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Combinatorics of distance doubling maps

by

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Abstract. We study the combinatorics of distance doubling maps on the circle $\mathbb{R}/\mathbb{Z}$ with prototypes $h(\beta) = 2\beta \bmod 1$ and $\overline{h}(\beta) = -2\beta \bmod 1$, representing the orientation preserving and orientation reversing case, respectively. In particular, we identify parts of the circle where the iterates $f^{\circ n}$ of a distance doubling map f exhibit "distance doubling behavior". The results include well known statements for h related to the structure of the Mandelbrot set M. For $\overline{h}$ they suggest some analogies to the structure of the tricorn, the "antiholomorphic Mandelbrot set".

1. Introduction. There is a very rich structure in the dynamics of the angle doubling map h on the unit circle, which was particularly studied in relation to the iteration of complex quadratic maps $p_c(z) = z^2 + c$ for given parameters $c \in \mathbb{C}$. Insights into the dynamics of h have been the base for an almost complete understanding of the combinatorial structure of the Mandelbrot set

$$M = \{c \in \mathbb{C} \mid p_c\text{-orbit of } c \text{ is bounded}\},$$

illustrated in Figure 1 (left). In the present paper we study the self-similar structure of distance doubling maps and give generalizations and extensions of statements known for the angle doubling map h. Relations to the Mandelbrot set and the tricorn are discussed.

Distance doubling maps. Let $\mathbb{T}$ be the unit circle, which we identify with $\mathbb{R}/\mathbb{Z} = [0, 1[$ via $\beta \leftrightarrow e^{2\pi\beta i}$, $\beta \in [0, 1[$. We say that a map f on $\mathbb{T}$ is *distance doubling* if it is of the form

$$f(\beta) = \pm 2\beta + \beta_0 \bmod 1 \quad \text{for some } \beta_0 \in \mathbb{T}. \tag{1.1}$$

If f is such a map, then indeed it doubles the inner distance on $\mathbb{T}$. For completeness, we show in Section 2 that conversely all maps with this property are characterized by (1.1). Depending on the sign in (1.1), a distance doubling map is either orientation preserving or orientation reversing. In the first case it is topologically conjugate to the *angle doubling map* h defined

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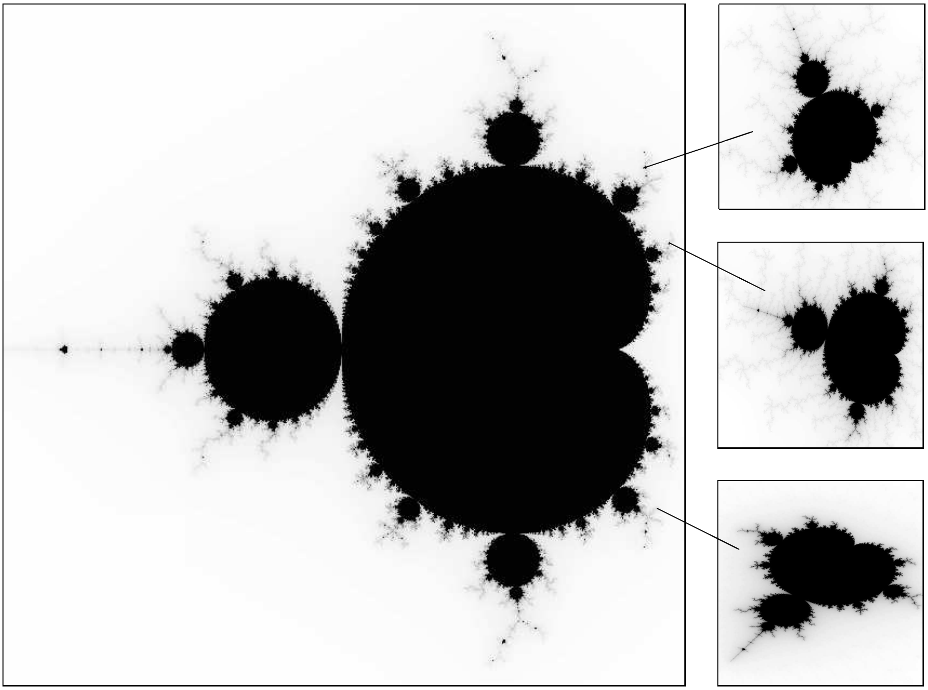

Fig. 1. Small “Mandelbrot sets” in the Mandelbrot set

by $h(\beta) = 2\beta \bmod 1$, and in the second case to the *angle “antidoubling” map* $\overline{h}$ defined by $\overline{h}(\beta) = -2\beta \bmod 1$ (cf. Proposition 2.1). This justifies concentrating on h and $\overline{h}$ in our discussion of distance doubling maps. First we try to shed some light on their relation to complex dynamics.

