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Virtual biquandles

by

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Abstract. We describe new approaches for constructing virtual knot invariants. The main background of this paper comes from formulating and bringing together the ideas of biquandle [KR], [FJK], the virtual quandle [Ma2], the ideas of quaternion biquandles by Roger Fenn and Andrew Bartholomew [BF], the concepts and properties of long virtual knots [Ma10], and other ideas in the interface between classical and virtual knot theory. In the present paper we present a new algebraic construction of virtual knot invariants, give various presentations of it, and study several examples. Several conjectures and unsolved problems are presented throughout the paper.

1. INTRODUCTION

1.1. Background. Virtual knot theory was proposed by Kauffman in 1996; see [KaV]. The combinatorial notion of virtual knot (1) is defined as an equivalence class of 4-valent plane diagrams (4-regular plane graphs with extra structure) where a new type of crossing (called virtual) is allowed. This theory can be regarded as a "projection" of knot theory in thickened surfaces $S_g \times \mathbb{R}$ (as studied in [JKS]). Regarded from this point of view, virtual crossings appear as artifacts of the diagram projection from S_g to $\mathbb{R}^2$. However, the rules for handling the virtual diagrams are motivated (in [KaV]) by the idea that one can generalize the notion of a knot diagram to its oriented Gauss code.

A Gauss code for a knot is a list of crossings encountered on traversing the knot diagram, with the signs of the crossings indicated, and whether they are over or under in the course of the traverse. Each crossing is encountered twice in such a traverse, and thus the Gauss code has each crossing label

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(1) In what follows, we use the generic term "knot" for both knots and links, unless otherwise specified.

appearing twice in the list. One can define Reidemeister moves on the Gauss codes, and thus abstract the knot theory from its planar diagrams.

Virtual knot theory is the theory of such Gauss codes, not necessarily realizable in the plane. When one takes such a non-realizable code, and attempts to draw a planar diagram, virtual crossings are needed to complete the connections in the plane. These crossings are artifacts of the planar projection. The rules for handling virtual knot diagrams are designed to make the representation of the virtual knot independent of the particular choice of virtual crossings that realizes the diagram. It turns out that these rules describe embeddings of knots and links in thickened surfaces, stabilized by the addition and subtraction of empty handles (i.e. the addition and subtraction of thickened 1-handles from the surface that do not have any part of the knot or link embedded in them) [KaV2, KaV4, Ma1, Ma8, Ma9, CKS, Kup].

Another approach to Gauss codes for knots and links is the use of Gauss diagrams as in [GPV]. In that paper by Goussarov, Polyak and Viro, the virtual knot theory, taken as all Gauss diagrams up to Reidemeister moves, was used to analyze the structure of Vassiliev invariants for classical and virtual knots. In both [KaV] and [GPV] it is proved that if two classical knots are equivalent in the virtual category [Kup], then they are equivalent in the classical category. Thus classical knot theory is properly embedded in virtual knot theory.

To date, many invariants of classical knots have been generalized to the virtual case; see [GPV, KaV, KR, Ma1, Ma2, Ma8, Ma9, Saw, SW]. In many cases, a classical invariant extends to an invariant of virtuals. In some cases one has an invariant of virtuals that is an extension of ideas from classical knot theory, but vanishes or is otherwise trivial for classical knots. An example of this is the polynomial invariant studied by Sawollek [Saw], by Silver and Williams [SW] and by Kauffman and Radford [KR] ([2]). This invariant is produced by the methods that give the classical Alexander polynomial, but it is an example of a zeroth order Alexander polynomial, and is trivial in the classical case and non-trivial in the virtual case. Such invariants are valuable for the study of virtual knots, since they promise the possibility of distinguishing classical from virtual knots in key cases. Other examples of this phenomenon can be found in [Ma2, Ma5].

On the other hand, some invariants evaluated on classical knots coincide with well known classical knot invariants (see [KaV, KaV2, KaV4, Ma3] on generalizations of the Jones polynomial, fundamental group, quandle and quantum link invariants). These invariants exhibit interesting phenomena on virtual knots and links: for instance, there exists a virtual knot K with "fundamental group" isomorphic to $\mathbb{Z}$ and Jones polynomial not equal to 1.

([2]) Later, we will describe another approach [Ma3] that leads to the same results.

This phenomenon immediately implies that the knot K is not classical, and underlines the difficulty of extracting the Jones polynomial from the fundamental group in the classical case. We know in principle that the fundamental group, plus peripheral information, determines the knot itself in the classical case. It is not known how to extract the Jones polynomial from this algebraic information.

Note that, for classical knots, the quandle is a generalization of the fundamental group with a geometric interpretation. In this paper we consider quandles of virtual knots, defined formally in terms of their diagrams. However, the formally defined fundamental group of a virtual knot can be interpreted as the fundamental group of the complement of the virtual knot in the one-point suspension of a thickened surface where this knot is presented.

Another phenomenon that does not appear in the classical case is long knots [Ma10]: if we break a virtual knot diagram at two different points and take them to the infinity, we may obtain two different long knots. We will discuss this subject later in the text.

Beyond the fundamental group and the quandle, there are two algebraic constructions defining virtual knot invariants: the biquandle [KaV2, KR, FJK] and the virtual quandle [Ma2]. Each of them has a number of realizations (representations). In this paper, we bring these two ideas together, and present a combined construction that allows the extraction of new algebraic invariants of virtual knots.

The general strategy of this paper is the following: One presents an algebraically defined invariant of knots (virtual knots, long knots, etc.). This invariant is defined axiomatically (the axioms correspond to invariance under diagrammatic moves). In this form, the invariant appears to have strength, but is difficult to work with. In order to manage the invariant, we may take its representation, say, I, into some category (e.g. groups) with operations defined in terms of that representation category. In this manner we can obtain, for example, the knot group and its generalizations. Or one can find a finite object (or category) G with operations satisfying the initial axioms. Then the set of homomorphisms $I(K) \to G$ is an invariant of the knot K. In particular, so is the *cardinality* of the set of such homomorphisms. Such homomorphisms can also be called *colorings* because they correspond to colorings of the diagram arcs (generators of $I(K)$) by elements of G.

In some cases, polynomial-type invariants emerge naturally from the algebra. In other cases it is useful to use algebraic and polynomial-type invariants together to extract information.

This paper does not pretend to describe all directions of virtual knot theory. For instance, we say a little about the Jones polynomial and do not describe its generalizations, the Khovanov complex [Ma11], the surface-bracket polynomial [KaV4], and the Ξ-polynomial [Ma3, Ma6]. Also, we do not touch the Vassiliev invariants for virtual links [KaV, GPV, Ma9]. All these concepts will be described in the book [Vbook] by the authors. Also, a list of unsolved problems concerning virtual knots can be found in [FKM].

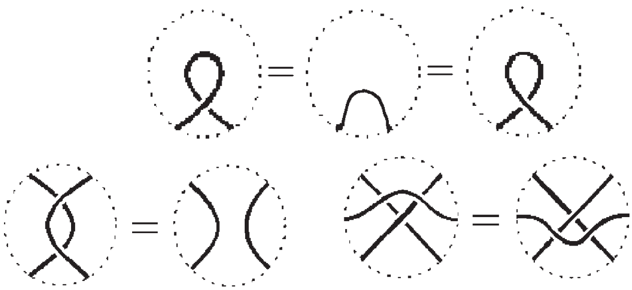
1.2. Basic definitions. We begin with the definition of a virtual knot according to [KaV].

Definition 1. A *virtual link diagram* is a 4-valent graph on the plane such that each crossing of it is either classical (i.e., one pair of opposite edges is selected to make an overcrossing; the other pair forms an undercrossing) or virtual (just marked by a circle).

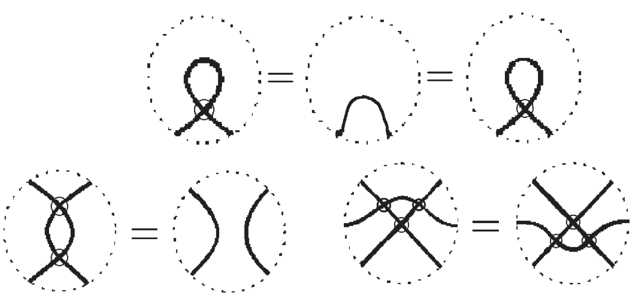
Definition 2. A *virtual knot* is an equivalence class of virtual knot diagrams modulo generalized Reidemeister moves.

The set of virtual Reidemeister moves consists of:

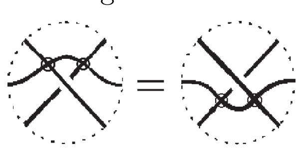
1. classical Reidemeister moves:



2. virtual versions of these moves (where all classical crossings are replaced with virtual ones):



3. the "semivirtual" version of the third Reidemeister move where two virtual crossings pass through a classical crossing:



The analogous version with two classical crossings and one virtual crossing is forbidden. There are two versions of these forbidden moves shown in Fig. 1.

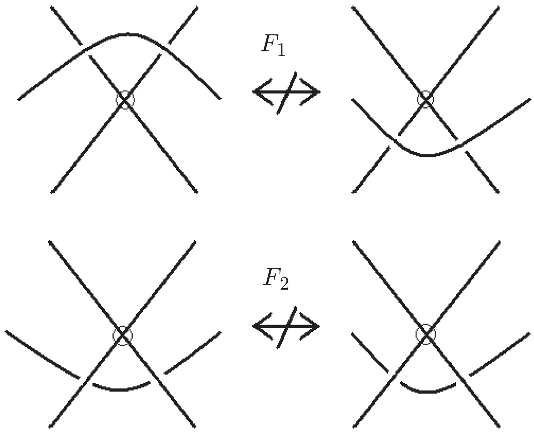


Fig. 1. The forbidden moves

It is proved in [GPV] (see also S. Nelson [Nel] and T. Kanenobu [Kan]) that if we include these two moves, each knot will be equivalent to the unknot. If we add only first of them (F_1), we will obtain what is called "welded knots" (see [FRR] and [Satoh]). In [FRR] Fenn, Rimányi and Rourke introduced the notion of the "welded braid group". This version of braids is the same as virtual braids with the forbidden move F_1 added in braid form. Satoh in [Satoh] showed that the concept of welded knots could be interpreted in terms of embeddings of tori in Euclidean four-space. This also leads to an interpretation of welded braids in terms of braidings of tubes in four-space.

One can easily consider *oriented virtual links* by endowing diagrams with an orientation.

As with classical knots, virtual knots can be obtained as closures of *virtual braids* first mentioned in the talk by Kauffman [Kau0], and in [KaV, KaV2, Ma1, KL1, KL2]. See also Vershinin [Ver] and Kamada [Kam]. Virtual braids are analogues of classical braids where some crossings are allowed to be virtual (and marked by a circle). The corresponding n-strand group has generators $\sigma_1, \ldots, \sigma_{n-1}$ (as in the classical case) and "virtual generators" $\zeta_1, \ldots, \zeta_{n-1}$ with obvious relations $\zeta_i^2 = 1$ (more details in [Ver, KaV, KL1]). The closure procedure is quite analogous to that for classical knots.

REMARK 1. All virtual (and classical) knots are oriented, unless otherwise specified.

The geometric approach to virtual knots is based on the following fact: virtual knots are isotopy classes of curves in “thickened surfaces” $S_g \times I$ up to “stabilization”, i.e. adding and removing handles. See the discussion of this viewpoint given earlier in this introduction.

2. BASIC CONSTRUCTION

Here we are going to recall the construction of the quandle and generalize it to the case of virtual knots.

2.1. The case of classical knots. For the classical case, there is a complete algebraic invariant of knots (complete up to reversed mirror images, and actually complete if one adds longitudinal information to the system), first proposed by D. Joyce [Joy] and S. V. Matveev [Mat]. Here is a sketch of the quandle construction. One considers a classical knot diagram, and encodes all arcs of it by letters. We use a formal algebraic structure with the operation $\circ$ for which the following condition holds: For each crossing X at which the arc a (we *do not look at its orientation*) lies on the right hand of the overcrossing oriented arc b and the arc c lies on the left hand, we write the formal relation

$$a \circ b = c. \tag{1}$$

It can be checked that the invariance of this structure (a formal algebra given by generators and relations) under the three Reidemeister moves implies the following algebraic axioms:

1. Idempotence: $\forall a : a \circ a = a$.
2. The existence of left inverses: $\forall b, c \,\exists! x : x \circ b = c$. This x is denoted by $c \,\overline{\circ}\, b$.
3. Right self-distributivity: $\forall a, b, c : (a \circ b) \circ c = (a \circ c) \circ (b \circ c)$.

