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Szpilrajn type theorem for concentration dimension

by

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Abstract. Let X be a locally compact, separable metric space. We prove that $\dim_{\mathrm{T}} X = \inf\{\dim_{\mathrm{L}} X' : X' \text{ is homeomorphic to } X\}$, where $\dim_{\mathrm{L}} X$ and $\dim_{\mathrm{T}} X$ stand for the concentration dimension and the topological dimension of X, respectively.

1. Introduction. The analytic theory of dimension of sets and measures is widely investigated and widely used, for example, in the geometric theory of sets and measures and the theory of dynamical systems. The most popular one is undoubtedly the Hausdorff dimension. Unfortunately, all known dimensions of even relatively simple sets are rather hard to calculate.

A. Lasota proposed to study a new concept of dimension of measures and sets defined by means of the Lévy concentration function (see [5]). This concept is also related to the mass distribution principle (see [4]). Some properties of this dimension—called concentration dimension—were given in [8]. In particular, it was proved that this dimension is strongly related to the Hausdorff dimension. More precisely, the Hausdorff dimension is always greater than or equal to the concentration dimension, and under suitable assumptions, they are equal. What is important, the concentration dimension seems to be easier to estimate or calculate (see [8, 9]).

The connection between Hausdorff dimension and topological dimension was made evident in the case of $\mathbb{R}^n$ by V. G. Nöbeling (see [10]) and in a more general setting by Szpilrajn in 1937 (see [11]). In this note we prove a similar result for concentration dimension. Note also that the relation between Hausdorff dimension and packing dimension was studied in [7].

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2. Notation and preliminaries. Throughout this paper (X,ϱ) denotes a locally compact, separable metric space. By $B(x,r)$ (resp. $B^o(x,r)$, $S(x,r)$) we denote the closed ball (resp. the open ball and sphere) in X with centre at x and radius r. For $A\subset X$, the symbols: $\mathrm{cl}A$, ∂A, $\mathrm{diam}\,A$ and 1_A stand for the closure, boundary, diameter and characteristic function of A, respectively. As usual, $\mathbb{R}$ stands for the set of all reals and $\mathbb{N}$ for the set of all positive integers. Moreover, $\mathbb{R}_+=[0,\infty)$ and $\overline{\mathbb{R}}_+=[0,\infty]$.

By $\mathcal{B}(X)$ we denote the σ-algebra of Borel subsets of X and by $\mathcal{M}(X)$ the family of all finite Borel measures on X. Moreover, $\mathcal{M}_1(X)$ denotes the family of all $\mu\in\mathcal{M}(X)$ such that $\mu(X)=1$, and $\mathcal{M}_{\le 1}(X)$ is the family of all measures $\mu\in\mathcal{M}(X)$ such that $0<\mu(X)\le 1$.

Given an arbitrary function $f:A\to\overline{\mathbb{R}}_+$, where A is a Borel subset of $\mathbb{R}$, we denote by $\mathcal{F}_f$ the set of all Borel measurable functions $\phi:A\to\overline{\mathbb{R}}_+$ such that $\phi(\lambda)\ge f(\lambda)$ for $\lambda\in A$. By the *upper integral* of f we mean the value

$$\overline{\int_A} f(\lambda)\,d\lambda=\inf_{\phi\in\mathcal{F}_f}\int_A \phi(\lambda)\,d\lambda.$$

LEMMA 1. *Let $f:(0,b]\to(0,\infty]$, $b>0$, be an arbitrary function. Then*

$$\overline{\int_0^b} f(\lambda)\,d\lambda>0.$$

Proof. This is standard and left to the reader.