Quadratic and “antiquadratic” dynamics. In analogy with the quadratic maps p_c, consider the antiholomorphic—we say *antiquadratic*—maps $\overline{p}_c$ defined on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by $\overline{p}_c(z) = \overline{z}^2 + c$. Clearly, their second iterates $\widetilde{p}_c(z) := \overline{p}_c^{\circ 2}(z) = (z^2+\overline{c})^2+c$ are polynomials of degree 4. For each $c \in \mathbb{C}$, the *filled-in Julia set* K_c (resp. $\overline{K}_c$, $\widetilde{K}_c$) of p_c (resp. $\overline{p}_c$, $\widetilde{p}_c$) is the set of all points whose p_c-orbit (resp. $\overline{p}_c$-orbit, $\widetilde{p}_c$-orbit) remains bounded. Its boundary, the *Julia set* J_c (resp. $\bar{J}_c$, $\widetilde{J}_c$), contains the points with the most interesting behavior with respect to p_c (resp. $\overline{p}_c$, $\widetilde{p}_c$). One easily sees that $\overline{K}_c = \widetilde{K}_c$, hence $\bar{J}_c = \widetilde{J}_c$, for each $c \in \mathbb{C}$. Note that K_c (resp. $\overline{K}_c$), and hence J_c (resp. $\bar{J}_c$), is connected iff $c \in K_c$ (resp. $c \in \overline{K}_c$). For the following see Douady and Hubbard [5] in the quadratic case and Nakane and Schleicher [13, 14] in the antiquadratic case.

Given some $c \in \mathbb{C}$, between sufficiently small neighborhoods of ∞ in the Riemann sphere there exists a unique conformal map fixing ∞, tangent to the identity there, and conjugating p_c and $p_0(z) = z^2$ (resp. $\widetilde{p}_c$ and

$\widetilde{p}_0(z) = z^4$). This has already been shown by Bötkher [2] in the general context of polynomials at the beginning of the last century. The map extends uniquely to a conformal conjugacy Φ_c (resp. $\bar{\Phi}_c$) onto a domain containing c if K_c (resp. $\overline{K}_c = \widetilde{K}_c$) is disconnected and onto the whole complement of K_c (resp. $\overline{K}_c = \widetilde{K}_c$) otherwise. In the latter case Φ_c (resp. $\bar{\Phi}_c$) maps $\mathbb{C} \setminus K_c$ (resp. $\mathbb{C} \setminus \overline{K}_c$) onto $\mathbb{C} \setminus \mathbb{D}$, where $\mathbb{D}$ denotes the unit disk. This allows us to define curves $R_c^\beta = \Phi_c^{-1}(\{re^{2\pi\beta i} \mid r \in]1, \infty[\})$ (resp. $\bar{R}_c^\beta = \bar{\Phi}_c^{-1}(\{re^{2\pi\beta i} \mid r \in]1, \infty[\})$ called *dynamic rays*, which have the invariance property

$$p_c(R_c^\beta) = R_c^{h(\beta)} \quad \text{(resp. } \overline{p}_c(\bar{R}_c^\beta) = \bar{R}_c^{\bar{h}(\beta)}). \tag{1.2}$$

This property indicates that the structure of J_c is strongly related to h, and similarly the structure of $\bar{J}_c$ to $\bar{h}$. The “antidoubling” map $\bar{h}$ plays a similar role for antiquadratic maps in the complex plane as h for quadratic maps.

A dynamic ray having exactly one accumulation point z in the Julia set is said to *land* at z. In case the Julia set is locally connected, each dynamic ray lands and the Julia set can be considered as the quotient of $\mathbb{T}$ with respect to the equivalence relation that identifies angles β_1, β_2 iff the corresponding dynamic rays land at the same point. (For a general discussion of the structure of polynomial Julia sets see [8].) Here we are only interested in the landing behavior of rays corresponding to periodic angles $\beta \in \mathbb{T}$ (periodic with respect to h or $\bar{h}$). The crucial point is that all dynamic rays for periodic angles land at a periodic point of the Julia set different from 0 and that only finitely many such rays land at the same periodic point. We come back to this fact when defining bifurcation chords.

Kneading sequences. In order to describe the dynamics of a distance doubling map f on $\mathbb{T}$ we look at the orbits of angles and how they differ. For this we use kneading sequences, first introduced by Milnor and Thurston [12] for interval maps. The idea is to divide $\mathbb{T}$ appropriately into two half-circles. The kneading sequence of an angle indicates in which half-circle each of its iterates lies.

