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Homological computations in the universal Steenrod algebra

by

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Abstract. We study the (bigraded) homology of the universal Steenrod algebra Q over the prime field $\mathbb{F}_2$, and we compute the groups $H_{s,s}(Q)$, $s \geq 0$, using some ideas and techniques of Koszul algebras developed by S. Priddy in [5], although we presently do not know whether or not Q is a Koszul algebra. We also provide an explicit formula for the coalgebra structure of the diagonal homology $D_*(Q) = \bigoplus_{s\geq 0} H_{s,s}(Q)$ and show that $D_*(Q)$ is isomorphic to the coalgebra of invariants Γ introduced by W. Singer in [6].

Introduction. It is a basic problem of homological algebra to compute the (co)homology of various augmented algebras. The purpose of this paper is to compute the diagonal homology $D_*(Q) = \bigoplus_{s\geq 0} H_{s,s}(Q)$ of the universal Steenrod algebra Q and provide a description of $D_*(Q)$ as a coalgebra in terms of invariant theory. Q is a graded algebra arising from algebraic topology, for it is the algebra of cohomology operations in the category $\mathcal{C}(2,\infty)$ of H_∞-ring spectra. It is an interesting object as it contains Λ, the lambda algebra introduced in [1], as a subalgebra, and the Steenrod algebra arises as a quotient of Q. Hence it would be nice to understand the cohomology algebra $H^*(Q) = \mathrm{Ext}_Q(\mathbb{F}_2, \mathbb{F}_2)$ and the homology $H_*(Q) = \mathrm{Tor}^Q(\mathbb{F}_2, \mathbb{F}_2)$, but these computational problems are presently unsolved. What makes such computation hard is the fact that Q is by no means locally finite and most of the methods developed by Priddy ([5]) do not apply. The second author succeeded in computing the diagonal cohomology of Q in [4].

In Section 1 of the present paper a description of the homology groups $H_{s,s}(Q)$, i.e. the groups $\mathrm{Tor}^Q_{s,s}(\mathbb{F}_2, \mathbb{F}_2)$, $s \in \mathbb{N}$, is provided, with an explicit formula for the coalgebra structure map of $D_*(Q)$. In [5] S. Priddy works under the hypothesis of local finiteness for the algebras involved, but the idea we borrow from his work in the proof of Theorem 1 does not depend

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on this assumption. On the contrary, this hypothesis is crucial in Theorem 5.3 of [5], hence we cannot apply that result to our case and deduce that $H_{s,s+h}(Q) = 0$ for $s, h \in \mathbb{N}$, although Q is a Poincaré–Birkhoff–Witt algebra.

In Section 2 we use the machinery of invariant theory (see e.g. [6]) to show that $D_*(Q)$ is isomorphic, as a coalgebra, to a certain coalgebra of invariants Γ introduced by W. Singer in [6].

1. The diagonal homology of Q. The universal Steenrod algebra Q at $p = 2$ can be presented as follows:

$$Q = \left\langle x_k \;\middle|\; x_{2k-1-n}x_k = \sum_j \binom{n-1-j}{j} x_{2k-1-j}x_{k-n+j} \right\rangle,$$

where $k \in \mathbb{Z}$ and $n \in \mathbb{N}_0$. The defining relations of the algebra Q are known as *generalized Adem relations*, since they are a generalization of the Adem relations for the mod 2 Steenrod algebra. A typical monomial $x_{i_1} \cdots x_{i_m}$ of Q is said to have *length* m and *total degree* $i_1 + \cdots + i_m$ (we put x_k in degree k). Such generalized Adem relations are homogeneous with respect to both total degree and length; hence Q is a bigraded algebra. For example x_k has bidegree $(1, k)$. A linear basis is given by the set $\mathcal{B}$ of all *admissible* monomials, i.e. monomials of the following type:

$$x_{i_1}x_{i_2} \cdots x_{i_m} : \; i_j \geq 2i_{j+1} \; \forall j = 1, \ldots, m-1.$$

Q is neither connected, nor of finite type: for example, the monomials $x_k x_{-k}$, $k \in \mathbb{N}_0$, are all admissible of length 2 and total degree zero. If $x_{i_1} \cdots x_{i_l} \in \mathcal{B}$, the string $I = (i_1, \ldots, i_l)$ will be called the *label* of $x_{i_1} \cdots x_{i_l}$ and we write x_I instead of $x_{i_1} \ldots x_{i_l}$. Q is also an augmented algebra, the augmentation $\varepsilon : Q \to \mathbb{F}_2$ is obtained by setting $\varepsilon(\alpha) = 0$ for each monomial α of positive length and the identity over $\mathbb{F}_2$. Let $J = \ker(\varepsilon)$ denote the augmentation ideal.