An algebraic structure satisfying these axioms is called a *quandle*. The quandle of an oriented knot or link is defined by generators and relations as above (plus the imposition of these axioms).

Now, this can be easily generalized [KaV] to the case of virtual knots: we allow arcs to pass through a virtual crossing and ignore the virtual intersections.

It is straightforward to check that this “virtualization” of the quandle is an invariant of virtual knots. Since the quandle plus longitudinal information is a complete invariant of classical knots, and since this information is preserved under virtual equivalence, one can conclude [KaV, GPV] that two classical knot diagrams are equivalent in the virtual category if and only if they are equivalent in the classical category.

2.2. Biquandles and a generalized Alexander polynomial $G_K(s,t)$. The *biquandle* [KaV2, FJK, CS] is an algebra associated with the diagram that is invariant (up to isomorphism) under the generalized Reidemeister moves for virtual knots and links. The operations in this algebra are motivated by the formation of labels for the edges of the diagram and the intended invariance under the moves. We will give the abstract definition of the biquandle after a discussion of these knot-theoretic issues. View Figure 2.

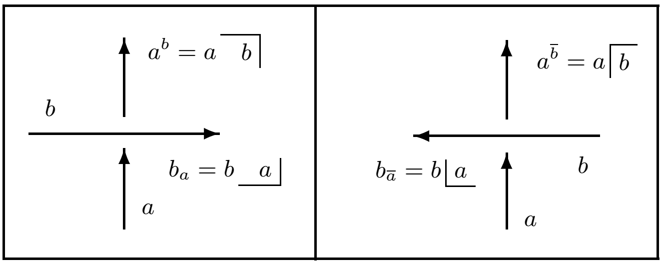


Fig. 2. Biquandle relations at a crossing

In this figure we have shown the format for the operations in a biquandle. The overcrossing arc has two labels, one on each side of the crossing. In a *biquandle* there is an algebra element labeling each *edge* of the diagram. An edge of the diagram corresponds to an edge of the underlying plane graph of that diagram.

Let the edges oriented toward a crossing in a diagram be called the *input* edges for the crossing, and the edges oriented away from the crossing be called the *output* edges for the crossing. Let a and b be the input edges for a positive crossing, with a the label of the undercrossing input and b the label on the overcrossing input. Then in the biquandle, we label the undercrossing output by

$$c = a^b$$

just as in the case of the quandle, but the overcrossing output is labeled

$$d = b_a.$$

We usually read a^b as "the undercrossing line a is acted upon by the overcrossing line b to produce the output $c = a^b$". In the same way, we can read b_a as "the overcrossing line b is operated on by the undercrossing line a to produce the output $d = b_a$". The biquandle labels for a negative crossing are similar but with an overline (denoting an operation of order two) placed on the letters. Thus in the case of the negative crossing, we would write

$$c = a^{\bar b}, \qquad d = b_{\bar a}.$$

To form the biquandle, $\mathrm{BQ}(K)$, we take one generator for each edge of the diagram and two relations at each crossing (as described above). This system of generators and relations is then regarded as encoding an algebra that is generated freely by the biquandle operations as concatenations of these symbols and subject to the biquandle algebra axioms. These axioms (which we will describe below) are a transcription in the biquandle language of the requirement that this algebra be invariant under Reidemeister moves on the diagram.

Another way to write this formalism for the biquandle is as follows:

$$a^b = a\,\overline{\;b\,}\rceil, \qquad a_b = a\,\underline{\;b\,}\rfloor, \qquad a^{\bar{b}} = a\lceil\overline{\,b\;}, \qquad a_{\bar{b}} = a\lfloor\underline{\,b\;}.$$

We call this the *operator formalism* for the biquandle. The operator formalism has advantages when one is performing calculations, since it is possible to maintain the formulas on a line rather than extending them up and down the page as in the exponential notation. On the other hand the exponential notation has intuitive familiarity and is good for displaying certain results.

The axioms for the biquandle are exactly the rules needed for invariance of this structure under the Reidemeister moves. Note that in analyzing invariance under Reidemeister moves, we visualize representative parts of link diagrams with biquandle labels on their edges. The primary labeling occurs at a crossing. At a positive crossing with over input b and under input a, the under output is labeled $a\,\overline{\;b\,}\rceil$ and the over output is labeled $b\,\underline{\;a\,}\rfloor$. At a negative crossing with over input b and under input a, the under output is labeled $a\lceil\overline{\,b\;}$ and the over output is labeled $b\lfloor\underline{\,a\;}$. At a virtual crossing there is no change in the labeling of the lines that cross one another.

Remark 2. Later in this paper, we shall generalize the biquandle to include operations at the virtual crossings.

Remark 3. A remark is in order about the relationship of the operator notations with the usual conventions for binary algebraic operations. Let $a*b = a^b = a\,\overline{\;b\,}\rceil$. We are asserting that the biquandle comes equipped with four binary operations of which one is $a*b$. Here is how these notations are related to the usual parenthesizations:

1. $(a*b)*c = (a^b)^c = a^{bc} = a\,\overline{\;b\,}\rceil\overline{\;c\,}\rceil$
2. $a*(b*c) = a^{b^c} = a\,\overline{\;b\;\overline{\;c\,}\rceil}\rceil$

From this the reader should see that the exponential and operator notations allow us to express biquandle equation with a minimum of parentheses.

In Figure 3 we illustrate the effect of these conventions and how they lead to the following algebraic transcription of the directly oriented second Reidemeister move:

$$a = a\,\overline{\;b\,}\rceil\lceil\overline{\,b\;\underline{\;a\,}\rfloor} \quad \text{or} \quad a = a^{b\overline{b_a}},$$

$$b = b\,\underline{\;a\,}\rfloor\lfloor\underline{\,a\;\overline{\;b\,}\rceil} \quad \text{or} \quad b = b_{a\overline{a^b}}.$$

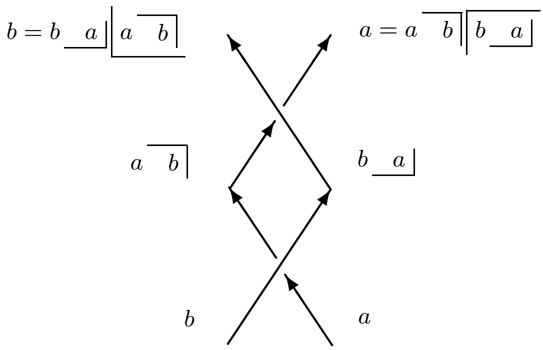
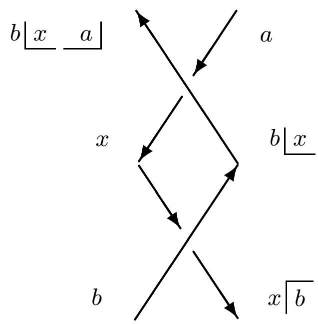


Fig. 3. Direct two move

The reverse oriented second Reidemeister move gives a different sort of identity, as shown in Figure 4. For the reverse oriented move, we must assert



$$\exists x \ni x = a\,\overline{b\lfloor\underline{x}}\rceil,\; a = x\lceil\overline{b} \text{ and } b = b\lfloor\underline{x}\;\underline{a}\rfloor$$

Fig. 4. Reverse two move

that given elements a and b in the biquandle, there exists an element x such that

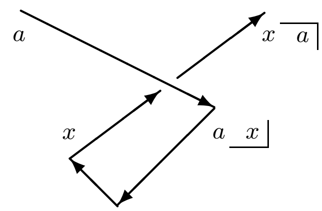
$$x = a\,\overline{b\lfloor\underline{x}}\rceil, \qquad a = x\lceil\overline{b}, \qquad b = b\lfloor\underline{x}\;\underline{a}\rfloor.$$

By reversing the arrows in Figure 4 we obtain a second statement for invariance under the type two move, saying the same thing with the operations reversed: Given elements a and b in the biquandle, there exists an element x such that

$$x = a\lceil\overline{b\,\overline{x}\rceil}, \qquad a = x\,\overline{b}\rceil, \qquad b = b\,\overline{x}\rceil\lfloor\underline{a}.$$

There is no necessary relation between the x in the first statement and the x in the second statement.

These assertions about the existence of x can be viewed as asserting the existence of fixed points for a certain operator. In this case such an operator is $F(x) = a\,\overline{b\,\underline{x}\rfloor}\rceil$. It is characteristic of certain axioms in the biquandle that they demand the existence of such fixed points. Another example is the axiom corresponding to the first Reidemeister move (one of them) as illustrated in Figure 5. This axiom states that given an element a in the biquandle, there exists an x in the biquandle such that $x = a\,\underline{x}\rfloor$ and that $a = x\,\overline{a}\rceil$. In this case the operator is $G(x) = a\,\underline{x}\rfloor$.



$\exists x \ni x = a\,\underline{x}\rfloor$ and $a = x\,\overline{a}\rceil$

Fig. 5. First move

It is unusual that an algebra would have axioms asserting the existence of fixed points with respect to operations involving its own elements. We plan to take up the study of this aspect of biquandles in a separate publication. For now it is worth remarking that a slight change in the axiomatic structure allows an easy definition of the free biquandle. The idea is this: Suppose that one has an axiom that states the existence of an x such that $x = a\,\underline{x}\rfloor$ for each a. Then we change the statement of the axiom by adding a new operation (unary in this case) to the algebra, call it $\mathrm{Fix}(a)$, such that $\mathrm{Fix}(a) = a\,\underline{\mathrm{Fix}(a)}\rfloor$. Existence of the fixed point follows from this property of the new operation, and we can describe the free biquandle on a set by taking all finite biquandle expressions in the elememts of the set, modulo these revised axioms for the biquandle.

The biquandle relations for invariance under the third Reidemeister move are shown in Figure 6. The version of the third Reidemeister move shown in this figure yields the following algebraic relations:

$$a\,\overline{b}\rceil\,\overline{c}\rceil = a\,\overline{c\,\underline{b}\rfloor}\rceil\,\overline{b\,\overline{c}\rceil}\rceil \quad \text{or} \quad a^{bc} = a^{c_b b^c},$$

$$c\,\underline{b}\rfloor\,\underline{a}\rfloor = c\,\underline{a\,\overline{b}\rceil}\rfloor\,\underline{b\,\underline{a}\rfloor}\rfloor \quad \text{or} \quad c_{ba} = c_{a^b b_a},$$

$$b\,\underline{a}\rfloor\,\overline{c\,\underline{a\,\overline{b}\rceil}\rfloor}\rceil = b\,\overline{c}\rceil\,\underline{a\,\overline{c\,\underline{b}\rfloor}\rceil}\rfloor \quad \text{or} \quad (b_a)^{c_{a^b}} = (b^c)_{a^{c_b}}.$$

The reader will note that if we replace the diagrams of Figure 6 with diagrams with all negative crossings then we will get a second triple of equations

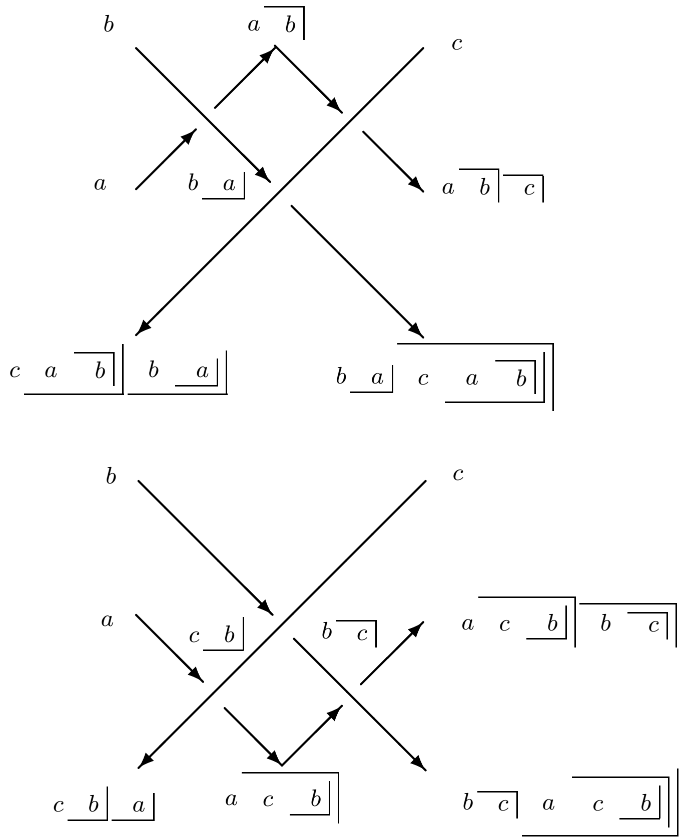


Fig. 6. Third move

identical to those above but with all right operator symbols replaced by the corresponding left operator symbols (equivalently: with all exponent literals replaced by their barred versions). Here are the operator versions of these equations; we refrain from writing the exponential versions because of the prolixity of barred variables.

