ such that when Player I plays these sets in succession, the strategy $\sigma$ always responds with a subset of $c$. Suppose the sets $a_n : n \in m$ have been built, and let $p \in M$ be the corresponding position of the game. Since $c \notin F_p$, there must be a set $a_m$ such that the strategy responds to the move $a_m$ by a subset of $c$. This concludes the inductive construction. In the end, the strategy $\sigma$ won the infinite play against the sequence $a_n : n \in \omega$ of Player I's challenges. Thus the set $\bigcup_n b_n$ it produced was not $J$-positive. This set is a subset of the set $c$ by the inductive construction, and therefore $c \notin J$ as required.

Now for the ZFC case, let $J$ be an analytic ideal disjoint from the P-point ultrafilter $U$. If $J$ can be separated from $U$ by an $F_\sigma$ set $K_0$, then the ideal $K$ generated by this set is still $F_\sigma$, still disjoint from $U$, and it includes $J$ as desired. If $J$ cannot be so separated, then the Kechris–Louveau–Woodin dichotomy [7, Theorem 21.22] shows that there is a perfect set $C \subset J \cap U$ such that $C \cap U$ is countable and dense in $C$. I will use it to construct a winning strategy for Player I in the P-point game, yielding a contradiction and completing the proof. Let $c_n : n \in \omega$ be an enumeration of the set $C \cap U$. Player I will win by playing sets $a_n \in C \cap U$ and on the side writing down finite initial segments $b'_n \subset a_n$ which include Player II's answer $b_n$ in such a way that

- $a_n$ contains $\bigcup_{i\in n} b'_i$ as an initial segment;
- $a_n \neq c_n$ and $c_n$ does not contain $\bigcup_{i\in n+1} b'_i$ as an initial segment.

This is easily possible. In the end, the set $\bigcup_{n\in\omega} b'_n \subset \omega$ is the limit of the sets $a_n \in C \cap U$, and therefore it belongs to $C$ by the first item, and it is not equal to any of the sets in $C \cap U$ by the second item. Consequently, it must belong to the ideal $J$, and since the set $\bigcup_{n\in\omega} b_n$ is included in it, it means that Player I won. ∎

Theorem 1.6 now follows easily. Suppose $P$ is a proper forcing, $P = P_I$ for some universally Baire $\sigma$-ideal on a Polish space $X$, $U$ is a P-point, $B \in P_I$ is a condition and $B \Vdash \dot{b} \subset \omega$ is a set. I must find a condition $C \subset B$ and a set $a \in U$ such that $C \Vdash \dot{b} \cap \check{a} = 0 \vee \check{a} \subset \dot{b}$. By strengthening the condition $B$ I may assume that there is a Borel function $f : B \to \mathcal{P}(\omega)$ such that $B \Vdash \dot{b} = \dot{f}(\dot{x}_{\text{gen}})$. Consider the set $J_0 = \{a \subset \omega : \exists C \subset B\ C \Vdash \check{a} \cap \dot{b} = 0 \vee C \Vdash \check{a} \subset \dot{b}\} = \{a \subset \omega : \{x \in B : f(x) \cap a = 0\} \notin I \vee \{x \in Ba \subset f(x)\} \notin I\}$. If it is not disjoint from the P-point $U$, then we are done. If $J_0 \cap U = 0$, then even the ideal $J$ generated by $J_0$ is disjoint from $U$. The ideal $J$ is universally Baire, and if the $\sigma$-ideal $I$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ then $J$ is in fact analytic. Claim 3.2 now shows that there is an $F_\sigma$ ideal $K \supset J$ disjoint from $U$. Claim 3.1 shows that there is a condition $C \subset B$ and a $K$-positive set $a \subset \omega$ such that $C \Vdash \check{a} \cap \dot{b} = 0$ or $C \Vdash \check{a} \subset \dot{b}$. This however contradicts the definition of the set $J_0 \subset K$!

**4. Proof of Example 1.7.** Suppose that the Continuum Hypothesis holds, and fix a Ramsey ultrafilter $U$. Consider the partial order $P_U$ consisting of those pruned trees $T \subset 2^{<\omega}$ such that there is a set $a \in U$ such that a node in $T$ is a split node if and only if its length is in the set $a$ ordered by inclusion. The forcing $P_U$ witnesses the conclusion of Example 1.7. It is clear that the generic real $\dot{x}_{\text{gen}}$, the union of the intersection of all trees in the generic filter, is a function in $2^\omega$ which is not constant on any set in the ultrafilter $U$. The forcing also has the Sacks property and adds no splitting reals.