DEFINITION 1.1. Let f be a distance doubling map on $\mathbb{T}$ and for $\alpha \in \mathbb{T}$ let β_1, β_2 be the two angles with $f(\beta_1) = f(\beta_2) = \alpha$. From the two open half-circles $]\beta_1, \beta_2[$ and $]\beta_2, \beta_1[$, taken in counter-clockwise direction, let $\mathbb{T}_0$ be the one containing α and $\mathbb{T}_1$ the other one. (In case $\alpha = \beta_1$ set $\mathbb{T}_0 =]\beta_1, \beta_2[$.) The *kneading sequence* of α with respect to f is the sequence $s_1 s_2 s_3 \ldots \in \{0, 1, *\}^{\mathbb{N}}$ defined by

$$s_i = \begin{cases} * & \text{if } f^{\circ i-1}(\alpha) \in \{\beta_1, \beta_2\}, \\ 0 & \text{if } f^{\circ i-1}(\alpha) \in \mathbb{T}_0, \\ 1 & \text{if } f^{\circ i-1}(\alpha) \in \mathbb{T}_1, \end{cases}$$

for $i = 1, 2, \ldots$, where $f^{\circ n}(\beta)$ denotes the nth iterate of $\beta \in \mathbb{T}$ with respect to f.

Note that kneading sequences of periodic angles are periodic and contain $*$. As a function of α the kneading sequence is discontinuous only at periodic angles. Here the changes of the dynamics take place.

EXAMPLE 1.2. With respect to h, the kneading sequence of $1/3$ is $0 * 0 * 0 * \ldots = \overline{0*}$. The counter-clockwise limit of the kneading sequence at $1/3$, i.e. the limit as the angle α tends to $1/3$ in counter-clockwise direction, is $000\ldots = \overline{0}$, while the clockwise limit at $1/3$ is $010101\ldots = \overline{01}$. There is a switch from the kneading sequence $\overline{0}$ ($= \overline{00}$) to the kneading sequence $\overline{01}$ at $1/3$ (see Figure 2). Such a switching behavior is typical for periodic

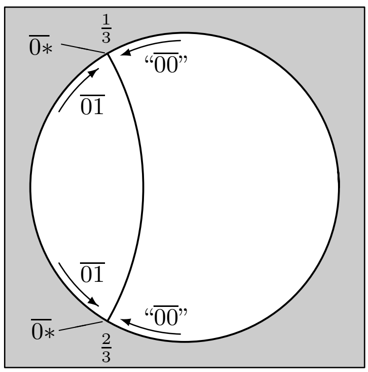

Fig. 2. Switch of the kneading sequence

angles of h as well as of $\overline{h}$, the only exceptions being the fixed points, where the kneading sequence is $\overline{*}$ and the clockwise and counter-clockwise limits coincide. For 0, the fixed point of h, they are $\overline{0}$, and for $0, \frac{1}{3}, \frac{2}{3}$, the fixed points of $\overline{h}$, they are $\overline{01}$.

Here we give the general statement. The rather simple proof is left to the reader.

LEMMA 1.3. *Let $\alpha \in \mathbb{T}$ be periodic with period $m > 1$. Then for each $N \in \mathbb{N}$ and $\varepsilon > 0$ sufficiently small the kneading sequences $s_1^{\pm} s_2^{\pm} \ldots$ of $\alpha \pm \varepsilon$ satisfy $s_n^+ \neq s_n^-$ whenever $n \leq N$ is a multiple of m, and $s_n^+ = s_n^-$ whenever $n \leq N$ is not a multiple of m. If $\alpha \in \mathbb{T}$ is neither periodic nor a fixed point, then for each $N \in \mathbb{N}$ and $\varepsilon > 0$ sufficiently small, $s_n^+ = s_n^-$ for all $n \leq N$.*

Bifurcation chords. To understand the dynamics it is helpful to look at pairs of angles, rather than at single periodic angles. We consider pairs of angles β_1, β_2 as *chords* $\beta_1\beta_2$ between these two angles, i.e. as straight lines in the unit disk bounded by $\mathbb{T}$ connecting both angles. (In pictures we draw the chords in a "hyperbolic" way, allowing us to show more chords.) For some map f on $\mathbb{T}$ the mth f-iterate of a chord $\beta_1\beta_2$ is defined as $f^{\circ m}(\beta_1\beta_2) = f^{\circ m}(\beta_1)f^{\circ m}(\beta_2)$. Further, we say that two chords *cross* each other if they intersect in exactly one interior point. Consequently, they do *not cross* each other if they have no common interior point (but possibly a common endpoint) or if they coincide.

Let Per^f denote the set of periodic angles of a distance doubling map f of period greater than 1, and $f^{-1}(\alpha)$ the chord connecting the two f-preimages of $\alpha \in \mathbb{T}$.

DEFINITION 1.4. A chord $\alpha\gamma$ with $\alpha, \gamma \in \mathrm{Per}^f$, $\alpha \neq \gamma$, is called a *bifurcation chord* if its iterates do not cross each other and do not cross $f^{-1}(\alpha)$ and $f^{-1}(\gamma)$.

It will become clear later that for orientation reversing maps f there exist bifurcation chords with endpoints of different periods. This is one of the astonishing differences from the classical (orientation preserving) case. The definition is motivated by the fact that in such a chord the dynamical properties of the angles in $\mathbb{T}$ "bifurcate". With respect to h, for example, the chord $\frac{1}{3}\frac{2}{3}$ is a bifurcation chord. The kneading sequence of $2/3$ is the same as that of $1/3$, while now the counter-clockwise limit of the kneading sequence is $\overline{01}$ and its clockwise limit is $\overline{0}$ (cf. Example 1.2). When passing the chord $\frac{1}{3}\frac{2}{3}$ the kneading sequence switches from $\overline{0}$ to $\overline{01}$ (see Figure 2).