$$a\overline{|b}\,\overline{|c} = a\overline{|c\,\underline{|b}}\,\overline{|b\,\overline{|c}}\,,$$

$$c\underline{|b}\,\underline{|a} = c\underline{|a\,\overline{|b}}\,\underline{|b\,\underline{|a}}\,,$$

$$b\underline{|a}\,\overline{|c\,\underline{|a\,\overline{|b}}} = b\overline{|c}\,\underline{|a\,\overline{|c\,\underline{|b}}}\,.$$

We now have a complete set of axioms, for it is a fact (see, e.g., [Knots, Ma1]) that the third Reidemeister move with the orientation shown in Figure 6 and either all positive crossings (as shown in that figure) or all negative crossings, is sufficient to generate all the other cases of third Reidemeister move just so long as we have both oriented forms of the second Reidemeister move. Consequently, we can now give the full definition of the biquandle.

Definition. A *biquandle* B is a set with four binary operations indicated by the conventions we have explained above: a^b , $a^{\bar{b}}$, a_b , $a_{\bar{b}}$. We shall refer to the operations with barred variables as the *left* operations and the operations without barred variables as the *right* operations. The biquandle is closed under these operations and the following axioms are satisfied:

1. For any elements a and b in B we have
$$a = a^{b\overline{b_a}}, \qquad b = b_{a\overline{a^b}}, \qquad a = a^{\bar{b}b_{\bar{a}}}, \qquad b = b_{\bar{a}a^{\bar{b}}}.$$
2. Given elements a and b in B, there exists an element x such that
$$x = a^{b_{\bar{x}}}, \qquad a = x^{\bar{b}}, \qquad b = b_{\bar{x}a}.$$
Given elements a and b in B, there exists an element x such that
$$x = a^{\overline{b_x}}, \qquad a = x^{b}, \qquad b = b_{x\bar{a}}.$$
3. For any a, b, c in B the following equations hold and the same equations hold when all right operations are replaced in these equations by left operations:
$$a^{bc} = a^{c_b b^c}, \qquad c_{ba} = c_{a^b b_a}, \qquad (b_a)^{c_{a^b}} = (b^c)_{a^{c_b}}.$$
4. Given an element a in B, there exists an x in the biquandle such that $x = a_x$ and $a = x^a$. Given an element a in B, there exists an x in the biquandle such that $x = a^{\bar{x}}$ and $a = x_{\bar{a}}$.

These axioms are transcriptions of the Reidemeister moves. The first axiom transcribes the directly oriented second Reidemeister move. The second axiom transcribes the reverse oriented Reidemeister move. The third axiom transcribes the third Reidemeister move as we have described it in Figure 6. The fourth axiom transcribes the first Reidemeister move. Much more work is needed in exploring these algebras and their applications to knot theory.

2.3. The Alexander biquandle. In order to realize a specific example of a biquandle structure, suppose that
$$a\,\overline{\;\;b\,}| = ta + vb, \qquad \underline{\;\;b\,}| = sa + ub,$$
where a, b, c are elements of a module M over a ring R and $t, s, v,\ u$ are in R. We use invariance under the Reidemeister moves to determine relations among these coefficients.

Taking the equation for the third Reidemeister move discussed above, we have

$$\begin{aligned} a\,\overline{\,b}\rceil\,\overline{\,c}\rceil &= t(ta+vb)+vc = t^2a+tvb+vc, \\ a\,\overline{\,c\,\underline{\,b}\rfloor}\Big\rceil\,\overline{\,b\,\overline{\,c}\rceil}\Big\rceil &= t(ta+v(sc+ub))+v(tb+vc) \\ &= t^2a+tv(u+1)b+v(ts+v)c. \end{aligned}$$

From this we see that we have a solution to the equation for the third Reidemeister move if $u=0$ and $v=1-st$. Assuming that t and s are invertible, it is not hard to see that the following equations not only solve this single Reideimeister move, but they give a biquandle structure, satisfying all the axioms:

$$\begin{aligned} a\,\overline{\,b}\rceil &= ta+(1-st)b, & a\,\underline{\,b}\rfloor &= sa, \\ a\lceil\overline{b\,} &= t^{-1}a+(1-s^{-1}t^{-1})b, & a\lfloor\underline{b\,} &= s^{-1}a. \end{aligned}$$

Thus we have a simple generalization of the Alexander quandle and we shall refer to this structure, with the equations given above, as the *Alexander Biquandle*.

Just as one can define the Alexander Module of a classical knot, we have the Alexander Biquandle of a virtual knot or link, obtained by taking one generator for each *edge* of the knot diagram and taking the module relations in the above linear form. Let $\mathrm{ABQ}(K)$ denote this module structure for an oriented link K. That is, $\mathrm{ABQ}(K)$ is the module generated by the edges of the diagram, modulo the submodule generated by the relations. This module then has a biquandle structure specified by the operations defined above for an Alexander Biquandle. We first construct the module and then note that it has a biquandle structure. See Figures 7–9 for an illustration of the Alexander Biquandle labelings at a crossing.

For example, consider the virtual knot in Figure 8. This knot gives rise to a biquandle with generators a, b, c, d and relations

$$a = d\,\overline{\,b}\rceil, \quad c = b\,\underline{\,d}\rfloor, \quad d = c\,\overline{\,a}\rceil, \quad b = a\,\underline{\,c}\rfloor.$$

Writing these out in $\mathrm{ABQ}(K)$, we have

$$a = td+(1-st)b, \quad c = sb, \quad d = tc+(1-st)a, \quad b = sa.$$

Eliminating c and b and rewriting, we find

$$a = td+(1-st)sa, \quad d = ts^2a+(1-st)a.$$

Note that these relations can be written directly from the diagram as indicated in Figure 9 if we perform the lower biquandle operations directly on the diagram. This is the most convenient algorithm for producing the relations.

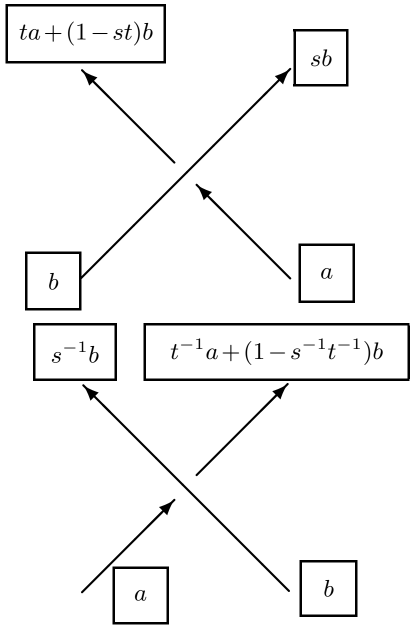


Fig. 7. Alexander Biquandle labeling at a crossing

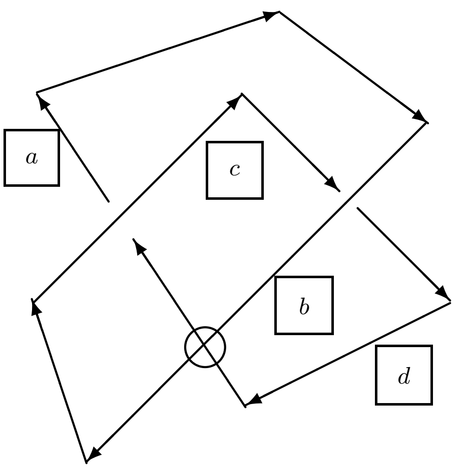


Fig. 8. A virtual knot fully labeled

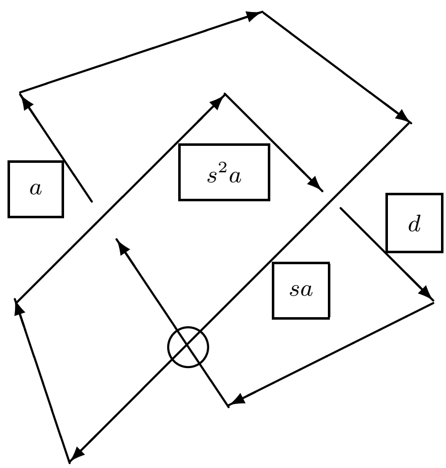


Fig. 9. A virtual knot with lower operations labeled

We can write these as a list of relations

$$(s - s^2t - 1)a + td = 0, \quad (s^2t + 1 - st)a - d = 0.$$

for the Alexander Biquandle as a module over $\mathbb{Z}[s, s^{-1}, t, t^{-1}]$. The relations can be expressed concisely with the matrix of coefficients of this system of equations:

$$M = \begin{bmatrix} s - s^2t - 1 & t \\ (s^2t + 1 - st) & -1 \end{bmatrix}.$$

The determinant of M is, up to multiples of $\pm s^i t^j$ for integers i and j, an invariant of the virtual knot or link K. We shall denote this determinant by $G_K(s,t)$ and call it the *generalized Alexander polynomial for* K.

A key fact about $G_K(s,t)$ is that $G_K(s,t) = 0$ *if* K *is equivalent to a classical diagram*. This is seen by noting that in a classical diagram one of the relations will be a consequence of the others. In this case we have

$$G_K = (1-s) + (s^2-1)t + (s-s^2)t^2,$$

which shows that the knot in question is non-trivial and non-classical.

Here is another example of the use of this polynomial. Let D denote the diagram in Figure 10. It is not hard to see that this virtual knot has

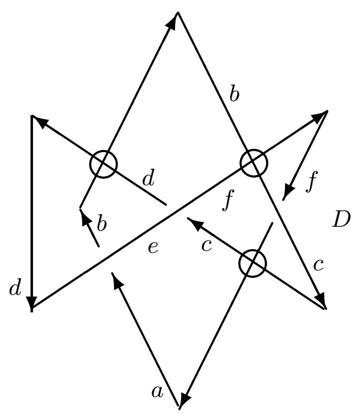


Fig. 10. Unit Jones polynomial, integer fundamental group

unit Jones polynomial, and that the fundamental group is isomorphic to the integers. The biquandle does detect the knottedness of D. The relations are

$$a\,\overline{\;d\;}\rceil = b, \quad d\,\underline{\;a\;}\rfloor = e, \quad c\,\overline{\;e\;}\rceil = d, \quad e\,\underline{\;c\;}\rfloor = f, \quad f\,\lceil\overline{\;b\;} = a, \quad b\,\lfloor\underline{\;f\;} = c,$$

from which we obtain the relations (eliminating c, e and f)

$$b = ta + (1-tv)d, \quad d = ts^{-1}b + (1-ts)sd, \quad a = t^{-1}s^2d + (1-t^{-1}s^{-1})b.$$

The determinant of this system is the generalized Alexander polynomial for D:

$$t^2(s^2-1) + t(s^{-1}+1-s-s^2) + (s-s^2).$$

This proves that D is a non-trivial virtual knot.

In fact the polynomial that we have computed is the same as the polynomial invariant of virtuals of Sawollek [Saw] and defined by an alternative method by Silver and Williams [SW] and, in a third way, by Manturov [Ma3], and given a state sum formulation by Kauffman and Radford [KR]. Sawollek defines a module structure essentially the same as our Alexander Biquandle. Silver and Williams first define a group. The Alexander Biquandle proceeds from taking the abelianization of the Silver–Williams group. Manturov uses the virtual quandle construction we shall describe in the next subsection.

