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Flat hierarchy

by

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Abstract. We consider the hierarchy flats, a combinatorial generalization of flat virtual links proposed by Louis Kauffman. An approach to constructing invariants for hierarchy flats is presented; several examples are given.

1. Introduction. One usually constructs invariants of knots and related objects by using planar projections. Namely, each planar knot diagram represents a quadrivalent graph endowed with a special structure at vertices. From this point of view, knots (and links) are generalisations of curves on the plane. In these terms, knots are equivalence classes of such embedded curves modulo so-called Reidemeister moves (which are local). One might endow these curves on the plane with more combinatorial structures that lead to links after Milnor [Mi], doodles [FT], virtual knots [KaV, KM], welded knots [Satoh], singular knots, etc. One more interesting problem comes from the virtual knot theory: one considers virtual knots up to crossing switches at classical crossings. This leads to what is called virtual knots up to Vassiliev invariants of order zero. They were considered by Carter, Kamada and Saito [CKS]. These objects admit a complete classification by using a rather simple algorithm (see e.g. [Ma1, Ma2, Kad]).

This theory can be considered according to the following combinatorial formalism.

1. The main objects are quadrivalent graphs on the plane endowed with a special structure at all crossings: classical crossings are marked by 1 and virtual crossings are marked by 2.
2. Two objects are called *equivalent* if one of them is obtained from the other by a sequence of variants of Reidemeister moves shown in Fig. 1.

   For the first Reidemeister move an arbitrary crossing label is available.

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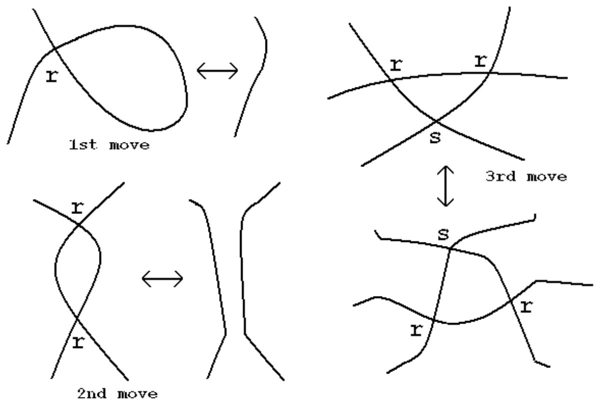

Fig. 1. Reidemeister moves for hierarchy flats

For the second Reidemeister move we require that both crossings are marked by the same number.

For the third Reidemeister move we require that the two labels on vertices going through are greater than or equal to the label of the fixed vertex.

The latter condition for the third Reidemeister moves means that virtual paths may pass through classical vertices, but not vice versa. As already mentioned, this theory has been completely classified geometrically. This theory with two types of flat crossings was called "flat virtual knot theory" in [KaV] and later called "virtual strings" (in a Gauss code formulation) in [Tur].

We can generalise the notion of flat virtuals (virtual strings) by passing to a *flat hierarchy*. That is, we use all positive integers to mark the vertices and use the same rules as above with regard to the order of the integers, giving an analogous equivalence relation. Namely, we allow the third Reidemeister move if we have two crossings of type j to go over a crossing of type i, where $j \geq i$. This structure will be called the *flat hierarchy*. Its objects will be called *hierarchy flats*.

The main problem is to classify such objects. The notion of the flat hierarchy (for an arbitrary oriented index set), and this problem of classification is due to L. H. Kauffman (see [FKM]).

The aim of the present article is to give an algebraic construction of an invariant for such objects and to present some examples how such invariants detect non-equivalent hierarchy flats.

2. Algebraic formalism. Here we shall deal only with oriented hierarchy flats for labels taken from $\mathbb{Z}_+$.

Given an oriented hierarchy flat diagram F. By an *arc* of it we mean a connected component of the set obtained from F by deleting all crossings. Enumerate all unicursal curves by integers from 1 to n for some n.

REMARK 1. In fact, the invariant we are going to construct works well for any partially ordered set.

Each crossing is characterised by three numbers: the label j, the number x of the curve coming from south-west to north-east, and the number y of the arc coming from south-east to north-west (see Fig. 2).

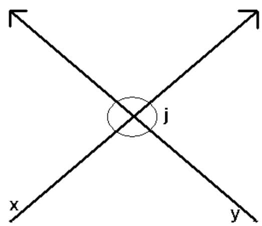

Fig. 2. A crossing

Now, let us associate with each arc an element a_i of the algebraic set M to be constructed.

After enumerating all arcs in such a way, we shall write down relations at crossings. They look as follows.

Having a crossing labelled by j between arcs $\#x$ and $\#y$, for the lower input a and b, we have the upper output $b - P^j_{xy}a$ and $a + P^j_{xy}b$, respectively.