Before we turn to the characterization of bifurcation chords, we try to give some idea how they are related to the dynamics of p_c (resp. $\overline{p}_c$). Assume that for two f-periodic angles $\beta_1, \beta_2 \in \mathbb{T}$ where $f = h$ (resp. $f = \overline{h}$) the corresponding dynamic rays land at the same periodic point z. Call β_1, β_2 *adjacent* if one of the open components of $\mathbb{C}$ bounded by the rays and z does not contain a periodic dynamic ray with landing point z. We fix adjacent β_1, β_2. By the invariance property (1.2), the dynamic rays corresponding to the nth iterates of β_1, β_2 both land at the nth iterate of z. Since all iterates of z are different from the critical point 0, the map p_c (resp. $\overline{p}_c$) preserves (resp. reverses) the circular order between dynamic rays landing at such an iterate of z. Hence all iterates of $\beta_1\beta_2$ are adjacent and, as a consequence of disjointness of dynamic rays, they do not cross each other.

Choose a longest chord $\delta_1\delta_2$ among them, i.e. one whose endpoints have maximal inner distance in $\mathbb{T}$. Further, let $\delta_1' = \delta_1 + \frac{1}{2} \bmod 1$ and $\delta_2' = \delta_2 + \frac{1}{2} \bmod 1$. By symmetry arguments one easily sees that the dynamic rays corresponding to δ_1' and δ_2' also land at the same point. Let $\alpha = f(\delta_1)$ and $\gamma = f(\delta_2)$. Since dynamic rays are disjoint, the iterates of $\alpha\gamma$ (coinciding with those of $\delta_1\delta_2$) also do not cross $\delta_1'\delta_2'$, thus, by the assumption on the distance of δ_1 and δ_2, they do not cross $f^{-1}(\alpha) = \delta_1\delta_1'$ and $f^{-1}(\gamma) = \delta_2\delta_2'$.

Hence $\alpha\gamma$ is a bifurcation chord. Therefore, if two dynamic rays $R_c^{\beta_1}, R_c^{\beta_2}$ land at the same point, then $\beta_1\beta_2$ is not necessarily a bifurcation chord itself, but there is one associated to the orbit of the point.

Our first main result characterizes the set Bif^f of all bifurcation chords of a distance doubling map f. In case f is orientation preserving (prototype h), this set consists of disjoint chords (Figure 3, left). In case f is orientation reversing (prototype $\overline{h}$), the situation is more delicate (Figure 3, right).

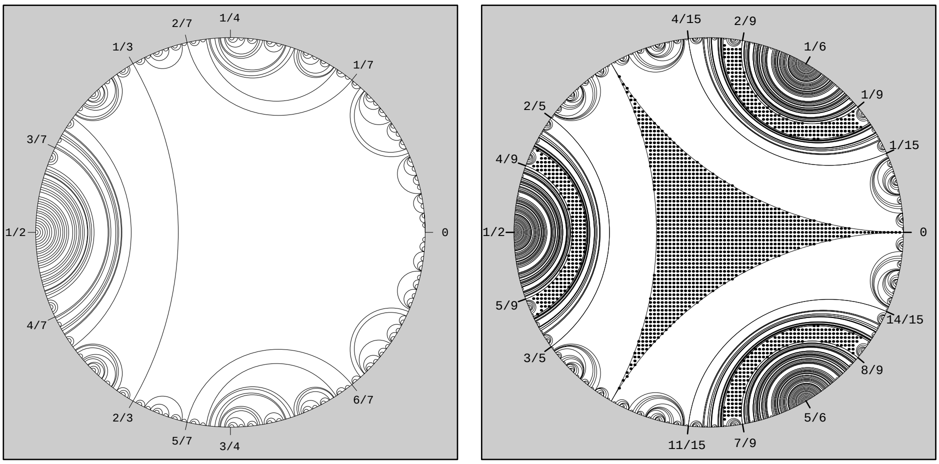

Fig. 3. The two prototypes of Bif^f: the abstract Mandelbrot set (left) and the abstract tricorn (right)

A set of disjoint chords is obtained when restricting to bifurcation chords with endpoints of equal period. But this time there are also bifurcation chords with one endpoint of odd period and the other of the double period. For each such chord there is a unique second one such that these two form a rectangle together with two bifurcation chords with endpoints of equal period. Finally, the three bifurcation chords connecting the three fixed points of f form a triangle, which we regard as a degenerate rectangle. The rectangles (indicated in gray in the picture) are disjoint from each other and disjoint from any other bifurcation chord of f.