The homology of Q is defined as $\mathrm{Tor}_*^Q(\mathbb{F}_2, \mathbb{F}_2)$, where Tor is computed in the category of graded Q-modules.

Let $\overline{B}(Q) = T(J) = \bigoplus_{s\in\mathbb{N}_0} \underbrace{J \otimes \cdots \otimes J}_{s}$. Thus $\overline{B}(Q)$ is generated by elements of the form $x_{I_1} \otimes \cdots \otimes x_{I_s}$, where $x_{I_j} \in J$. Such elements are written simply as

$$[x_{I_1}|\cdots|x_{I_s}] = [x_{i_1} \cdots x_{i_{t_1}}|x_{i_{t_1+1}} \cdots x_{i_{t_2}}|\cdots|x_{t_{s-1}+1} \cdots x_{t_s}]$$

and are trigraded: s is the *homological degree*, $t = t_s$ is the *length* and $u = \sum_{k=1}^{t_s} i_k$ is the *total degree*, which we usually disregard. Let $\overline{B}_s(Q)_t$ denote the submodule generated by elements of bidegree (s, t). We define a map

$$\partial_s : \overline{B}_s(Q) \to \overline{B}_{s-1}(Q)$$

by setting

$$\partial_s([x_{I_1}|\cdots|x_{I_s}]) = \sum_{j=1}^{s-1}[x_{I_1}|\cdots|x_{I_j}x_{I_{j+1}}|\cdots x_{I_s}],$$

which is a differential for $\overline{B}(Q)$. The chain complex $(\overline{B}(Q),\partial)$ is known as the *reduced bar construction*, and from homological algebra we know that it computes the homology of Q, i.e. $H(Q) = \mathrm{Tor}^Q(\mathbb{F}_2,\mathbb{F}_2)$ is the homology of $(\overline{B}(Q),\partial)$. We set $D_k(Q) = H_{k,k}(Q)$. The direct sum $D_*(Q) = \bigoplus_{k\geq 0} D_k(Q)$ is the *diagonal* homology of Q. In order to compute $D_*(Q)$, we notice that

$$\begin{aligned} D_k(Q) &= H_{k,k}(Q) \\ &= \frac{\ker[\partial_k : \overline{B}_k(Q)_k \to \overline{B}_{k-1}(Q)_k]}{\mathrm{im}[\partial_{k+1} : \overline{B}_{k+1}(Q)_k \to \overline{B}_k(Q)_k]} \\ &= \ker[\partial_k : \overline{B}_k(Q)_k \to \overline{B}_{k-1}(Q)_k] \end{aligned}$$

since there exist no non-zero $(k+1)$-chains of length k (a $(k+1)$-chain has length at least $k+1$). Hence, each element of $D_k(Q)$ is uniquely represented by a cycle of $\overline{B}_k(Q)_k$ of the form $\sum_I f_I[x_{i_1}|\cdots|x_{i_k}]$, where $I = (i_1,\ldots,i_k)$ and $x_{i_j} \in J$. Observe that $D_0(Q) = \mathbb{F}_2$ and $D_1(Q)$ is the $\mathbb{F}_2$-vector space with $\{[x_k] \mid k \in \mathbb{Z}\}$ as a basis, because $\overline{B}_0(Q) = \mathbb{F}_2$ and $\partial : \overline{B}_1(Q) = J \to \overline{B}_0(Q)$ is the zero homomorphism. In the following statement we describe the groups $D_m(Q) = \ker\partial_m$ in terms of cycles.