We end this discussion of the Alexander Biquandle with two examples that show clearly its limitations. View Figure 11. In this figure we illus-

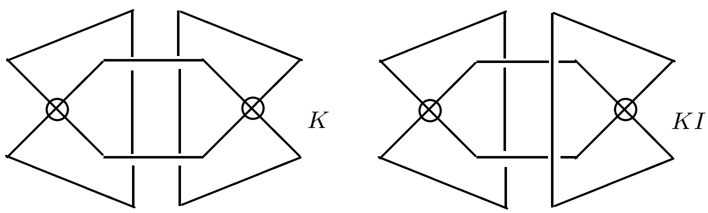


Fig. 11. The Knot K and the Kishino Diagram KI

trate two diagrams labeled K and KI. It is not hard to calculate that both $G_K(s,t)$ and $G_{KI}(s,t)$ are equal to zero. However, the Alexander Biquandle of K is non-trivial—calculation shows that it is isomorphic to the free module over $\mathbb{Z}[s, s^{-1}, t, t^{-1}]$ generated by elements a and b subject to the relation

$$(s^{-1} - t - 1)(a - b) = 0.$$

Thus K represents a non-trivial virtual knot. This shows that it is possible for a non-trivial virtual diagram to be a connected sum of two trivial virtual diagrams, and it shows that the Alexander Biquandle can sometimes be more powerful than the polynomial invariant G. However, the diagram KI also has trivial Alexander Biquandle. This knot is proved to be knotted virtual by Kishino and Satoh [KS], Bartholomew and Fenn [BF], Manturov [Ma8], Kadokami [Kad], Dye and Kauffman [KaV4] and others (see also the first problem in [FKM]).

2.4. Virtual quandles

Definition 3. A *virtual quandle* is a quandle $(M, \circ)$ endowed with a unary operation f such that:

1. f is invertible, the inverse operation being denoted by f^{-1};
2. $\circ$ is distributive with respect to f:

$$\forall a, b \in M : f(a) \circ f(b) = f(a \circ b). \tag{2}$$

REMARK 4. Equation (2) implies that for all $a, b \in M$:

$$f^{-1}(a) \circ f^{-1}(b) = f^{-1}(a \circ b),$$
$$f(a) \,\overline{\circ}\, f(b) = f(a \,\overline{\circ}\, b),$$
$$f^{-1}(a) \,\overline{\circ}\, f^{-1}(b) = f^{-1}(a \,\overline{\circ}\, b).$$

Given a virtual link diagram L, we construct its virtual quandle $Q(L)$ as follows.

We say that a diagram has *proper arcs* if there are no circles (circular arcs) in the diagram. See Figure 13. We can choose a diagram L' in such a way that it is divided into long arcs in a proper way. (A *long arc* is an arc of the diagram that may contain virtual crossings.) Such diagrams are called *proper*. It is clear that a proper diagram with m crossings has m long arcs. We do need the fact that each long arc has two different final crossing points. For some diagrams this is not true. However, this can easily be accomplished by slight deformations of the diagram; see Fig. 12.

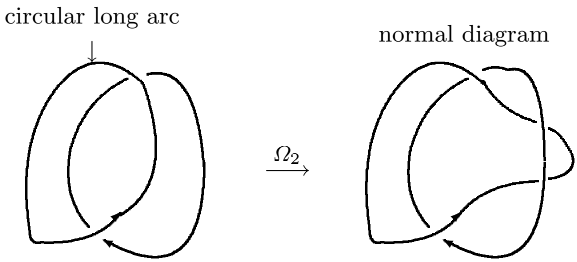


Fig. 12. Reconstructing a link diagram in a proper way

DEFINITION 4. An *arc* of L' is an oriented interval in the diagram not containing undercrossings or virtual crossings.

EXAMPLE 1. The knot shown in Fig. 13 has 3 classical crossings, 3 arcs (a_1 and a_2; b_1 and b_2; c), and 5 virtual arcs (a_1, a_2, b_1, b_2, c).

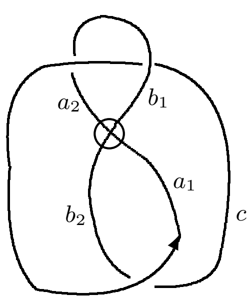


Fig. 13. A knot diagram and its arcs

The invariant $Q(L)$ is now constructed as follows. Consider all arcs a_i, $i = 1, \ldots, n$, of the diagram L'. Consider the set of formal words $X(L')$ obtained inductively from a_i by using $\circ, \bar{\circ}, f, f^{-1}$. In order to construct $Q(L')$ we will factorize $X(L')$ by the equivalence relation generated by the general axioms shown below, plus the consequences of the specific relations incurred from the diagram.

Axiomatically, for each $a, b, c \in X(L')$ we identify:

$$\begin{aligned}
&f^{-1}(f(a)) \sim f(f^{-1}(a)) \sim a;\\
&(a \circ b)\,\bar{\circ}\, b \sim a;\\
&(a\,\bar{\circ}\, b) \circ b \sim a;\\
&a \sim a \circ a;\\
&(a \circ b) \circ c \sim (a \circ c) \circ (b \circ c);\\
&f(a \circ b) \sim f(a) \circ f(b).
\end{aligned}$$

With the equivalence generated by these axioms, we get a "free" virtual quandle with generators $a_1, \ldots, a_n$. We then add the extra equivalences from the structure of L':

For each classical crossing we write the relation (1) just as in the classical case. For each virtual crossing V we also write relations. Let $a_{j_1}, a_{j_2}, a_{j_3}, a_{j_4}$ be the four arcs incident to V as shown in Fig. 14.

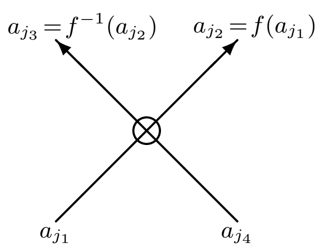


Fig. 14. Relation for a virtual crossing

Then, let us write the relations

$$a_{j_2} = f(a_{j_1}) \tag{3}$$

and

$$a_{j_3} = f(a_{j_4}). \tag{4}$$

So, the virtual quandle $Q(L)$ is the quandle generated by all arcs a_i, $i = 1, \ldots, n$, all linear relations at classical vertices and all relations (3), (4) at virtual vertices.

THEOREM 1. *The quandle $Q(L)$ is a link invariant.*

Proof. First, note that two proper diagrams generate isotopic virtual links if and only if one can be deformed to the other by using a sequence of virtual Reidemeister moves. Indeed, if during isotopy a circular long link occurs, we can modify the isotopy by applying the first classical Reidemeister move to this long arc and subdividing it into two parts.

We have to show that $Q(L)$ is invariant under virtual Reidemeister moves. The invariance of $Q(L)$ under all classical moves is checked in the same way as that of the ordinary quandle. Let us check now the invariance of Q under purely virtual Reidemeister moves.

The first virtual Reidemeister move is shown in Fig. 15. In the initial local picture we have one local generator a. Here we just add a new generator b and two coinciding relations: $b = f(a)$. Thus, it does not change the virtual quandle at all.

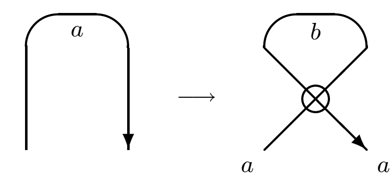


Fig. 15. Invariance of Q under the 1st virtual move

The case of inverse orientation at the crossings gives us $b = f^{-1}(a)$ that does not change the situation.

For each next relation, we will check only one case of arc orientation.

The second move (see Fig. 16) adds two generators c and d and two pairs of coinciding relations $c = f(a)$ and $d = f^{-1}(b)$. Thus, the quandle Q stays the same. In the case of the third Reidemeister move we have six "exterior

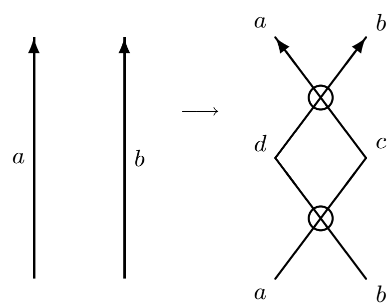


Fig. 16. Invariance of Q under the 2nd virtual move

arcs": three incoming (a, b, c) and three outgoing (p, q, r); see Fig. 17. In both cases we have $p = f^2(a)$, $q = b$, $r = f^{-2}(c)$. The three interior arcs are expressed in terms of a, b, c, and give no other relations.

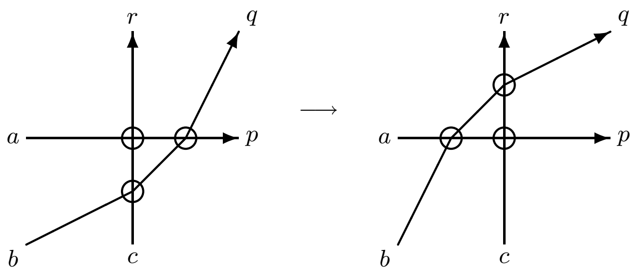


Fig. 17. Invariance of Q under the 3rd virtual move

Finally, let us check the mixed move. We will only check the version of it shown in Fig. 18.

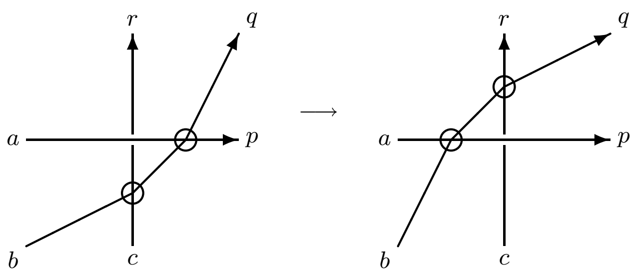


Fig. 18. Invariance of Q under the mixed move

In both pictures we have three incoming edges a, b, c and three outgoing edges p, q, r. In the first case we have the relations $p = f(a)$, $q = b$, $r = f^{-1}(c) \circ a$. In the second case we have $p = f(a)$, $q = b$, $r = f^{-1}(c \circ f(a))$.

The distributivity relation $f(x \circ y) = f(x) \circ f(y)$ implies $f^{-1}(c) \circ a = f^{-1}(c \circ f(a))$. Hence, two virtual quandles before the mixed move and after the mixed move coincide.

The other cases of the mixed move lead to other relations all equivalent to $f(x \circ y) = f(x) \circ f(y)$.

This completes the proof of the theorem. ∎

This leads to a construction of a polynomial invariant (the "easier" part of the paper [Ma3]). First, we take $a \circ b$ to be $ta + (1 - t)b$ and $f(a)$ to be sa, where t and s are independent commuting variables. After this, the normalized determinant of this matrix gives an invariant.

It was proved recently by R. Fenn (considering virtual knots as closures of virtual braids; this idea was inspired by H. Morton) that this polynomial in fact coincides with that proposed by Sawollek and Silver–Williams and with that coming from the Alexander Biquandle (up to a variable change) as discussed by Kauffman and Radford. Fenn's method also proves that we get nothing new if we try to use the linear biquandle structure at classical crossings and linear automorphism f at virtual crossings.

In the present paper, we will construct a more general invariant in Section 5 (which works for the case of colored links).

2.5. Jones polynomial for virtual knots and involutory quandles. We use a generalization of the bracket state summation model for the Jones polynomial to extend it to virtual knots and links. We call a diagram in the plane *purely virtual* if the only crossings in the diagram are virtual crossings. Each purely virtual diagram is equivalent by virtual moves to a disjoint collection of circles in the plane.