THEOREM 1 (Bifurcation Theorem). *Let f be a distance doubling map on $\mathbb{T}$.*

(i) *For each $\alpha \in \mathrm{Per}^f$ there exists a unique $\theta = \Theta^f(\alpha)$ of the same period such that $\alpha\theta$ forms a bifurcation chord. The kneading sequences of α and θ coincide and are different from those of all β in the smaller arc between α and θ. Two chords $\alpha_1\Theta^f(\alpha_1)$ and $\alpha_2\Theta^f(\alpha_2)$ either coincide or are disjoint.*

(ii) *For f orientation preserving,* $\mathrm{Bif}^f = \{\alpha\Theta^f(\alpha) \mid \alpha \in \mathrm{Per}^f\}$.

(iii) *For f orientation reversing and $\alpha \in \mathrm{Per}^f$ of odd period n, there exists a unique $\lambda = \Lambda^f(\alpha)$ of period $2n$ such that $\alpha\lambda$ forms a bifurcation chord. Moreover, $\Theta^f \circ \Lambda^f = \Lambda^f \circ \Theta^f$, and so $\Theta^f(\alpha)\Theta^f(\lambda)$ is a bifurcation chord as well. The rectangle with sides $\alpha\Theta^f(\alpha)$, $\Theta^f(\alpha)\Theta^f(\lambda)$, $\lambda\Theta^f(\lambda)$ and $\alpha\lambda$ is disjoint from any other bifurcation chord of f. A bifurcation chord $\beta\Theta^f(\beta)$ with endpoints of period $2n$ is a side of such a rectangle iff n is odd and $f^{\circ n}(\beta) = \Theta^f(\beta)$.*

(iv) *For f orientation reversing,* Bif^f *is the union of the set* $\{\alpha\Theta^f(\alpha) \mid \alpha \in \mathrm{Per}^f\}$, *the set* $\{\alpha\Lambda^f(\alpha) \mid \alpha \in \mathrm{Per}^f$ *of odd period*$\}$ *and the set of the three chords connecting the three fixed points of f.*

An immediate consequence of Theorem 1(i) is that Θ^f maps the set Per^f onto itself and that $\Theta^f \circ \Theta^f$ is the identity map. We call a bifurcation chord of f *free* if it is disjoint from any other bifurcation chord of f. For orientation preserving maps, assertions (i) and (ii) state that all bifurcation chords are free. For orientation reversing maps f, a bifurcation chord is free iff it is not part of a rectangle. Hence, by (iii), a free chord has two endpoints of equal even period and, in case their period is $2n$ with n odd, the last assertion of (iii) implies that they are not interchanged by $f^{\circ n}$. The remaining bifurcation chords with endpoints of even period determine non-degenerate rectangles: Each rectangle contains exactly one such chord, which we call the *regular side* of the rectangle; the other three sides, which have at least one endpoint of odd period, are referred to as *irregular*. Naturally, the sides of the triangle spanned by the fixed points of f are also considered to be irregular.

Mandelbrot set and tricorn. The left part of Figure 3 illustrating Bif^h is well known. It reflects the combinatorial structure of the Mandelbrot set M (cf. Figure 1, left), a fact first described by Lavaurs in [9]. In order to recall the exact statement, we introduce two more notions.

For a distance doubling map f let $\mathbb{D}^f$ be the topological space obtained from the disk $\mathbb{D}$ by contracting to a point each chord in the closure of the set of free bifurcation chords. (A sequence of chords converges to a (possibly degenerate) chord if the sequences of endpoints converge to the endpoints of the chord.) Each free bifurcation chord $\alpha\gamma$ (i.e. each bifurcation chord if f is orientation preserving) divides $\mathbb{D}$ into two parts. The part of $\mathbb{D}^f$ obtained from the smaller closed part of $\mathbb{D}$ is denoted by $\mathbb{D}^f(\alpha\gamma)$.

$\mathbb{D}^h$ is Douady's pinched-disk model [4] based on the following celebrated results by Douady and Hubbard [5] (cf. also [9, 10, 15, 7]): The map $c \mapsto \Phi_c(c)$ provides a conformal isomorphism from $\mathbb{C} \setminus M$ onto $\mathbb{C} \setminus \mathbb{D}$ and *parameter rays* R^β defined similarly to the dynamic rays. Two parameter rays R^α, R^γ, where α and γ are h-periodic, land at the same point of M if and only if $\alpha\gamma$ is a bifurcation chord. (More precisely, those common landing points coincide with the roots of hyperbolic components different from the main one containing 0.) R^α and R^γ and their landing point $c(\alpha\gamma)$ divide the complex plane into two open parts. For $c \in M$, the dynamic rays R_c^α and R_c^γ land at the same point iff c is in the part not containing 0 or $c = c(\alpha\gamma)$. This yields the following

Description of the Mandelbrot set. There exists a continuous map Π from the Mandelbrot set M onto $\mathbb{D}^h$ with connected preimages $\Pi^{-1}(a)$ for all $a \in \mathbb{D}^h$ and with the following property for each $\alpha\gamma \in \mathrm{Bif}^h$: For $c \in M$, the dynamic rays R_c^α, R_c^γ land at the same point iff $\Pi(c) \in \mathbb{D}^h(\alpha\gamma)$.