THEOREM 1. *Let $x = \sum_{I\in\mathcal{C}} f_I[x_{i_1}|\cdots|x_{i_m}] \in \overline{B}_m(Q)_m$, where $\mathcal{C}$ is a suitable set of labels. Then x is a cycle if and only if for each j $(1 \leq j \leq m-1)$ and each $(k_1,\ldots,k_{j-1}) \in \mathbb{Z}^{j-1}$, $(k_{j+2},\ldots,k_m) \in \mathbb{Z}^{m-j-1}$, the following condition holds*:

$$\sum_I f_I x_{i_j} x_{i_{j+1}} = 0,$$

where the summation runs over all $I \in \mathcal{C}$ such that $(i_1,\ldots,i_{j-1}) = (k_1,\ldots,k_{j-1})$ and $(i_{j+2},\ldots,i_m) = (k_{j+2},\ldots,k_m)$.

Proof. Let us compute ∂x:

$$\partial x = \sum_I \sum_{j=1}^{m-1} f_I[x_{i_1}|\cdots|x_{i_j}x_{i_{j+1}}|\cdots|x_{i_m}],$$

where $[x_{i_1}|\cdots|x_{i_j}x_{i_{j+1}}|\cdots|x_{i_m}] \in \overline{B}_{m-1}(Q)_m$. Hence $\partial x = 0$ if and only if for each j $(1 \leq j \leq m-1)$ and each $(k_1,\ldots,k_{j-1}) \in \mathbb{Z}^{j-1}$, we have

$$\sum_I f_I[x_{i_1}|\cdots|x_{i_j}x_{i_{j+1}}|\cdots|x_{i_m}] = 0,$$

where the summation is taken over all I such that $(i_1,\ldots,i_{j-1}) = (k_1,\ldots,k_{j-1})$ and $(i_{j+2},\ldots,i_m) = (k_{j+2},\ldots,k_m)$. But this is equivalent to $\sum_I f_I x_{i_j} x_{i_{j+1}} = 0$, summed over the same values of I. ∎

COROLLARY 2. *Suppose that* $x = \sum_{I\in\mathcal{C}} f_I[x_{i_1}|\cdots|x_{i_m}]$ *is a cycle of* $\overline{B}_m(Q)_m$. *For each* $K = (k_1,\dots,k_q) \in \mathbb{Z}^q$ *and* $K' = (k_{q+1},\dots,k_m) \in \mathbb{Z}^{m-q}$, *let*

$$x_K = \sum_I f_I[x_{i_{q+1}}|\cdots|x_{i_m}],$$

where the summation runs over the labels $I \in \mathcal{C}$ *such that* $i_1 = k_1,\dots,$ $i_q = k_q$, *and*

$$x_{K'} = \sum_I f_I[x_{i_1}|\cdots|x_{i_q}],$$

where the summation runs over the labels $I \in \mathcal{C}$ *such that* $i_{q+1} = k_{q+1},\dots,$ $i_m = k_m$. *Then* x_K *is a cycle of* $\overline{B}_{m-q}(Q)_{m-q}$ *and* $x_{K'}$ *is a cycle of* $\overline{B}_q(Q)_q$.

Proof. We have

$$\begin{aligned}\partial x_{K'} &= \sum_{j=1}^{q-1} \sum_{(i_{q+1},\dots,i_m)=K'} f_I[x_{i_1}|\cdots|x_{i_j}x_{i_{j+1}}|\cdots|x_{i_q}]\\ &= \sum_{j=1}^{q-1}\sum [x_{l_1}|\cdots|x_{l_{j-1}}|\sum_I f_I x_{i_j}x_{i_{j+1}}|x_{l_{j+2}}|\cdots|x_{l_q}],\end{aligned}$$

where the second summation is over all $(l_1,\dots,l_{j-1}) \in \mathbb{Z}^{j-1}$, $(l_{j+1},\dots,l_q)$ $\in \mathbb{Z}^{q-j}$ and the innermost summation is over all I such that

$$(i_1,\dots,i_{j-1}) = (l_1,\dots,l_{j-1}),$$
$$(i_{j+2},\dots,i_q,i_{q+1},\dots,i_m) = (l_{j+2},\dots,l_q,k_{q+1},\dots,k_m).$$

Since $\partial x = 0$, Theorem 1 implies that each of the summations over I is zero. Hence $\partial x_{K'} = 0$. A similar argument shows $\partial x_K = 0$. This completes the proof. ∎

EXAMPLE 1. Set

$$R_{k,n} = x_{2k-1-n}x_k + \sum_j \binom{n-1-j}{j} x_{2k-1-j}x_{k-n+j};$$

then $\{R_{k,n}\}_{k\in\mathbb{Z},n\in\mathbb{N}_0}$ is a (linear) $\mathbb{F}_2$-basis of $D_2(Q)$.