Given a link diagram K, a state S of this diagram is obtained by choosing a smoothing for each crossing in the diagram and labeling that smoothing with either A or A^{-1} according to the convention that a counterclockwise rotation of the overcrossing line sweeps two regions labeled A, and that a smoothing that connects the A regions is labeled by the letter A, or diagrammatically,

Fig. 19

Then, given a state S, one has the evaluation $\langle K|S\rangle$ equal to the product of the labels at the smoothings, and one has the evaluation $\|S\|$ equal to the number of loops in the state (the smoothings produce purely virtual diagrams). One then has the formula

$$\langle K\rangle = \sum_S \langle K|S\rangle d^{\|S\|-1}, \tag{5}$$

where the summation runs over the states S of the diagram K, and $d = -A^2 - A^{-2}$. This state summation is invariant under all classical and virtual moves except the first Reidemeister move. The bracket polynomial is normalized to an invariant $f_K(A)$ of all the moves by the formula $f_K(A) = (-A^3)^{-w(K)}\langle K\rangle$ where $w(K)$ is the *writhe* of the (now) oriented diagram K. The writhe is the sum of the orientation signs (± 1) of the crossings of the diagram. The Jones polynomial $V_K(t)$ is given in terms of this model by the formula

$$V_K(t) = f_K(t^{-1/4}). \tag{6}$$

The reader should note that this definition is a direct generalization to the virtual category of the state sum model for the original Jones polynomial. It is straightforward to verify the invariances stated above. In this way one has the Jones polynomial for virtual knots and links (see [KaV]).

In terms of the interpretation of virtual knots as stabilized classes of embeddings of circles into thickened surfaces, our definition coincides with the simplest version of the Jones polynomial for links in thickened surfaces. In that version one counts all the loops in a state the same way, with no regard for their isotopy class in the surface. It is this equal treatment that makes the invariance under handle stabilization work. With this generalized version of the Jones polynomial, one has again the problem of finding a geometric/topological interpretation of this invariant. There is no fully satisfactory topological interpretation of the original Jones polynomial and the problem is inherited by this generalization.

In [KaV2], the following theorem was proved.

THEOREM 2. *For each non-trivial classical knot diagram K of one component there is a corresponding non-trivial virtual knot diagram* $\mathrm{Virt}(K)$ *with unit Jones polynomial.*

This theorem is a key ingredient in the problems involving virtual knots. Here is a sketch of its proof. The proof uses two invariants of classical knots and links that generalize to arbitrary virtual knots and links. These invariants are the *Jones polynomial* and the *involutory quandle*; we denote the latter by $\mathrm{IQ}(K)$ for a knot or link K.

Given a crossing i in a link diagram, we define $s(i)$ to be the result of *switching* that crossing so that the undercrossing arc becomes an overcrossing arc and vice versa. We also define the *virtualization* $v(i)$ of the crossing by the local replacement indicated in Figure 20. In this figure we illustrate how in the virtualization of the crossing the original crossing is replaced by a crossing that is flanked by two virtual crossings.

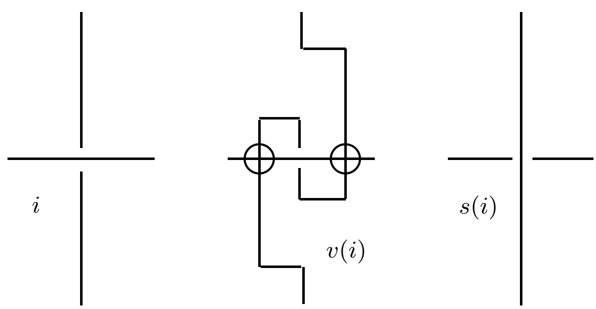


Fig. 20. Switching and virtualizing a crossing

Proof of Theorem 2. Suppose that K is a (virtual or classical) diagram with a classical crossing labeled i. Let $K^{v(i)}$ be the diagram obtained from K by virtualizing the crossing i while leaving the rest of the diagram just as before. Let $K^{s(i)}$ be the diagram obtained from K by switching the crossing

i while leaving the rest of the diagram just as before. Then it follows directly from the definition of the Jones polynomial that

$$V_{K^{s(i)}}(t) = V_{K^{v(i)}}(t).$$

As far as the Jones polynomial is concerned, switching a crossing and virtualizing a crossing look the same.

The involutory quandle [Knots] is an algebraic invariant equivalent to the fundamental group of the double branched cover of a knot or link in the classical case. In this algebraic system one associates a generator of the algebra $\mathrm{IQ}(K)$ to each arc of the diagram K and there is a relation of the form $c = ab$ at each crossing, where ab denotes the (non-associative) algebra product of a and b in $\mathrm{IQ}(K)$. See Figure 21. In this figure we have illustrated through the local relations the fact that

$$\mathrm{IQ}(K^{v(i)}) = \mathrm{IQ}(K).$$

As far as the involutory quandle is concerned, the original crossing and the virtualized crossing look the same.

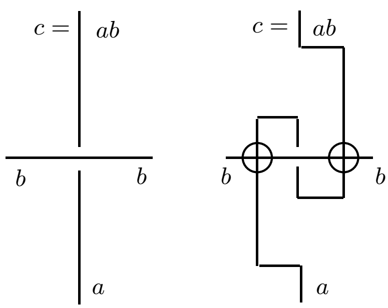


Fig. 21. $\mathrm{IQ}(\mathrm{Virt}(K)) = \mathrm{IQ}(K)$

If a classical knot is actually knotted, then its involutory quandle is non-trivial [W]. Hence if we start with a non-trivial classical knot, we can virtualize any subset of its crossings to obtain a virtual knot that is still non-trivial. There is a subset A of the crossings of a classical knot K such that the knot SK obtained by switching these crossings is an unknot. Let $\mathrm{Virt}(K)$ denote the virtual diagram obtained from A by virtualizing the crossings in the subset A. By the above discussion the Jones polynomial of $\mathrm{Virt}(K)$ is the same as the Jones polynomial of SK, and this is 1 since SK is unknotted. On the other hand, the IQ of $\mathrm{Virt}(K)$ is the same as the IQ of K, and hence if K is knotted, then so is $\mathrm{Virt}(K)$. We have shown that $\mathrm{Virt}(K)$ is a non-trivial virtual knot with unit Jones polynomial. This completes the proof of the theorem. ∎

If there exists a classical knot with unit Jones polynomial, then one of the knots $\mathrm{Virt}(K)$ produced by this theorem may be equivalent to a classical knot. It is an intricate task to verify that specific examples of $\mathrm{Virt}(K)$ are not classical; a very special partial case was considered in [SW2]. This has led to an investigation of new invariants for virtual knots. In this investigation a number of issues appear. One can examine the combinatorial generalization of the fundamental group (or quandle) of the virtual knot and sometimes one can prove by pure algebra that the resulting group is not classical. This is related to observations by Silver and Williams [SW], by Manturov [Ma6, Ma8] and by Satoh [Satoh] showing that the fundamental group of a virtual knot can be interpreted as the fundamental group of the complement of a torus embedded in four-dimensional Euclidean space.

A very fruitful line of new invariants comes about by examining the biquandles and virtual quandles. Flat virtual diagrams are seldom trivial. If we can verify that the flat knot $F(\mathrm{Virt}(K))$ is non-trivial, then $\mathrm{Virt}(K)$ is non-classical. In this way the search for classical knots with unit Jones polynomial expands to the exploration of the structure of the infinite collection of virtual knots with unit Jones polynomial; for a detailed description of this problem, see [FKM].

Another way of putting Theorem 2 is as follows: In the arena of knots in thickened surfaces there are many examples of knots with unit Jones polynomial. Might one of these be equivalent via handle stabilization to a classical knot? In [Kup] Kuperberg shows the uniqueness of the embedding of minimal genus in the stable class for a given virtual link. The minimal embedding genus can be strictly less than the number of virtual crossings in a diagram for the link. There are many problems associated with this phenomenon.

There are generalizations of the Jones polynomial that involve the surface representation of virtual knots. To begin with, one can keep track of the isotopy classes of the curves in the state expansion of the bracket polynomial for a knot embedded in a surface. This gives a surface bracket polynomial [KaV4] that can be used in tandem with Kuperberg's results [Kup] to determine the minimal surface embedding genus for some virtual knots and links. In [KaV4] this method is used to show that the Kishino diagram has genus two. Another approach [Ma8] uses a relative of the surface bracket polynomial, and includes in the equivalence relation of the curves in the states, the stabilization of the surfaces themselves. In this way the Manturov invariant becomes an element of a module over the Laurent polynomial ring in one variable, and is strictly stronger than the original extension of the Jones polynomial to virtuals. We do not yet know the relative strengths of these two methods.

2.6. A common construction—the virtual biquandle. Consider an oriented virtual link diagram L and divide it into arcs. Write down the biquandle relations at each classical crossing, and the virtual quandle relations at each virtual crossing.

The full list of operations and axioms for this object, the *virtual biquandle* (making this related algebra an invariant of virtual knots and links), is as follows:

1. We have operations $\urcorner, \lrcorner, \ulcorner, \llcorner$ as above and the invertible operation f acting on the whole algebra.
2. Biquandle operations satisfy all conditions described previously.
3. The operation f ia a biquandle automorphism. Thus, for instance, $f(a\overline{\ b}|) = f(a)\overline{\ f(b)}|$; the same statements are true for $\lrcorner, \ulcorner, \llcorner$.

The object defined in this manner is called the *virtual biquandle of the link*. Now, we consider a larger context, admitting more general constructions at virtual crossings.

Let us now give the definition of the *formal virtual biquandle* as some algebraic object satisfying certain axioms; along the same lines, we shall define the virtual biquandle of a link by interpreting the biquandle operations according to the crossing structure.

Let us now write down the axioms in the general case, when virtual crossings are endowed with binary operations $a.b$ and $b|a$ (see Fig. 22).

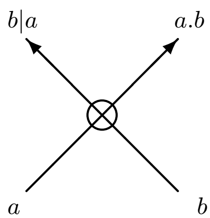


Fig. 22. Binary relation at a virtual crossing

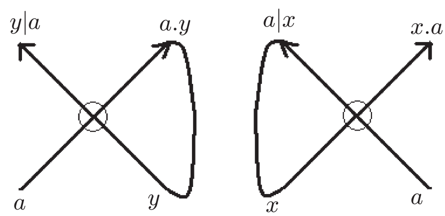


Fig. 23. Relations coming from the first virtual move

The axioms for the classical case are the same as above.

The axioms for purely virtual moves are:

1. The first Reidemeister move (see Fig. 23):

$$\forall a\ \exists x : \begin{cases} cx.a = a, \\ a|x = x, \end{cases} \tag{7}$$

$$\forall a\ \exists y : \begin{cases} a.y = y, \\ y|a = a. \end{cases} \tag{8}$$

2. The second oriented Reidemeister move (see the left part of Fig. 24):

$$\forall a, b : (a.b)|(b|a) = a \text{ and } (b|a).(a.b) = b. \tag{9}$$

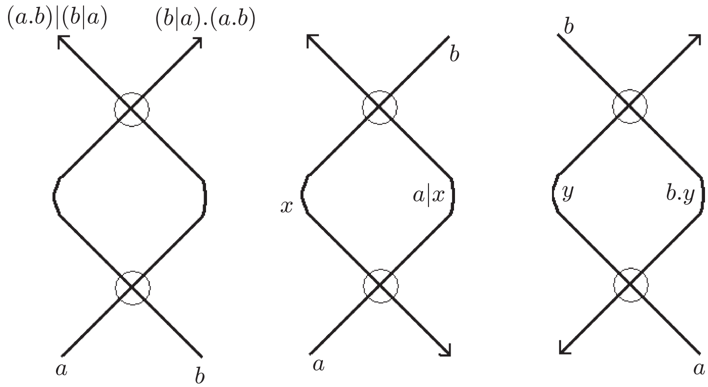


Fig. 24. Relations coming from the second virtual move

The second unoriented Reidemeister moves (see the central and right parts of Fig. 24):

$$\begin{cases} \forall a, b : \exists! x : b|(a|x) = x; (a|x).b = a, \\ \forall a, b : \exists! y : (y|b) = a; (b.y)|a = b. \end{cases} \tag{10}$$

3. The third Reidemeister move (see Fig. 25, upper and lower pictures). The two cases are:

$$\forall a, b, c : \begin{cases} (a.b).c = (a.(c|b)).(b.c), \\ (b|a).(c|(a.b)) = (b.c)|(a.(c|b)), \\ (c|(a.b))|(b|a) = (c|b)|a, \end{cases} \tag{11}$$

and

$$\forall a, b, c : \exists! x, y : \begin{cases} (c|(b|(a|x))) = x, \\ b.(c.(a.y)) = y, \\ (a|x).b = (a.y)|c, \\ y|a = (b|(a|x)).c, \\ x.a = (c.(a.y))|b. \end{cases} \tag{12}$$

Remark 5. In Figs. 24 and 25 we label only some arcs; all other arcs can be labeled according to the rule shown in Fig. 22.