If M is locally connected, as is conjectured by many researchers in complex dynamics, Π can be chosen as a homeomorphism. (Then the restriction of Π^{-1} to the boundary of M is uniquely determined.)

The question arises whether there is an object in the complex plane related to $\mathrm{Bif}^{\overline{h}}$ in a similar way as the Mandelbrot set to Bif^h. With the above considerations in mind the natural candidate is the *tricorn*

$$\overline{M} = \{c \in \mathbb{C} \mid \overline{p}_c\text{-orbit of } c \text{ is bounded}\}$$

shown in Figure 4 (left). It is connected like M (see Nakane [13]), but it

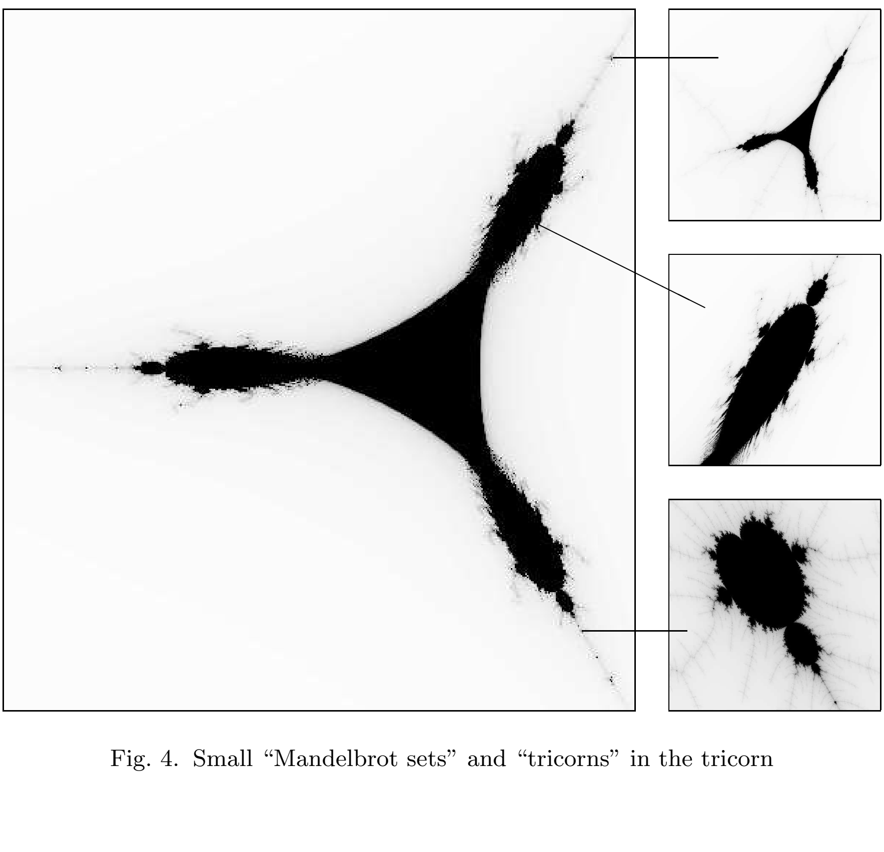

Fig. 4. Small “Mandelbrot sets” and “tricorns” in the tricorn

fails to be locally connected (see Nakane and Schleicher [14]). Note that the interest in the set $\overline{M}$ goes back to Crowe, Hasson, Rippon and Strain-Clark [3], who called it the Mandelbar set, Winters [17] and Milnor [11].

The map $c \mapsto \bar{\Phi}_c(c)$ is a well defined homeomorphism from $\mathbb{C} \setminus \overline{M}$ onto $\mathbb{C} \setminus \mathbb{D}$, allowing one to define parameter rays as in the quadratic case. But now the dependence on c is not complex-analytic any more (see Nakane [13]). This is the reason why the landing behavior of parameter rays is not well understood, even for periodic angles. Note that in the quadratic case complex analyticity is substantial in the proof of the landing of "periodic" parameter rays. Nevertheless we believe that the following is true.

Conjecture (Description of the tricorn). There exists a continuous map Π from the tricorn onto $\mathbb{D}^{\bar{h}}$ with connected preimages $\Pi^{-1}(a)$ for all $a \in \mathbb{D}^{\bar{h}}$ and with the following property for each free bifurcation chord $\alpha\gamma$: For $c \in \overline{M}$, the dynamic rays $\bar{R}_c^{\alpha}, \bar{R}_c^{\gamma}$ land at the same point iff $\Pi(c) \in \mathbb{D}^{\bar{h}}(\alpha\gamma)$.