Given a measure $\mu\in\mathcal{M}_1(X)$ we define the *lower* and *upper concentration dimension* of μ by the formulas

$$\underline{\dim}_{\mathrm{L}}\,\mu=\liminf_{r\to 0}\frac{\log Q_\mu(r)}{\log r},\qquad \overline{\dim}_{\mathrm{L}}\,\mu=\limsup_{r\to 0}\frac{\log Q_\mu(r)}{\log r},$$

where

$$Q_\mu(r)=\sup\{\mu(A):\mathrm{diam}A\le r,\ A\in\mathcal{B}(X)\}\quad \text{for } r>0.$$

Recall that Q_μ is the well known *Lévy concentration function* frequently used in the theory of random variables (see [5]).

The *concentration dimension* of X is defined by the formula

$$\dim_{\mathrm{L}} X=\sup_{\mu\in\mathcal{M}_1(X)}\underline{\dim}_{\mathrm{L}}\,\mu. \tag{1}$$

For $A\subset X$ and $s,\delta>0$ define

$$\mathcal{H}^s_\delta(A)=\inf\Big\{\sum_{i=1}^\infty(\mathrm{diam}\,U_i)^s: A\subset\bigcup_{i=1}^\infty U_i \text{ and } \mathrm{diam}\,U_i\le\delta\Big\}$$

and

$$\mathcal{H}^s(A)=\lim_{\delta\to 0}\mathcal{H}^s_\delta(A).$$

The restriction of $\mathcal{H}^s$ to the σ-algebra of $\mathcal{H}^s$-measurable sets is called the *s-dimensional Hausdorff measure*. Note that all Borel sets are $\mathcal{H}^s$-measurable. The value

$$\dim_{\mathrm{H}} A = \inf\{s > 0 : \mathcal{H}^s(A) = 0\}$$

is called the *Hausdorff dimension* of the set A. As usual we set $\inf \emptyset = +\infty$.

The *Hausdorff dimension* of a measure $\mu \in \mathcal{M}_1(X)$ is defined by the formula

$$\dim_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}} A : A \in \mathcal{B}(X),\ \mu(A) = 1\}. \tag{2}$$

Finally, recall that if X is a separable metric space, the three principal topological dimensions (small inductive dimension, large inductive dimension and covering dimension) are equal (see [2]). We denote this common value by $\dim_{\mathrm{T}} X$ and call it the *topological dimension* of X. The value $\dim_{\mathrm{T}} X$ is an integer greater than or equal to -1, or ∞. It can be defined by the following recurrent scheme:

(i) $\dim_{\mathrm{T}} X = -1$ if and only if $X = \emptyset$;

(ii) $\dim_{\mathrm{T}} X \le n$, $n = 0, 1, \ldots,$ if for every $x \in X$ and every neighbourhood U of x there is a neighbourhood V of x such that $V \subset U$ and $\dim_{\mathrm{T}} \partial V \le n - 1$;

(iii) $\dim_{\mathrm{T}} X = n$ if and only if $\dim_{\mathrm{T}} X \le n$ and it is not true that $\dim_{\mathrm{T}} X \le n - 1$;

(iv) $\dim_{\mathrm{T}} X = \infty$ if $\dim_{\mathrm{T}} X \ge n$ for every $n \in \mathbb{N}$.

LEMMA 2. *If* $\dim_{\mathrm{T}} X \ge d+1$, *where d is an integer greater than or equal to* -1, *then there exist* $x_0 \in X$ *and* $\lambda_0 > 0$ *such that* $\dim_{\mathrm{T}} S(x_0, \lambda) \ge d$ *for every* $\lambda \in (0, \lambda_0]$.

Proof. Suppose, for a contradiction, that for every $x_0 \in X$ and $\lambda_0 > 0$ there exists $\lambda \in (0, \lambda_0]$ such that $\dim_{\mathrm{T}} S(x_0, \lambda) \le d - 1$. Then by the definition of topological dimension we have $\dim_{\mathrm{T}} X \le d$, which is impossible. ■

3. Results

PROPOSITION 3. *For every* $\mu \in \mathcal{M}_1(X)$ *we have* $\dim_{\mathrm{H}} \mu \ge \underline{\dim}_{\mathrm{L}}\, \mu$.