EXAMPLE 2. The chain $z = [x_3|x_3|x_2] + [x_5|x_1|x_2] + [x_5|x_3|x_0]$ is a cycle of $\overline{B}_3(Q)_3$ since $x_3x_2 = 0$, $x_1x_2 = x_3x_0$, $x_3x_3 = x_5x_1$, $x_5x_3 = 0$. More generally, for each $h \in \mathbb{Z}$,

$$z(h) = [x_{4h-5}|x_{2h-1}|x_h] + [x_{4h-3}|x_{2h-3}|x_h] + [x_{4h-3}|x_{2h-1}|x_{h-2}]$$

is a cycle of $\overline{B}_3(Q)_3$, as $x_{2k-1}x_k = 0$, $x_{2k-2}x_k = x_{2k-1}x_{k-1}$, $x_{2k-3}x_k = x_{2k-1}x_{k-2}$. Another example is $z' = [x_{4h-3}|x_{2h-1}|x_h]$, since $4h-3 = 2(2h-1)-1$. More generally, for a fixed $k \in \mathbb{N}_0$ and for each $m \in \mathbb{Z}$,

$$z'(m) = [x_{2^k(m-1)+1}|x_{2^{k-1}(m-1)+1}|\cdots|x_{2(m-1)+1}|x_m]$$

is a cycle of $\overline{B}_{k+1}(Q)_{k+1}$.

EXAMPLE 3. Using the relations $x_7x_5 = x_9x_3$ and $x_6x_6 = x_{11}x_1 + x_{10}x_2 + x_9x_3$, we can exhibit the following cycle in $\overline{B}_4(Q)_4$:

$$z'' = [x_7|x_5|x_3|x_2] + [x_9|x_3|x_3|x_2] + [x_9|x_5|x_1|x_2] + [x_9|x_5|x_3|x_0].$$

Corollary 2 allows us to endow the diagonal homology of Q with a coalgebra structure described in the following statement.

THEOREM 3. *For each non-negative integer m, let ψ_m be the map*

$$\psi_m : D_m(Q) \to \bigoplus_{p+q=m} (D_p(Q) \otimes D_q(Q)),$$

$$x = \sum_I f_I[x_{i_1}|x_{i_2}|\cdots|x_{i_m}] \mapsto x \otimes 1 + 1 \otimes x + \sum x' \otimes x'',$$

where the cycles x' and x'' are obtained by splitting all the summands of x and suitably grouping the common terms. Then the sequence $\{\psi_m\}_{m\in\mathbb{N}_0}$ defines a coproduct in $D_*(Q)$.

In other words, the coalgebra structure of $D_*(Q)$ is the obvious one.

EXAMPLE 4. If x is the cycle in $\overline{B}_4(Q)_4$ of Example 3, then

$$\begin{aligned}\psi(x) = {} & [x_7] \otimes [x_5|x_3|x_2] + [x_9] \otimes ([x_3|x_3|x_2] + [x_5|x_1|x_2] + [x_5|x_3|x_0]) \\ & + ([x_7|x_5|x_3] + [x_9|x_3|x_3] + [x_9|x_5|x_1]) \otimes [x_2] \\ & + [x_9|x_5|x_3] \otimes [x_0] + ([x_7|x_5] + [x_9|x_3]) \otimes [x_3|x_2] \\ & + [x_9|x_5] \otimes ([x_1|x_2] + [x_3|x_0]).\end{aligned}$$

It is quite obvious that the groups of the form $H_{s,t}(Q)$, with $s > t$, all vanish, while it is still unclear what happens when $s < t$. But at least we know that $H_{1,2}(Q) = 0$, since each cycle of the form

$$\sum_{(i_1,i_2)\in\mathcal{D}} f_{(i_1,i_2)}[x_{i_1}x_{i_2}] \in \overline{B}_1(Q)_2$$