For the semivirtual move we have the pictures in Fig. 26 and relations:

$$\forall a, b, c : \begin{cases} (a\,\overline{\;\;b\,}|)|c = (a|(c.b))\,\overline{\;\;b|c\,}|, \\ (b\,\underline{\;\;a\,}|)|(c.(a\,\overline{\;\;b\,}|))) = (b|c)\,\underline{\;\;a|(c.b)\,}|, \\ c.(a\,\overline{\;\;b\,}|).(b\,\underline{\;\;a\,}|) = (c.b).a \end{cases} \tag{13}$$

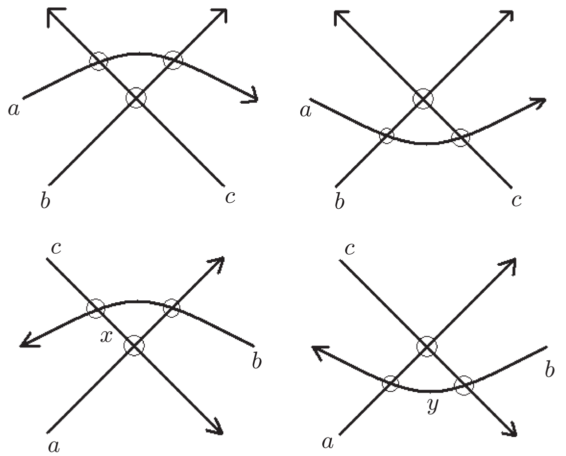


Fig. 25. Relations coming from the third virtual move

and

$$\text{(14)} \qquad \forall a,b,c : \exists! x,y : \begin{cases} (b|(a\lfloor\underline{x}\,)).c = x, \\ a.(b.(c\,\underline{y}\rfloor)) = y, \\ (a\lfloor\underline{x}\,).b = y\lfloor\underline{c}\,, \\ c|(b|(a\lfloor\underline{x}\,)) = (b.(c\lceil\overline{y}\,))|a, \\ x\lceil\overline{a}\, = (c\lceil\overline{y}\,)|b. \end{cases}$$

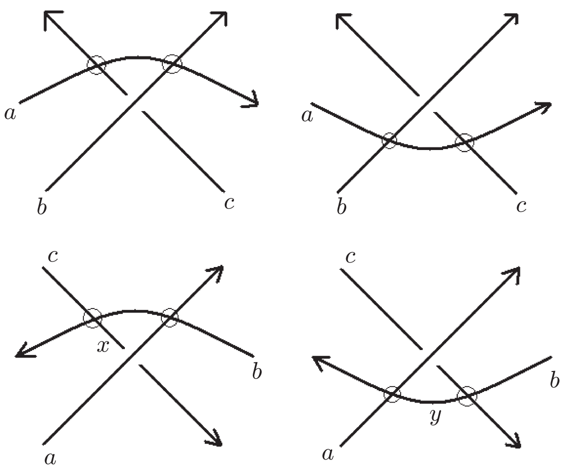


Fig. 26. Relations coming from the mixed move

The proof is trivial by design. It is left to the reader.

This common structure should have a lot of realizations. We indicate one of them in the following section.

3. PRESENTATIONS

In this section, we describe several explicit constructions of the virtual quandle and biquandle invariants described above, using specific algebraic structures for the representation.

3.1. A presentation for virtual biquandles. Virtual biquandles admit the following linear representation: at classical crossings we have the Alexander Biquandle operations (with generators t, s) and set $(a.b) = (1+qb)a$, $(b'a) = (1-qa)b$ for some new generator q that commutes with t, s such that $q^2 = (t-1)q = (s-1)q = 0$. We are going to investigate other models of the virtual biquandle with binary operations at virtual crossings.

We shall discuss other presentations of the virtual biquandle construction in our further publications. Later in this section, we deal with long quandles, virtual quandles, and some generalizations of them.

3.2. Formal power functions and conjugation. For the virtual quandles, one can take a group together with the following two possible operations:

1. $a \circ b = b^n a b^{-n}$, $f(a) = qaq^{-1}$, where q is a new generator for this group, and n is an integer;
2. $a \circ b = ba^{-1}b$, $f(a) = qaq^{-1}$ or $f(a) = qa^{-1}q$, where q is a new generator for the group.

In this way, we associate a group to each virtual knot in two possible ways.

One can also associate a finite group G with the virtual quandle operations defined as above, and look at the set of all maps $\Gamma(L) \to G$, where $\Gamma(L)$ is the virtual knot quandle, and G is a finite virtual quandle. The set of such mappings is finite for any finite group (we can fix the images of generators of the quandle thus defining the map completely), so the number of "colorings", i.e., mappings, is finite and invariant under generalized Reidemeister moves. For more details see [Ma2].

These ideas can also be used for the case of biquandles and virtual biquandles. More precisely, one can say the following: if we have an abstract knot invariant $\mathcal{G}$ constructed according to some axioms (quandle, virtual quandle or biquandle) and a finite object G satisfying the same axioms, then the cardinality of the set of mappings $\mathcal{G}(L) \to G$ is a knot invariant.

Thus there is a way to search for finite-valued invariants (colorings) by finding *finite* sets satisfying such axioms.

3.3. The quaternionic biquandle. It turns out that there are beautiful, explicit biquandle constructions that give powerful results. One class of such constructions come from the quaternions. This idea and the following formulae are due to R. A. Fenn and A. Bartholomew [BF].

The quaternionic biquandle is defined by the following operations:

$$a\,\overline{\;b\,}\rceil = j\cdot a + (1+i)\cdot b, \qquad a\lceil\overline{b\;} = j\cdot a + (1-i)\cdot b,$$
$$a\,\underline{\;b\,}\rfloor = -j\cdot a + (1+i)\cdot b, \qquad a\lfloor\underline{b\;} = -j\cdot a + (1-i)\cdot b.$$

where $i^2 = j^2 = k^2 = ijk = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$ in the associative, non-commutative algebra of the quaternions. The elements $a, b, \ldots$ are in a module over the ring of integer quaternions.

Amazingly, one can verify that these operations satisfy the axioms for the biquandle.

For more about quaternionic and other non-commutative biquandles, see [BF].

Let us look at what happens with the Kishino knot (see labeling shown in Fig. 27).

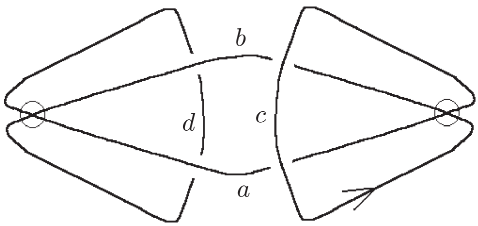


Fig. 27. The Kishino knot

Let us first consider the rightmost crossings (that deal with arcs a, b, c). We have:

$$c\,\underline{\;a\,}\rfloor = -j\cdot c + (1+i)a, \qquad a\,\overline{\;c\,}\rceil = j\cdot a + (1+i)\cdot c.$$

Now,

$$a\,\overline{\;c\,}\rceil\lfloor\underline{c\;\;\underline{\;a\,}\rfloor} = c \quad\text{and}\quad c\,\underline{\;a\,}\rfloor\lceil\overline{a\;\;\overline{\;c\,}\rceil} = b$$

give us

$$b = j(-jc + (1+i)a) + (1-i)(ja + (1+i)c) = 3c + 2(j-k)a, \tag{15}$$
$$c = -j(ja + (1+i)c) + (1-i)(-jc + (1+i)a) = 3a - 2(j-k)c. \tag{16}$$

It follows that $3a = 3b$.

Now, let us write the relations for the left crossings. We have:

$$b\lfloor\underline{d\;} = -jb + (1-i)d, \qquad d\lceil\overline{b\;} = jd + (1-i)b.$$

So,

$$d\lceil\overline{b\;}\;\underline{\;b\;\lfloor\underline{d\;}\,}\rfloor = a \quad\text{and}\quad b\lfloor\underline{d\;}\;\overline{\;d\;\lceil\overline{b\;}\,}\rceil = d.$$

Hence,

$$a = -j(jd + (1-i)b) + (1+i)(-jb + (1-i)d) = 3d - 2(j+k)b, \tag{17}$$
$$d = j(-jb + (1-i)d) + (1+i)(jd + (1-i)b) = 3b + 2(j+k)d. \tag{18}$$

From these equations we can also conclude that $3a = 3b$.

Let us show that the module we get is not just the module generated by a where $a = b$. Indeed, by tensoring the above equations with $\mathbb{Z}_3$, we get

$$b = 2(j-k)a, \quad c = -2(j-k)c, \quad a = -2(j+k)b, \quad d = 2(j+k)d.$$

Thus, b is out of picture, and we get

$$c = -2(j-k)c, \quad a = 2ia, \quad d = 2(j+k)d.$$

But this implies $(1-2i)a = 0$ from which we deduce

$$2a = 5a = (1+2i)(1-2i)a = 0,$$

whence $a = 0$.

Therefore we are left with

$$c = 2(j-k)c, \quad d = 2(j+k)d,$$

and this is certainly a non-trivial module over $\mathbb{Z}_3$.

Thus, this linear biquandle is not isomorphic to the free one-dimensional linear space over integer quaternions. From this we conclude that the Kishino diagram is a non-trivial virtual knot. The proof given above is a simplification of the method used by Fenn and Bartholomew. In any case, this proof is perhaps the most direct verification for the detection of the Kishino diagram!

4. INFINITE-DIMENSIONAL LIE ALGEBRAS

Here we propose one new method to obtain quandles, biquandles, virtual quandles, etc. (see [Ma4]). For simplicity, we will deal with quandles. We know that quandles can be well defined on discrete groups by putting, say, $a \circ b = bab^{-1}$. The question is: is it possible to arrange a quandle operation which would act on a geometric group (Lie group)? Well, each of the relations

$$\begin{cases} a \circ b = bab^{-1}, \\ a \circ b = b^n ab^{-n}, \\ a \circ b = ba^{-1}b \end{cases} \tag{19}$$

works well in *any* group (i.e. satisfies all the quandle axioms), but what we actually want to do is to construct a group by generators and relations.

This problem for geometrical groups is quite difficult, so we may imagine we already have a group with such an operation, and this group G is good, it is a Lie group. Thinking in this way, we see that this group should have the corresponding Lie algebra $\mathfrak{g}$ (which *can* in fact be constructed by using generators and relations), and the group is connected with the algebra by the exponential mapping $\exp : \mathfrak{g} \to G$. So, it would be nice to understand the quandle relations of type (19), say, the relation $a \circ b = bab^{-1}$. In the algebra, this relation would be

$$\mathfrak{a} \circ \mathfrak{b} = \log(\exp(\mathfrak{a}) \exp(\mathfrak{b}) \exp(\mathfrak{c})), \tag{20}$$

where log is the inverse operation $G \to \mathfrak{g}$ to the exponential map (more precisely, this operation is defined in the vicinity of the group unit, however, we will operate with formulae and look what happens).

It turns out that the operation $a, b \mapsto \log(\exp(a) \exp(b))$ can be defined in terms of the Lie algebra, i.e., it can be expressed in commutators.

The required statement is called the *Baker–Campbell–Hausdorff theorem*, and the coefficients are given by a beautiful formula due to Dynkin [Dyn]:

$$\ln(e^x e^y) \sum \frac{(-1)^{k-1}}{k} \frac{1}{p_1!q_1! \cdots p_k!q_k!} (x^{p_1} y^{q_1} \cdots x^{p_k} y^{q_k})^\circ, \tag{21}$$

where $p_i q_i > 0$ for $i \leq k$; here the function $\cdot^\circ$ is defined on formal non-commutative monomials in variables x_j by the rule

$$(x_{i_1} \cdots x_{i_k})^\circ = \frac{1}{k} [\ldots [[x_{i_1}, x_{i_2}], x_{i_3}], \ldots, x_{i_k}] \tag{22}$$

and extended linearly to their linear combinations.

Thus, for any classical (or virtual) link L we construct an infinite-dimensional Lie algebra $\mathrm{Li}(L)$ as follows. (All this can be done, if we, for instance, take into account virtual crossings as well; we can just add a new generator $\mathfrak{q}$ and define

$$f(\mathfrak{a}) = \log(\exp(\mathfrak{q}) \exp(\mathfrak{a}) \exp(-\mathfrak{q})).$$

We take its arcs as generators of the free infinite-dimensional Lie algebra, and factorize the resulting algebra subject to the relation (20) taking into account formula (21).