Now, what is P^j_{xy}? They are elements of the abstract algebra to be constructed. This (non-commutative) algebra together with its left action on the set of arcs should have the following properties:

1. For all i, x, y we have $P^i_{xy} = P^i_{yx}$.
2. For all $i \leq j$ and for all x, y, z, t we have $P^i_{xy}P^j_{zt} = 0$.
3. For all $i > j$ and for all x, y, z, t we have $P^i_{xy}P^j_{zt}a = P^i_{xy}P^j_{zt}b$, where a and b play the role of labels of some arcs of the diagram (regardless of the lower indices).

The last condition can be viewed as follows: when multiplying some label a by some $P^i_{xy}P^j_{zt}$, we just forget about this a and introduce the notation $P^i_{xy}P^j_{zt}*$, where $*$ means some arc.

Thus, we have constructed an algebra of P^i_{xy}'s and its left action on arcs by multiplication.

Note that for the case of one-component link, we use just P^i's without any lower indices.

So, with each n-hierarchy flat F, we associate a module over the ring generated by P^i's with the relations described above.

Denote this module by $M(F)$.

THEOREM 1 (Main Theorem). *$M(F)$ is an invariant of hierarchy flats.*

3. Proof. In order to prove the main theorem, we have to consider all Reidemeister moves. We are going to show the following: having a part of the diagram, we can mark its input edges, i.e. edges oriented inwards, and show that all output edges can be expressed in terms of inputs (as well as all interior edges); moreover, for two diagrams which differ by a Reidemeister move, the expression of outputs in terms of inputs is the same. We shall consider only one case of the first move, one case of the second move and one case of the third move, leaving the remaining cases to the reader.

For the first Reidemeister move, we deal with only one component and one crossing, thus we can write simply P instead of P^i_{pp}. See Fig. 3.

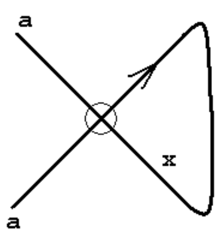

Fig. 3. The first Reidemeister move

We see that the input a is connected with the middle edge x and the output edge by the following relations:

$$a + Px = x, \qquad x - Pa = b.$$

Thus, we have $(1 - P)x = a$, so $x = (1 + P)x$; moreover, $b = a(1 + P) - Pa = a$.

Now, let us consider the second Reidemeister move. By definitions, both crossings have the same label, and in both cases we deal with the same arcs. So, we can again write P^i without any lower indices. See Fig. 4.

Having the two input edges a, b, we see that $x = a + P^i b, y = b - P^i a$, $c = x - Py = a + P^i b - P^i P^i b + P^i P^i a = a, d = y + P^i x = b$.

Finally, in the third case, let us consider Figure 5. We have three inputs (denoted by *italic* letters), labels at vertices ($r \geq s$), and numbers of components (i, j, k) typed in **bold face**.

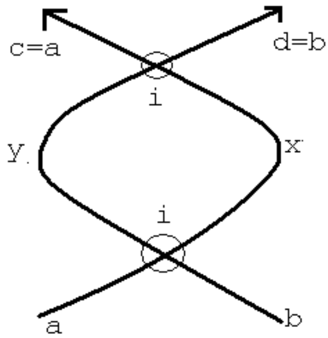

Fig. 4. The second Reidemeister move

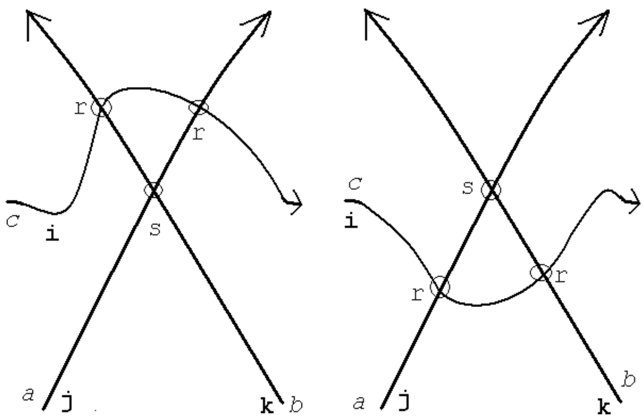

Fig. 5. The third Reidemeister move

In the first cases, the outputs are

$$a + P^s_{jk} b - P^r_{ij}(c + P^r_{ik}(b - P^s_{jk} a)),$$
$$b - P^s_{jk} a - P^r_{ik} c,$$
$$c + P^r_{ik}(b - P^s_{jk} a) + P^r_{ij}(a + P^s_{jk} b).$$

In the second case, we have:

$$a + P^r_{ij} c + P^s_{jk}(b - P^r_{ik} c),$$
$$b - P^r_{ik}(c + P^r_{ij} a) - P^s_{jk}(a - P^r_{ij} c),$$
$$c + P^r_{ij} a + P^r_{ik} b.$$

Comparing these results with respect to the multiplication axioms, we see that these outputs are indeed the same. The remaining cases can be checked analogously. Thus we complete the proof.