In the following we call Bif^f the *abstract Mandelbrot set* for $f = h$ and the *abstract tricorn* for $f = \overline{h}$. The reader will see later on why we consider the set of all bifurcation chords also for $f = \overline{h}$, although the relationship to the tricorn conjectured above relies on the set of free ones.

Tuning for distance doubling maps. One of the main objectives of this paper is to localize self-similar structures in the system of distance doubling maps. By self-similarity we mean the existence of subsets of $\mathbb{T}$ on which some iterate $f^{\circ n}$ of a distance doubling map f behaves like some distance doubling map on $\mathbb{T}$. It will turn out that such subsets are always Cantor sets, i.e. totally disconnected, closed and perfect subsets of $\mathbb{T}$.

The complement $\mathbb{T} \setminus C$ of a Cantor set C consists of countably many open intervals. We denote the set of chords connecting the endpoints of these open intervals by $\mathcal{B}_C$. Contracting each chord of $\mathcal{B}_C$ to a point turns C into a topological circle which we denote by $\mathbb{T}_C$. Thus $\mathcal{B}_C$ can be seen either as a set of chords as defined above, or as a set of points in $\mathbb{T}_C$. Each Cantor set C we are interested in is closely related to some iterate $f^{\circ n}$ of a distance doubling map f, therefore we call it an $f^{\circ n}$-set. The following definitions (cf. [1]) make this precise.

Definition 1.5. Finitely many subsets of $\mathbb{T}$ are said to be *weakly unlinked* if for any two of them there exist angles β_1, β_2 such that the interval $[\beta_1, \beta_2]$ contains one of the sets and $[\beta_2, \beta_1]$ the other one. They are called *unlinked* if they are weakly unlinked and mutually disjoint.

If sets A and B are weakly unlinked, they have at most two angles in common, and chords with endpoints in A do not cross chords with endpoints in B.

Definition 1.6. Let f be a distance doubling map on $\mathbb{T}$. An $f^{\circ n}$-invariant Cantor set $C \subset \mathbb{T}$ is called an *$f^{\circ n}$-set* if

(i) the sets $C, f(C), f^{\circ 2}(C), \ldots, f^{\circ n-1}(C)$ are weakly unlinked,
(ii) $f^{\circ n-1}(C)$ is invariant under $\beta \mapsto \beta + \frac{1}{2} \bmod 1$.

For each $f^{\circ n}$-set C there is a unique longest chord B_C in $\mathcal{B}_C$ (connecting the two angles in C with the largest distance from each other). It turns out that B_C is always a bifurcation chord and that it determines the $f^{\circ n}$-set uniquely. More precisely, there is a one-to-one correspondence between bifurcation chords of f and $f^{\circ n}$-sets. This characterization of $f^{\circ n}$-sets is given in the second central theorem below. Moreover, the theorem states that the $f^{\circ n}$-sets are the "loci of self-similarity": Each $f^{\circ n}$-set C carries a small copy of one of the pictures in Figure 3. The properties of its longest chord B_C determine which one. If B_C is a free bifurcation chord or an irregular side of a rectangle, then an abstract Mandelbrot set is included. If and only if B_C is the regular side of a rectangle—which is equivalent to C being an $f^{\circ n}$-set with f orientation reversing and n odd—an abstract tricorn occurs.

In the case $f = h$, all bifurcation chords are free, hence only abstract Mandelbrot sets are included. The corresponding phenomenon in the complex plane of small copies of M in the Mandelbrot set M, illustrated by Figure 1, is well known as Douady–Hubbard tuning (cf. [6]). Here the location of small copies of M can be understood on the abstract level. In the abstract tricorn there are copies of the abstract Mandelbrot set and the abstract tricorn, and equally $\overline{M}$ contains small copies of M and $\overline{M}$ (cf. Figure 4). However, the exact relationship between the copies on the abstract and the concrete level is not established here.

Theorem 2 (Similarity Theorem). *Let f be a distance doubling map on the circle $\mathbb{T}$.*

(i) *The assignment $C \mapsto B = B_C = \beta_1\beta_2$ provides a one-to-one correspondence between $f^{\circ n}$-sets C with $n \in \mathbb{N}$, and bifurcation chords $B \in \mathrm{Bif}^f$ of f.*
(ii) *A classification of $f^{\circ n}$-sets in terms of B is given as follows:*

$$\begin{aligned} B \textit{ is free} &\Leftrightarrow \beta_1, \beta_2 \textit{ have period } n \\ &\Rightarrow f^{\circ n} \textit{ preserves orientation}, \\ B \textit{ is regular} &\Leftrightarrow \beta_1, \beta_2 \textit{ have period } 2n \\ &\Leftrightarrow f^{\circ n} \textit{ reverses orientation}, \\ B \textit{ is irregular} &\Leftrightarrow \beta_1, \beta_2 \textit{ have minimum period } n/2 \\ &\Rightarrow f^{\circ n} \textit{ preserves orientation.} \end{aligned}$$

With $g = \left\{ \begin{array}{ll} h & \textit{if } B \textit{ is free or irregular} \\ \overline{h} & \textit{if } B \textit{ is regular} \end{array} \right\}$ *the following holds*:

(iii) *There exists a unique orientation preserving map* $\pi_C : \mathbb{T}_C \to \mathbb{T}$ *mapping the chord* B *to* 0 *and conjugating* $f^{\circ n}$ *to* g.