The infinite-dimensional Lie algebra obtained is a link invariant that is not the most pleasant to work with, however, it admits a simplification. Thus, for instance, we can take just $a \circ b = a + [a, b]$.

Decree $a \circ b$ to be $a + [a, b]$ (actually, one should take $a - [a, b]$, but we choose this operation for simplicity). Then the first axiom holds by definition:

$$[a, a] = a + 0 = a.$$

As for the second axiom, the reverse operation exists by the formula

$$a \,\bar{\circ}\, b = a - [a, b] + [[a, b], b] - [[[a, b], b], b] + \cdots$$

without worrying a lot whether this formula converges. Even if it has infinitely many members, we can write all in the finite manner: namely, instead of writing $p \,\bar{\circ}\, q = r$, we write $r \circ q = p$. There is another way to handle the situation: we put

$$a \circ b = a + \varepsilon[a, b],$$

where ε is just a new generator in the basic ring. In this case there would be no problem with convergence.

Now, for the third relation we need

$$(a \circ b) \circ c = (a \circ c) \circ (b \circ c).$$

So, we have the relation

$$\begin{aligned} &a + [a,b] + [a,c] + [[a,b],c] \\ &\qquad = a + [a,c] + [a,b] + [a,[b,c]] + [[a,c],[b,c]] + [[a,b],b] \end{aligned}$$

in any of these cases, with or without ε.

Taking into account the Jacobi identity, we see that the equation above is equivalent to

$$[[a,c],[b,c]] = 0.$$

This means that we must factorize only by third (not second!) commutators of the form $[[a,c],[b,c]]$. Thus, we should not cut our series at some finite place. This leads to some interesting results.

Denote by $\mathrm{Li}(K)$ the invariant obtained (with $a \circ b = a + \varepsilon[a,b]$ at classical crossings and no structure at virtual crossings).

Moreover, this idea together with the Baker–Campbell–Hausdorff formula leads to many other realizations and models for quandles.

Having this approach, one may search for particular representations which certainly lead to new invariants of classical and virtual knots.

Let us consider the $\mathrm{Li}(K)$ invariant (without any structure at virtual crossings). For instance, the trefoil and the figure eight knot give us a finite-dimensional Lie algebra, whence the $(5,2)$-torus knot Lie algebra is finite-dimensional.

Indeed, for the trefoil, we have three arcs a, b, c and three vertices where we have to write down the commutators. Obviously, we get

$$a + \varepsilon[a,b] = c$$

and two other relations obtained by permuting this one cyclically. Thus, we can express all first commutators as linear combinations of generators. So, this algebra cannot be infinite-dimensional.

For the figure eight knot, we have four arcs and three commutators, one of them is expressed twice. Namely, view Figure 28.

We see that the first crossing gives us $d \circ c = a$, the second one gives $b \circ d = a$, the third one gives $b \circ a = c$, and the fourth one gives $d \circ b = c$. So, we have expressions for three of six commutators. Thus we have

$$\varepsilon[b,d] = a - b, \varepsilon[d,b] = c - d,$$

which easily leads to the finite-dimensionality of the algebra.

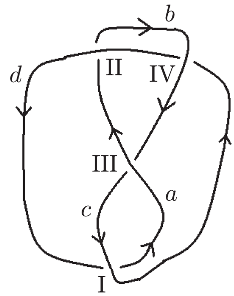


Fig. 28. The labeled figure eight knot

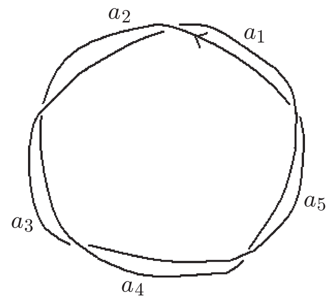


Fig. 29. The $(5,2)$-torus knot

However, if we take the $(5,2)$-torus knot, we have five generators a_1, a_2, a_3, a_4, a_5 and five cyclic relations

$$\varepsilon[a_1, a_2] = a_3 - a_1, \quad \varepsilon[a_2, a_3] = a_4 - a_2, \quad \varepsilon[a_3, a_4] = a_5 - a_3,$$
$$\varepsilon[a_4, a_5] = a_1 - a_4, \quad \varepsilon[a_5, a_1] = a_2 - a_5;$$

see Figure 29.

The commutator $[a_1, a_3]$ is not expressible in the terms of linear combinations of a_i's. It is also easy to show that the elements

$$a_1,\ [a_1, a_3],\ [[a_1, a_3], a_3],\ [[[a_1, a_3], a_3], a_3]$$

represent linearly independent elements. Thus, we obtain a finite-dimensional algebra.

5. COLORED VIRTUAL LINKS AND THEIR INVARIANTS

It is not difficult to show that the virtual quandle construction generalizes to the case of colored links.

We will not write down all the axioms for the multicomponent virtual quandle, we are just going to represent one linear model for oriented virtual links.

Here we give a generalization of the work by Manturov initiated in [Ma7, Ma3]. Namely, consider an n-component link L. Let us fix n generators $t_1, \ldots, t_n$ and n generators $s_1, \ldots, s_n$. We are going to associate elements of the module over $\mathbb{Z}[s_1, \ldots, s_n, t_1, \ldots, t_n]$ to the arcs of the diagram, whence the module itself will be an invariant. The relations for virtual crossings look as follows: while passing through the ith component from the left to the right, we multiply the element associated to the arc "before" by s_i. If we go from the right to the left, we multiply by the corresponding s_j^{-1}.

At classical crossings, we do the following. Suppose we have a crossing where the ith component goes over, and the jth component goes under (see Fig. 30). Suppose the arc going over is b, the arc lying on the right is a, and that on the left is c. Then the relation looks like

$$t_i a + (1 - t_j)b = c.$$

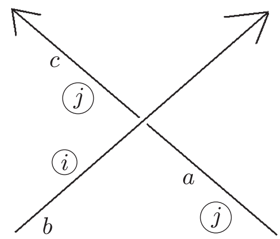


Fig. 30. Relation at a crossing for colored links

For negative crossings we write down exactly the same relation just as in Fig. 30 with the orientation of the jth arc reversed.

Denote the resulting module by $M(L)$. The invariance of this module under Reidemeister moves goes straightforwardly. It turns out that this leads to a polynomial invariant. This invariant is a generalization of that proposed in [Ma3] (there we use one variable t and many variables s).

First, we label all arcs of the diagram by monomials in $s_1, \ldots, s_n$ (this will correspond to the operations at virtual crossings). We do it as follows. First, we deal only with proper diagrams. Thus, we can associate each crossing with a long arc. After that, we associate 1 with each "first" arc outgoing directly from a crossing. After that, we can associate monomials in s_i's to all arcs in the way described above. View Fig. 31.

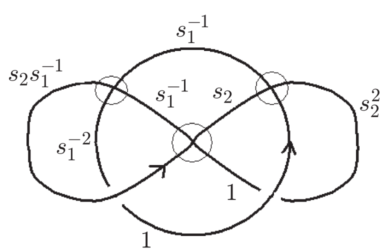


Fig. 31. Two-component link completely labeled

Now, we construct a matrix according to the relations at classical crossings. Namely, each row of our matrix represents a crossing, each column of the matrix represents a long arc. The matrix element is going to be the incidence between them. More specifically, suppose we have positive crossing number i with incoming edge number j having label P (this edge lies on the component p), and overcrossing edge number k having label Q (this edge lies on the component q). Then the ith row of our matrix should consist of at most three elements, more precisely, it is equal to the sum of three rows, each of them having only one non-zero element. One of them has element i equal to 1, another one has element j equal to $-t_q P$, and the last one has element k equal to $(t_p - 1)Q$. In the case when we have a negative crossing, we will have three rows, one of them having t_q at position i, another one having $-P$ at position j and the last one having $(t_p - 1)Q$ at position k.

The determinant of the matrix does not change while renumbering crossings and long arcs correspondingly.

Thus, the determinant is well defined on proper link diagrams. Let us denote by κ this function on diagrams.

The invariance check is quite similar to that performed in [Ma3], so we can show only the most difficult case, namely, the invariance under the third Reidemeister move.

Namely, consider the two diagrams shown in Figure 32.

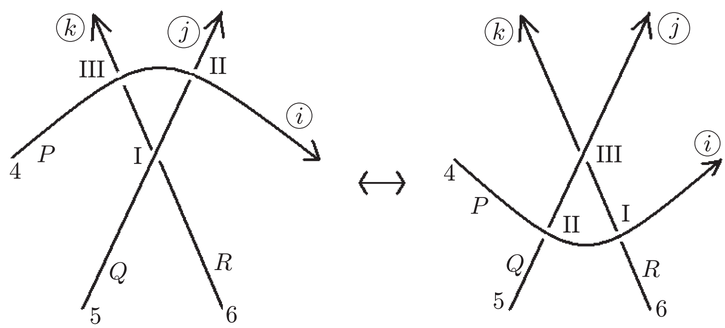


Fig. 32. Labeling for the third Reidemeister move

Here all numbers of crossings are indicated by Roman numbers, all numbers of arcs are indicated by Arabic numbers, numbers of components are encircled, and monomials corresponding to incoming arcs are marked by letters P, Q, R.

In the first case we have the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & (t_k-1)Q & -t_jR & 0\dots0 \\ 0 & 1 & 0 & (t_j-1)P & -t_iQ & 0 & 0\dots0 \\ -t_i & 0 & 1 & (t_k-1)P & 0 & 0 & 0\dots0 \\ 0 & & & & & & \\ \vdots & & & * & & & \\ 0 & & & & & & \end{pmatrix}.$$

The second matrix looks like

$$\begin{pmatrix} 1 & 0 & 0 & (t_k-1)P & 0 & -t_iR & 0\dots0 \\ 0 & 1 & 0 & (t_j-1)P & -t_iQ & 0 & 0\dots0 \\ -t_j & (t_k-1) & 1 & 0 & 0 & 0 & 0\dots0 \\ 0 & & & & & & \\ \vdots & & & * & & & \\ 0 & & & & & & \end{pmatrix}.$$

As we see, the second rows of these matrices coincide. So, let us look at the third rows. For the first matrix, add the first row multiplied by t_i to the third row. For the second matrix, add the first row multiplied by t_j and the second row multiplied by $1-t_k$ to the third row. We will get two matrices for which the third rows coincide. Namely, it looks like

$$(0, 0, 1, (t_k-1)P, t_i(t_k-1)Q, -t_it_jR, 0, \dots, 0).$$

Now, the first column of the new matrices consists of the only non-zero element, namely, 1 at position $(1,1)$. So, we can easily make the first rows of these matrices equal. Thus, we have proved that the determinants of these matrices coincide.

As for the first Reidemeister move, it might multiply the determinant by a monomial in t_i's. Thus, we have proved the following

THEOREM 3. *The polynomial κ is invariant under Reidemeister moves up to multiplication by powers of t_i's.*

The colored link invariant can be used for knots (not links) as well: one should just take cabling and care about the proper normalization of the invariant.

6. LONG KNOTS AND THEIR INVARIANTS

It is well known that long classical knots (non-compact knots in $\mathbb{R}^3$ lying on the straight line outside some big circle) are just the same as ordinary classical knots. For a proof see e.g. [Ma1].

The results and main ideas of the present section were sketched in the book [Ma1]; see also [Ma10]. The two main arguments that can be taken into account in the theory of "long" virtual knots and could not be used before are the following:

1. One can indicate the *initial* and the *final arcs* (which are not compact) of the diagram representing some two fixed elements of the quandle; the elements corresponding to them are invariant under generalized Reidemeister moves.

2. One can take two different quandle-like operations at vertices depending on which arc is "before" and which is "after" according to the orientation of a long knot.

As shown in [GPV], the procedure of breaking a virtual knot is not well defined: breaking the same knot diagram at different points, we may obtain different long knots. Moreover, a virtual unknot diagram broken at some point can generate a non-trivial long knot diagram. The aim of this section is to construct invariants of long virtual knots that "feel" the breaking point.