(where $\mathcal{D}$ is a suitable set of indices) is the boundary of the chain

$$\sum_{(i_1,i_2)\in\mathcal{D}} f_{(i_1,i_2)}[x_{i_1}|x_{i_2}] \in \overline{B}_2(Q)_2.$$

2. Invariant theory. For each $n \geq 1$, let $P_n = \mathbb{F}_2[t_1, \ldots, t_n]$ be the polynomial ring in n indeterminates, which are assigned degree 1, and let $Q_{n,0}$ be its Euler class, that is, the product of all the elements of degree 1 in P_n. There is a standard action of the general linear group $\mathrm{GL}_n(\mathbb{F}_2)$ and its upper triangular subgroup T_n on P_n, and we consider the following rings of invariants:

$$P_n^{T_n} = \mathbb{F}_2[V_1, \ldots, V_n], \qquad P_n^{\mathrm{GL}_n} = \mathbb{F}_2[Q_{n,0}, Q_{n,1}, \ldots, Q_{n,n-1}]$$

and their localizations $\varDelta_n$ and $\varGamma_n$ obtained by formally inverting the Euler class,

$$\varDelta_n = P_n^{T_n}[Q_{n,0}^{-1}] = \mathbb{F}_2[V_1^{\pm 1}, \dots, V_n^{\pm 1}],$$
$$\varGamma_n = P_n^{\mathrm{GL}_n}[Q_{n,0}^{-1}] = \mathbb{F}_2[Q_{n,0}^{\pm 1}, Q_{n,1}, \dots, Q_{n,n-1}].$$

Here we follow the notation of [6] and, as in [6], we set

$$\varDelta = \bigoplus_{n\geq 0} \varDelta_n, \qquad \varGamma = \bigoplus_{n\geq 0} \varGamma_n,$$

where $\varDelta_0 = \varGamma_0 = \mathbb{F}_2$. As shown in [6], $\varDelta$ is a coalgebra with comultiplication $\psi : \varDelta \to \varDelta \otimes \varDelta$ induced by the maps

$$\psi_{p,q} : \varDelta_{p+q} \to \varDelta_p \otimes \varDelta_q \quad (p, q \in \mathbb{N}_0)$$

defined by setting

$$\psi_{p,q}(v_1^{i_1} \cdots v_{p+q}^{i_{p+q}}) = v_1^{i_1} \cdots v_p^{i_p} \otimes v_1^{i_{p+1}} \cdots v_q^{i_{p+q}}$$

and $\varGamma$ is a subcoalgebra of $\varDelta$. As an example, if we set $n = p + q$, we have

$$\psi_{p,q}(Q_{n,s}) = \sum_{j\geq 0} Q_{p,0}^{2^q-2^j} Q_{p,s-j}^{2^j} \otimes Q_{q,j}.$$

We now borrow some ideas from N. H. V. Hung ([2]) and apply them to Q. Let us consider the $\mathbb{F}_2$-linear maps

$$\pi_{m,q} : \varDelta_m \to \overline{B}_{m-1}(Q)_m \qquad (m \geq 2,\ q = 1, \dots, m-1)$$

defined by setting

$$\pi_{m,q}(v_1^{i_1} \cdots v_m^{i_m}) = [x_{i_1+1}|\cdots|x_{i_{q-1}+1}|x_{i_q+1}x_{i_{q+1}+1}|\cdots|x_{i_m+1}].$$

We remark that, in particular, $\ker \pi_{2,1} = \varGamma_2$. In fact, as shown in [3] (though in a slightly different context), the elements of $\varGamma_2$ correspond to the generalized Adem relations under the map $\pi_{2,1}$. For example

$$\pi_{2,1}(Q_{2,0}^n) = \pi_{2,1}(v_1^{2n}v_2^n) = x_{2n+1}x_{n+1} = 0,$$
$$\pi_{2,1}(Q_{2,1}) = \pi_{2,1}(v_1^2 + v_1v_2) = x_3x_1 + x_2x_2 = 0.$$

Lemma 4. *For any $m \geq 2$ and $q = 1, \dots, m-1$, we have*

$$\ker \pi_{m,q} = \varDelta_{q-1} \otimes \varGamma_2 \otimes \varDelta_{m-q-1}.$$