Proof. We use an argument similar to that of Proposition 1.2 in [8] (see also [3, Chapter 3]). Let $A \in \mathcal{B}(X)$ be such that $\mu(A) = 1$. Set $d = \underline{\dim}_{\mathrm{L}}\, \mu$. If $d = 0$, the statement of Proposition 3 is obvious. Suppose now that $0 < d \le \infty$ and choose a positive number $s < d$. Define

$$\omega(r) = \frac{\log Q_\mu(r)}{\log r} \quad \text{for } r > 0.$$

Then obviously

$$Q_\mu(r) = r^{\omega(r)} \quad \text{and} \quad \liminf_{r \to 0} \omega(r) > s.$$

Let $r_0 \in (0,1)$ be such that $\omega(r) > s$ for every $r \in (0, r_0)$. Fix $r \in (0, r_0)$ and let $\{U_i\}$ be an arbitrary cover of A satisfying $\operatorname{diam} U_i \le r$, $i \in \mathbb{N}$. We have

$$1 = \mu(A) \le \sum \mu(U_i) \le \sum (\operatorname{diam} U_i)^s.$$

Therefore $\mathcal{H}^s_r(A) \ge 1$ for $r \in (0, r_0)$. Consequently, $\mathcal{H}^s(A) \ge 1$ and so $\dim_{\mathrm{H}} A \ge s$. Since $s < d$ was arbitrary, the statement follows. ■

From (1), (2) and Proposition 3 we immediately obtain

COROLLARY 4. $\dim_{\mathrm{H}} X \ge \dim_{\mathrm{L}} X$.

PROPOSITION 5. *Suppose that* $\dim_{\mathrm{T}} X \ge d$, *where* $d \in \mathbb{N} \cup \{0\}$. *Then there exists a Borel measure* $\mu \in \mathcal{M}_{\le 1}(X)$ *such that*

$$\mu(B(x,r)) \le r^d \quad \textit{for every } x \in X \textit{ and } r > 0. \tag{3}$$

Proof. We use induction on d. For $d = 0$ condition (3) obviously holds for every measure $\mu \in \mathcal{M}_{\le 1}(X)$. Assume that the statement holds for $d = k$. We will prove it for $d = k+1$. By Lemma 2 there exist $x_0 \in X$ and $\lambda_0 > 0$ such that $\dim_{\mathrm{T}} S(x_0, \lambda) \ge k$ for every $\lambda \in (0, \lambda_0]$. We can assume that $\lambda_0 < 1$ and $B(x_0, \lambda)$ is compact. Fix $\lambda \in (0, \lambda_0]$ and set $X_\lambda = S(x_0, \lambda)$. By the induction hypothesis there exists a nontrivial Borel measure $\widetilde{\mu}_\lambda$ on X_λ such that

$$\widetilde{\mu}_\lambda(X_\lambda) \le 1 \quad \text{and} \quad \widetilde{\mu}_\lambda(B_\lambda(x,r)) \le r^k \quad \text{for every } x \in X_\lambda \text{ and } r > 0,$$

where $B_\lambda(x,r)$ stands for the closed ball in X_λ with centre at $x \in X_\lambda$ and radius r. Define the measure $\mu_\lambda : \mathcal{B}(X) \to [0,1]$ by the formula

$$\mu_\lambda(A) = \widetilde{\mu}_\lambda(A \cap X_\lambda) \quad \text{for } A \in \mathcal{B}(X).$$

Clearly $\mu_\lambda \in \mathcal{M}_{\le 1}(X)$, $\operatorname{supp} \mu_\lambda \subset S(x_0, \lambda)$ and

$$\mu_\lambda(B(x,r)) \le 2^k r^k \quad \text{for every } x \in X \text{ and } r > 0. \tag{4}$$

Define

$$\alpha_{n,i} = \sup\left\{\mu_\lambda(X) : \lambda \in \left(\frac{(i-1)\lambda_0}{n}, \frac{i\lambda_0}{n}\right]\right\} \quad \text{for } n \in \mathbb{N},\ i = 1, \ldots, n.$$