Instead of the somewhat slippery argument for this latter statement, I will prove a closely related fact. Consider the symmetric Sacks forcing $P$ of [13]. It consists of those pruned trees $T \subset 2^{<\omega}$ such that there is an infinite set $a \subset \omega$ such that a node in $T$ is a splitnode if and only if its length is in the set $a$, ordered by inclusion. It is not difficult to see that the forcing $P$ splits into a two-step iteration, $P = Q * P_{\dot{U}}$, where $Q$ is the ordering of infinite subsets of $\omega$ with modulo finite inclusion, and $\dot{U}$ is the $Q$-name for the Ramsey ultrafilter added by $Q$. A standard fusion argument directly transferred from the usual Sacks forcing case shows that the symmetric Sacks forcing has the Sacks property. It is significantly harder to show that $P$ adds no splitting reals; it follows for example from the upcoming work of [15]. Now, summing up, it is clear that in the $Q$ extension, there is a forcing, namely $P_U$, which has the Sacks property and adds no splitting reals, and adds a function from $\omega$ to 2 which is not constant on any set in the Ramsey ultrafilter $U$.

**5. Applications of the main theorems.** Theorems 1.5 and 1.6 can be used in two directions: to ensure that certain forcings preserve P-points, and to prove that other forcings do not preserve P-points. In this brief section I will give examples of both.

An important and well studied class of forcings consists of the quotient forcings obtained from ideals on a Polish space $X$ generated by a compact collection of compact sets in the hyperspace $K(X)$ [17, Theorem 4.1.8]; this is a slight generalization of the fairly common limsup infinity tree forcings of [14]. These quotient forcings do not add splitting reals and have the weak Laver property; therefore, they preserve P-points. Their countable products are more difficult to analyze. However, a simple fusion argument shows that the products possess the weak Laver property, and a subtle combinatorial argument [15] shows that the products do not add splitting reals. Theorem 1.6 then implies the conclusion:

Proposition 5.1. *The countable product of quotient forcings of $\sigma$-ideals generated by a compact collection of compact sets preserves P-points.*

The methods of [15] show that many other forcings, including the wide Silver forcing, symmetric Sacks forcing [13], and the $E_0$ and $E_2$ forcings [17, Section 4.7], do not add splitting reals. The forcings just named all have the weak Laver property, and therefore, by Theorem 1.6, they also preserve P-points. This is perhaps not quite surprising, but a direct proof seems to be out of reach.

As an example of the application in the opposite direction, let me include

Proposition 5.2. (CH) *If $P$ is a forcing adding a bounded eventually different real, then $P$ fails to preserve P-points.*

Note that every bounding forcing making the set of all ground model reals meager falls into this category essentially by [2, Theorem 2.4.7]. Thus, for example, forcing with an ideal associated with a Ramsey capacity is bounding and adds no splitting reals [17, Theorem 4.3.25], but it must destroy a P-point. On the other hand, the Blass–Shelah forcing makes the set of ground model reals meager, it is not bounding, and it preserves P-points.

*Proof.* It will be enough to show that $P$ fails the weak Laver property. Suppose $\dot{g}$ and $f$ are a $P$-name and a function in $\omega^\omega$ respectively such that $P \Vdash \dot{g} < \check{f}$ and for every ground model function $h \in \omega^\omega$, $\dot{g} \cap \check{h}$ is finite. Let $\omega = \bigcup_n b_n$ be a partition of $\omega$ into finite sets of the respective size $2^n$, let $\bar{f}(n)$ be the set $\pi_{i \in b_n} f(i)$ and let $\bar{g} \in \Pi_n \bar{f}(n)$ be the name for the function in the extension defined by $\bar{g}(n) = \dot{g} \restriction \check{b}(n)$. I claim that $\bar{f}, \bar{g}$ witness the failure of the weak Laver property.

Indeed, if $a \subset \omega$ were an infinite set, $h$ a ground model function on $a$ such that $h(n)$ is a subset of $\bar{f}(n)$ of size $< 2^n$ and $p \in P$ a condition forcing $\forall n \in a\ \bar{g}(n) \in \check{h}(n)$, one could find surjections $u_n : b_n \to h(n)$ for every number $n \in a$, find a function $k \in \omega^\omega$ such that $k(i) = u_n(i)(i)$ for every $n \in a$ and every $i \in b_n$, and conclude that $p \Vdash \check{k} \cap \dot{g}$ is infinite. This contradicts the assumptions on the name $\dot{g}$. ∎

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