Proof. We consider the map

$$f = (\psi_{q-1,2} \otimes 1) \circ \psi_{q+1,m-q-1} : \varDelta_m \to \varDelta_{q-1} \otimes \varDelta_2 \otimes \varDelta_{m-q-1}$$

and define, for each $t \in \mathbb{N}$,

$$\alpha_t : \varDelta_t \to \overline{B}_t(Q)_t, \qquad v_1^{i_1} \cdots v_t^{i_t} \mapsto [x_{i_1+1}|\cdots|x_{i_t+1}].$$

Clearly

$$\pi_{m,q} = (\alpha_{q-1} \otimes \pi_{2,1} \otimes \alpha_{m-q-1}) \circ f,$$

and the maps f, α_t are both linear $\mathbb{F}_2$-isomorphisms. The result now follows. ■

PROPOSITION 5. *For each $m \geq 2$, we have*

$$\bigcap_{q=1}^{m-1} \ker \pi_{m,q} = \Gamma_m.$$

Proof. We simply use the fact that $\mathrm{GL}_m(\mathbb{F}_2)$ is generated by all the matrices of the form

$$B = \begin{pmatrix} I_{q-1} & O & O \\ O & A & O \\ O & O & I_{m-q-1} \end{pmatrix},$$

where $A \in \mathrm{GL}_2(\mathbb{F}_2)$. Thus

$$\Gamma_m = \bigcap_{q=1}^{m-1} \Delta_{q-1} \otimes \Gamma_2 \otimes \Delta_{m-q-1} = \bigcap_{q=1}^{m-1} \ker \pi_{m,q}$$

as claimed. ■

We now look at the commutative diagram

$$\begin{tikzcd} & \overline{B}_m(Q)_m \arrow[dl, "\beta_m"'] \arrow[d, "\partial_{m,q}"] \\ \Delta_m \arrow[r, "\pi_{m,q}"'] & \overline{B}_{m-1}(Q)_m \end{tikzcd}$$

where β_m and $\partial_{m,q}$ are defined by setting

$$\beta_m[x_{j_1}|\cdots|x_{j_m}] = v_1^{j_1-1}\cdots v_m^{j_m-1}$$

and

$$\partial_{m,q}[x_{j_1}|\cdots|x_{j_m}] = [x_{j_1}|\cdots|x_{j_q}x_{j_{q+1}}|\cdots|x_{j_m}].$$

We remark that the β_m's induce a map of coalgebras

$$\beta : \bigoplus_m \overline{B}_m(Q)_m \to \Delta.$$

Clearly each β_m is a linear isomorphism. We are now ready to state and prove the following result.

THEOREM 6. *The diagonal homology $D_*(Q)$ of the algebra Q and the subcoalgebra Γ of Δ are isomorphic as coalgebras.*

Proof. We have already pointed out that a chain $x \in \overline{B}_m(Q)_m$ is a cycle (i.e. the unique representative of an element in $D_m(Q)$) if and only if $\partial_{m,q}(x) = 0$ for each q, i.e. if and only if $\pi_{m,q}\beta_m(x) = 0$ for each q, i.e. if and only if $\beta_m(x) \in \Gamma_m$. Therefore β_m induces, by restriction, an isomorphism

$$\overline{\beta}_m : D_m(Q) = H_{m,m}(Q) = \ker \partial_m \to \Gamma_m$$

of coalgebras. ■

EXAMPLE 5. Another basis for $D_2(Q)$ is given by the set

$$\mathcal{B} = \{c_{s,k} \mid k \in \mathbb{Z},\ s \in \mathbb{N}_0\},$$

where

$$c_{s,k} = \sum_{i=0}^{s} \binom{s}{i} [x_{2k-1-i} | x_{k-s+i}].$$

These cycles correspond to the elements $Q_{2,0}^{k-s-1} Q_{2,1}^{s}$, which form a basis of Γ_2. The cycles $z(h) \in D_3(Q)$ and $z'(m) \in D_{k+1}(Q)$ of Example 2 correspond to the invariants $Q_{3,0}^{h-3} Q_{3,1}^{2}$ and $Q_{k+1,0}^{m-1}$ respectively. Finally, the invariant $Q_{4,0}^{-1} Q_{4,1}^{2}$ corresponds to the cycle z'' of Example 3.

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