Let

$$\nu_n = \frac{\lambda_0}{n} \sum_{i=1}^{n} \mu_{n,i} \quad \text{for } n \in \mathbb{N}, \tag{5}$$

where $\mu_{n,i} = \mu_{\lambda_{n,i}}$ with $\lambda_{n,i} \in ((i-1)\lambda_0/n, i\lambda_0/n]$ and such that

$$\mu_{\lambda_{n,i}}(X) \ge \alpha_{n,i}/2. \tag{6}$$

Set $K = B(x_0, \lambda_0)$. Clearly $\operatorname{supp} \nu_n \subset K$ and $\nu_n \in \mathcal{M}_{\le 1}(X)$. By (5) and (6) we have

$$2\nu_n(X) = 2\nu_n(K) = \frac{2\lambda_0}{n} \sum_{i=1}^{n} \mu_{n,i}(K) \ge \frac{\lambda_0}{n} \sum_{i=1}^{n} \alpha_{n,i}. \tag{7}$$

Consider the function $\phi : (0, \lambda_0] \to (0, \infty)$ given by

$$\phi(\lambda) = \sum_{i=1}^{n} \alpha_{n,i} \cdot 1_{((i-1)\lambda_0/n,\ i\lambda_0/n]}(\lambda).$$

Clearly ϕ is Borel measurable and $\phi(\lambda) \ge \mu_\lambda(X)$ for $\lambda \in (0, \lambda_0]$. Thus by (7), the definition of the upper integral and Lemma 1 we have

$$2\nu_n(K) \ge \int_0^{\lambda_0} \phi(\lambda)\, d\lambda \ge \overline{\int_0^{\lambda_0}} \mu_\lambda(X)\, d\lambda > 0. \tag{8}$$

Consider the sequence $(\mu_n)_{n \ge 1} \subset \mathcal{M}_1(X)$ given by the formula $\mu_n = \nu_n/\nu_n(X)$, $n \in \mathbb{N}$. Since $\nu_n \in \mathcal{M}_{\le 1}(X)$ we can choose a sequence $(m_n)_{n \ge 1}$ such that $\nu_{m_n}(X) \to a$ as $n \to \infty$. From (8) it follows that $a > 0$. Since the supports of ν_n are contained in a compact set K, passing to a subsequence if necessary, we can assume that $(\mu_{m_n})_{n \ge 1}$ converges weakly to some $\mu_* \in \mathcal{M}_1(X)$. It suffices to verify that the measure $\mu = a\mu_*/2^{k+1}$ satisfies (3) with $d = k + 1$. To this end, fix $x \in X$ and $r > 0$ and consider the ball $B(x, r)$. For $n \in \mathbb{N}$ define

$$\underline{i}(n) = \min J_n \quad \text{and} \quad \bar{i}(n) = \max J_n,$$

where

$$J_n = \{1 \le i \le n : B(x, r) \cap S(x_0, \lambda_{n,i}) \ne \emptyset\}.$$

If $J_n = \emptyset$ we set $\underline{i}(n) = \bar{i}(n) = 0$. It can be verified that

$$\frac{\lambda_0}{n}(\bar{i}(n) - \underline{i}(n)) \le 2r + \frac{\lambda_0}{n}. \tag{9}$$

Further, by (5) and the construction of the measure $\mu_{n,i}$ we have

$$\nu_n(B(x, r)) = \frac{\lambda_0}{n} \sum_{i=1}^{n} \mu_{n,i}(B(x, r)) = \frac{\lambda_0}{n} \sum_{i=\underline{i}(n)}^{\bar{i}(n)} \mu_{n,i}(B(x, r)).$$

Now, using (4) and (9) we obtain

$$\nu_n(B(x, r)) \le \frac{\lambda_0}{n} 2^k r^k (\bar{i}(n) - \underline{i}(n) + 1) \le 2^{k+1} r^{k+1} + \frac{\lambda_0}{n} 2^{k+1} r^k. \tag{10}$$