4. Examples. It is obvious that for the trivial circle we have the free module with one generator and for the k-component "trivial link" we have the free module with k generators. It is easy to see that if we take the trefoil shadow with three crossings, for which two of three labels coincide, we obtain a diagram which is equal to the trivial one (see Fig. 6).

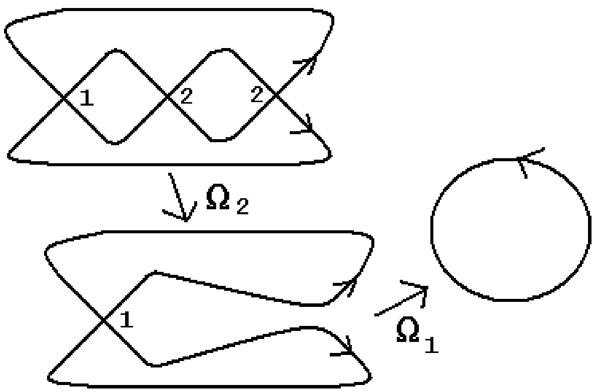

Fig. 6. A trivial flat

However, if we take three different labels, the diagram will not be equivalent to the trivial one (see Fig. 7).

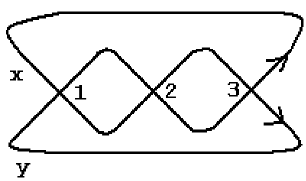

Fig. 7. A non-trivial flat

Namely, if we denote the two leftmost arcs by x and y, as shown in Fig. 7, we obviously see that y is expressible in terms of x, but the module M is not free.

More precisely, since we have only one component, we will use the notation P^j instead of P^j_{11}. Also, we shall factorize our module by all triple relations of the type $P^iP^jP^k = 0$. The aim is to show that even after such a factorization, the module turns out to be non-trivial. Thus, it is non-trivial from the very beginning.

We have the pair (x, y) which is transformed to $(y - P^1x, x + P^1y)$ after the first crossing. Then, the second crossing gives us $(x + (P^1 - P^2)y -$

$P^2P^1*, y + (P^2 - P^1)x + P^2P^1*)$. Finally, after the third crossing we get

$$(y + (P^2 - P^1 - P^3)x + (P^2P^1 + P^3P^2 - P^3P^1)y,$$
$$x + (P^1 - P^2 + P^3)y + (P^2P^1 + P^3P^2 - P^3P^1)x),$$

and the latter should be equal to (x, y).

Denote $P^1 - P^2 + P^3$ by $\mathcal{P}$ and denote $P^2P^1 + P^3P^2 - P^3P^1$ by $\mathcal{Q}$. Thus we get

$$y - \mathcal{P}x + \mathcal{Q}* = x, \quad x + \mathcal{P}y + \mathcal{Q}* = y.$$

Thus, $x = y - \mathcal{P}y - \mathcal{Q}*$, from which we get $y - \mathcal{P}y + \mathcal{P}\mathcal{P}y + \mathcal{Q}* = y - \mathcal{P}y - \mathcal{Q}*$. Hence $\mathcal{P}\mathcal{P}* + 2\mathcal{Q}* = 0$. Taking into account that $\mathcal{P}^2 = -\mathcal{Q}$, this implies $\mathcal{Q}* = 0$, which is a non-trivial relation in our module.

Thus, the hierarchy flat diagram shown in Fig. 7 is not trivial.

Analogously, one can prove that the "Borromean rings" with labelling shown in Fig. 8 are not equivalent to the flat diagram consisting of three simple disjoint circles.

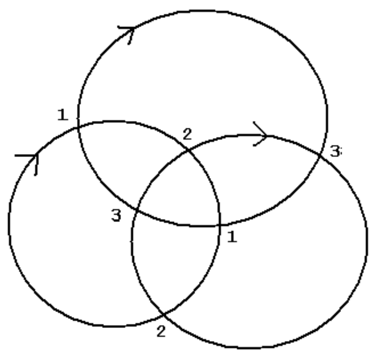

Fig. 8. Borromean rings are a non-trivial flat

To see that, it is sufficient to consider the module factorized by all relations of the type $P^iP^jP^k = 0$ and represent the Borromean rings as a closure of the 3-strand braid.

5. Post scriptum. Virtual strings and knots in thickened surfaces are closely connected to interesting algebraic structures [Tur]: Goldman's algebra and Turaev's bialgebra, whose quantizations lead to skein-modules of knots in thickened surfaces. It turns out that virtual flats admit similar structures.

We are going to discuss such structures in our future publications.

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