REMARK 6. Throughout the present section, we deal only with long virtual knots, not links.

REMARK 7. We shall never indicate the orientation of the long knot assuming it to be oriented from left to right.

NOTATION. Throughout this section, R will denote the field of rational functions in one (real) variable t: $R = \mathbb{Q}(t)$.

Now, let us define the virtual long knot.

DEFINITION 5. By a *long virtual knot diagram* we mean a smooth immersion f of the oriented line L_x, $x \in (-\infty, +\infty)$, in $\mathbb{R}^2$, such that:

1. outside some big circle, we have $f(t) = (t, 0)$;
2. each intersection point is double and transverse;
3. each intersection point is endowed with classical or virtual crossing structure.

DEFINITION 6. A *long virtual knot* is an equivalence class of long virtual knot diagrams modulo generalized Reidemeister moves.

DEFINITION 7. A *long quandle* is a set Q equipped with two binary operations $\circ$ and $*$ and one unary operation $f(\cdot)$ such that $(Q, \circ, f)$ is a virtual quandle and $(Q, *, f)$ is a virtual quandle and the following relations hold: The reverse operation for $\circ$ is $\overline{\circ}$ and the reverse operation for $*$ is $\overline{*}$;

$$\forall a, b, c \in Q : (a \circ b) * c = (a * c) \circ (b * c),$$

$$\forall a, b, c \in Q : (a * b) \circ c = (a \circ c) * (b \circ c)$$

(new distributivity relations); and

$$\forall x, a, b \in Q : x\alpha(a \circ b) = x\alpha(a * b)$$

$$\forall x, a, b \in Q : x\beta(a \,\overline{\circ}\, b) = x\beta(a \,\overline{*}\, b)$$

(strange relations), where α and β are some operations from the list $\circ, *, \overline{\circ}, \overline{*}$.

REMARK 8. It might seem that the last two relations hold only in the case when $\circ$ coincides with $*$. However, the equation $(a \circ b) = c$ has the only relation in a and not in b! As will be shown later, there are non-trivial algebraic presentations of the long quandle.

Consider a diagram $\overline{K}$ of a virtual knot and arcs of it. Fix the initial arc a and the final arc b.

Now, we construct the long quandle of it by the following rule. First, we take all arcs of it including a and b and consider the *free long quandle*: just by using formal operations $\circ, *, \overline{\circ}, \overline{*}, f$ factorized only by the quandle relations (together with the new distributivity relations and strange relations).

After this, we factorize by relations at crossings. At each virtual crossing, we do just the same as in the case of virtual quandle. At each classical crossing we write the relation either with $\circ$ or with $*$, namely, if the overcrossing is passed *before* the undercrossing (with respect to the orientation of the knot) then we use the operation $\circ$ (respectively, $\overline{\circ}$); otherwise we use $*$ (respectively, $\overline{*}$).

After this factorization, we obtain an algebraic object M equipped with the five operations $\circ, \overline{\circ}, *, \overline{*}$, f and two selected elements a, b.

Definition 8. Denote the resulting object by $Q_{\mathrm{L}}(\overline{K})$.

Let $\overline{K}$ be a diagram of a long knot K. Call $Q_{\mathrm{L}}(\overline{K})$ *the long quandle* of K.

Obviously, for the *long unknot* U (represented by a line without any crossings) we see that the elements $a, b \in Q_{\mathrm{L}}(U)$ representing the initial and the final arc should be equal.

Theorem 4. *The quandle Q_{L} together with selected elements a, b is invariant with respect to generalized Reidemeister moves.*

Proof. The proof is quite analogous to the invariance proof of the virtual quandle. Therefore, the details will only be sketched.

The invariance under purely virtual moves and the semivirtual move goes as in the classical case: we deal only with f and one of the operations $*$ or $\circ$. Only one of the operators $*, \circ$ appears when applying the first or the second classical Reidemeister move.

So, the most interesting case is the third classical Reidemeister move. In fact, it is sufficient to consider the four cases $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d})$ shown in Fig. 33. In some cases this means the distributivity of the operations $\circ, *, \overline{\circ}, \overline{*}$ with respect to each other.

In each of the four cases everything is OK with p and q (p does not change and q is affected by p in the same manner on the right and on the left). So, one should only check the transformation for r.

In each picture, at each crossing we put some operation α, β or γ. This means one of the operations $\circ, *, \overline{\circ}, \overline{*}$ (which are going to be applied to the arc below to obtain the corresponding arc above).

Consider case a. Each α, β, γ is a "multiplication" $\circ$ or $*$ (the operations $\overline{\circ}, \overline{*}$ are thus "divisions").

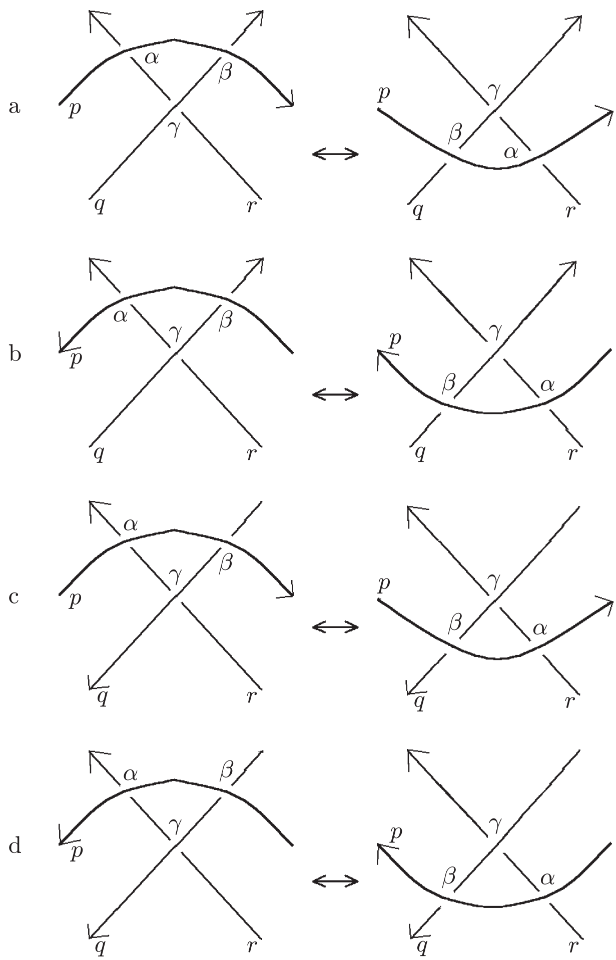


Fig. 33. Checking the invariance under Ω_3

Thus, at the upper left corner we shall have $(r\gamma q)\alpha p$ in the left picture and $(r\alpha p)\gamma(q\beta p)$ in the right picture. But, by definition,

$$(r\gamma q)\alpha p = (r\alpha p)\gamma(q\alpha p).$$

The latter expression equals $(r\alpha p)\gamma(q\beta p)$ according to the "new relation" (because both β and α are "multiplications").

Now, let us turn to case b. Here γ is a "multiplication" and α, β are "divisions". Thus, the same equality takes place:

$$(r\gamma q)\alpha p = (r\alpha p)\gamma(q\alpha p) = (r\alpha p)\gamma(q\beta p).$$

The same equality holds for the cases shown in pictures c and d: the only important thing is that α and β are either both multiplications (as in case c) or both divisions (as in case d). The remaining part of the statement follows straightforwardly. ∎

The (non-trivial) virtual knot in Figure 34 is the connected sum of two unknots. In particular, this means that the corresponding long virtual knots are not trivial.

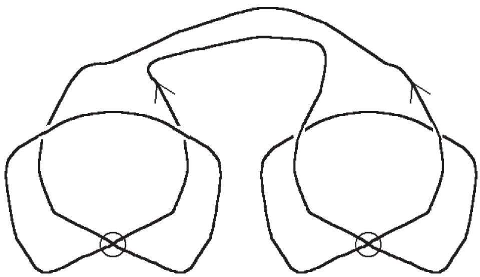


Fig. 34. The Kishino knot

Fig. 35. Two long virtual knots obtained by breaking the unknot

Consider the unknots shown in both parts of Fig. 35. Let us show that they are not isotopic to the trivial knot. To do this, we will use the presentation of the long virtual quandle to the module over $\mathbb{Z}_{16}$ by

$$a \circ b = 5a - 4b, \quad a * b = 9a - 8b, \quad f(x) = 3 \cdot x.$$

It can be readily checked that these relations satisfy all axioms of the long quandle.

Let us show that for none of these two knots $a = b$. Indeed, for the first knot (Fig. 35.a), denote by c the next arc after a. Then we have

$$9a - 8 \cdot (3c) = c, 5b - 4 \cdot (3c) = c \Rightarrow b = 9a.$$

For the second knot (Fig. 35.b), denote by c the upper (shortest) arc. We have

$$5 \cdot (3b) - 4a = c, 9 \cdot (3a) - 8b = c \Rightarrow b = 9a.$$

As we see, in none of these cases $a = b$. Moreover, the expressions of b via a are different. Thus, none of the two long knots shown in Fig. 35.a and Fig. 35.b is trivial.

If we take another field, say, $\mathbb{Z}_{25}$ with the operations $a \circ b = 6a - 5b$, $a*b = 11a-10b$, $f(x) = 3x$, we shall see that the two knots from Figures 35.a and 35.b are indeed different: in the first case we obtain $a = 11b$, in the second case we obtain $a = 21b$.

In fact, the linear model for long quandles allows us to prove even more, namely, we can show that the knots 35.a and 35.b do not commute (this implies the non-triviality and non-classicality of each of them together with their non-equivalence).

Later, the same effect was established by D. Silver and S. Williams by using non-commutative structures in biquandles.

7. LONG KNOT FLAT QUANDLES

There is an interesting question to consider: virtual knots modulo classical crossing change. These objects are called *virtual flats* (for more details see, e.g., [Ma1, Ma6]). They are classified geometrically, moreover, they lead to powerful invariants of virtual knots [Ma6, Tur]. However their algebraic classification and invariants are worth studying (which was performed in [Tur, HK]), because this may lead to the construction of skein algebras (see e.g. [Ma1]) for virtual knots.

It turns out that this plan can be performed somehow by using the ideas described in the previous section.

As in [FJK] and [BF], one can consider linear biquandles over non-commutative rings, e.g., over quaternions. Namely, having a classical crossing with two inputs, one writes down two outputs depending on the first ones linearly by means of some matrix A. Here we do not pay attention to virtual crossings ([3]). Obviously, the second (classical) Reidemeister move requires invertibility of A, and the third Reidemeister move requires some equation of Yang–Baxter type, namely,

$$A_1A_2A_1 = A_2A_1A_2, \tag{23}$$

where A_1 is a 3×3-matrix consisting of blocks A (of size 2×2) and 1 (of size 1×1), and A_1 is the matrix consisting of blocks 1 and A.

What have the equations got to do with *long virtual knots*? It turns out that we can use two different matrices, say, A and B, for different types of crossings, namely, A for the early overcrossing and B for the late overcrossing. In this case, we get the equations (23) for A_1, A_2 and analogous ones for B_1, B_2 (as before). Besides this, we get one more equation for the third Reidemeister move which corresponds to the "strange relation" in the case of long quandles. Namely,

$$A_1A_2B_1 = B_2A_1A_2. \tag{24}$$

([3]) This is an interesting object to be discussed.

Besides the obvious solution $B = A$, one can also take the solution $B = A^{-1}$. It is clear that all conditions described above hold (this is left to the reader as a simple exercise). What do we get in this case? Obviously, each "bad crossing" (with late overcrossing) is operated on by B (or B^{-1}), so we will have the same result as in the case of the inverse crossing (A^{-1} or A).

Finally, we get the invariant of the "descending" long knot with the same shadow, obtained by using simply the Fenn approach with the matrix A. This is going to be an invariant under generalized Reidemeister moves, so this is an invariant of flats.

This agrees with the following general statement due to V. G. Turaev: the mapping associating to a flat long virtual link diagram the corresponding *ascending* long virtual link diagram is well defined. Thus it allows one to use any long virtual knot invariants for recognizing flat long virtual knots.

So, we can take each of the solutions presented in [BF] and derive long virtual flat invariants from them.

Whether these flat long virtual knot invariants can be used as a basis for some skein module is still to be discovered.

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