Since $(\mu_{m_n})_{n \ge 1}$ converges weakly to μ_*, by the Alexandrov theorem (see [1]) for every $\eta > 0$ we have

$$\mu_*(B(x,r)) \le \mu_*(B^o(x,r+\eta)) \le \liminf_{n\to\infty} \mu_{m_n}(B^o(x,r+\eta)) \le \liminf_{n\to\infty} \frac{\nu_{m_n}(B(x,r+\eta))}{\nu_{m_n}(X)}.$$

Consequently, by (10) and the choice of the subsequence $(m_n)_{n\ge 1}$ we have

$$\mu_*(B(x,r)) \le 2^{k+1}a^{-1}(r+\eta)^{k+1},$$

and since $\eta > 0$ was arbitrary, we obtain $\mu_*(B(x,r)) \le 2^{k+1}a^{-1}r^{k+1}$. Keeping in mind the definition of μ we obtain $\mu(B(x,r)) \le r^{k+1}$. Since $x \in X$ and $r > 0$ were arbitrary, the proof is complete. ∎

Proposition 6. *Let X be a locally compact, separable metric space. Then there exists a measure $\mu_* \in \mathcal{M}_1(X)$ such that*

$$\underline{\dim}_{\mathrm{L}}\, \mu_* \ge \dim_{\mathrm{T}} X.$$

Proof. We can assume that $X \ne \emptyset$. Set $d = \dim_{\mathrm{T}} X$. By Proposition 5 there exists a measure $\mu \in \mathcal{M}_{\le 1}(X)$ such that $\mu(B(x,r)) \le r^d$ for every $x \in X$ and $r > 0$. Define $\mu_* = \mu/\mu(X)$. Clearly $\mu_* \in \mathcal{M}_1(X)$ and

$$\mu_*(B(x,r)) \le (\mu(X))^{-1}r^d \quad \text{for every } x \in X \text{ and } r > 0.$$

Hence $Q_{\mu_*}(r) \le (\mu(X))^{-1}r^d$ for $r > 0$ and consequently

$$\underline{\dim}_{\mathrm{L}}\, \mu_* = \liminf_{r\to 0} \frac{\ln Q_{\mu_*}(r)}{\ln r} \ge \liminf_{r\to 0} \frac{d\ln r - \ln \mu(X)}{\ln r} = d. \ ∎$$

As a consequence of Proposition 6 we immediately obtain

Corollary 7. *Let X be as in Proposition 6. Then*

$$\dim_{\mathrm{L}} X \ge \dim_{\mathrm{T}} X.$$

Corollary 8 (Szpilrajn [11]). *Let X be as above. Then*

$$\dim_{\mathrm{H}} X \ge \dim_{\mathrm{T}} X.$$

Proof. From the inequality $\dim_{\mathrm{H}} X \ge \dim_{\mathrm{H}} \mu$ for every $\mu \in \mathcal{M}_1(X)$, Proposition 4 and the definition of the concentration dimension of X it follows that $\dim_{\mathrm{H}} X \ge \dim_{\mathrm{L}} X$. From this and Corollary 7 the statement follows. ∎

Theorem 9. *Let X be a locally compact, separable metric space. Then*

$$\dim_{\mathrm{T}} X = \inf\{\dim_{\mathrm{L}} X' : X' \textit{ is homeomorphic to } X\}.$$

Proof. Set $d = \dim_{\mathrm{T}} X$. We can assume that $d < \infty$. By Proposition 6 for every X' homeomorphic to X, we have

$$\dim_{\mathrm{L}} X' \ge d. \tag{11}$$

On the other hand, it follows from [6, Theorem VII.5] that if we let X' range over all the spaces homeomorphic to X, then

$$\inf\{\dim_{\mathrm{H}} X'\} = d. \tag{12}$$

The assertion now follows from Corollary 4 and relations (11) and (12). ∎

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