(iv) *A bifurcation chord* S *with endpoints different from those of* B *has either both endpoints or no endpoint in* C, *and* π_C *maps the set* Bif_C^f *of bifurcation chords of* f *with endpoints in* C *bijectively onto* Bif^g.

The above pictures and what is known about the Mandelbrot set suggest that free chords in the abstract tricorn correspond to small Mandelbrot sets in the tricorn, and rectangles to small tricorns. As chords in the abstract Mandelbrot set characterize the roots of small Mandelbrot sets, the regular chord of a rectangle seems to be related to the "root" of the corresponding small tricorn. Each of the three irregular chords corresponds to the "root" of one of the small Mandelbrot sets springing forth directly from the main component of the small tricorn. But now, as shown in [3] for the main component of the original tricorn, these "roots" do not seem to be single points any more but curves.

Organization of the paper. In Section 2 we classify distance doubling maps and their periodic angles. In Sections 3–6 we derive several preliminary results in preparation for the proof of the main theorems. The behavior of chords and finite sets under iteration is studied in Section 3, while we construct and investigate very special chord sets using backward iteration in Sections 4 and 5. $f^{\circ n}$-sets are examined in Section 6. Finally, in the last two sections we put all the pieces together to prove Theorems 1 and 2. Below, f will always denote an arbitrary but fixed distance doubling map.

2. Distance doubling maps. Let d be the distance on $\mathbb{T}$ determined by arc length: according to the identification we have made,

$$d(\beta_1, \beta_2) = \min\{(\beta_1 - \beta_2) \bmod 1, (\beta_2 - \beta_1) \bmod 1\}. \tag{2.1}$$

To justify the notion of distance doubling map defined in (1.1), we show that such maps are equally characterized by the property

$$d(f(\beta_1), f(\beta_2)) = 2d(\beta_1, \beta_2) \quad \text{for } d(\beta_1, \beta_2) \leq 1/4. \tag{2.2}$$

The implication (1.1)⇒(2.2) is obvious. Clearly, the class of maps defined by (2.2) remains the same when $1/4$ is replaced with any number $\varepsilon < 1/4$. Moreover, it is easily seen that a map f satisfying (2.2) is continuous and (locally) orientation preserving or orientation reversing, hence a two-fold covering map. In order to see that such a map is of the form (1.1), one only needs to show that it has a fixed point.

If $\beta \in \mathbb{T}$ is not a fixed point, then for some δ with sufficiently small absolute value we have

$$d(\beta+\delta, f(\beta+\delta)) = d(\beta+\delta, f(\beta)+2\delta) < d(\beta, f(\beta)) \quad \text{or}$$
$$d(\beta+\delta, f(\beta+\delta)) = d(\beta+\delta, f(\beta)-2\delta) < d(\beta, f(\beta)),$$

for f orientation preserving or orientation reversing, respectively. (All angles are to be considered modulo 1.) Since $\mathbb{T}$ is compact, there exists an angle whose distance to its image is minimal. By the above inequalities this distance is 0, providing an angle β_0 fixed by f.

Remark. Obviously, h has the unique fixed point 0, and $\overline{h}$ has the three fixed points $0, 1/3$ and $2/3$. A conjugacy maps fixed points to fixed points. So on the one hand h and $\overline{h}$ cannot be conjugate, and on the other hand an orientation preserving conjugacy between h and h must be the identity, and between $\overline{h}$ and $\overline{h}$ the identity, rotation by $1/3$, or rotation by $2/3$. Finally, note that both h and $\overline{h}$ commute with the map $\beta \mapsto 1-\beta$, implying that the corresponding kneading sequences are invariant with respect to this map.

This remark and the above considerations provide the following

Proposition 2.1 (Two prototypes of distance doubling maps). *A map f on the circle satisfies* (2.2) *iff it is of the form* (1.1). *In this case*, *f is topologically conjugate either to the angle doubling map h or to the angle "antidoubling" map $\overline{h}$. There is a unique orientation preserving conjugacy to h if f is orientation preserving*, *and an orientation preserving conjugacy to $\overline{h}$ which is unique up to rotation by $1/3$ or $2/3$ if f is orientation reversing. All conjugacies are isometries.* ∎

Note that h was comprehensively studied in [7]. However, it does not require much additional effort to include h into our discussion. We state two obvious but useful equalities for $\beta \in \mathbb{T}$:

$$\overline{h}^{\circ m}(\beta) = h^{\circ m}(\beta) \qquad \text{for } m \text{ even,} \tag{2.3}$$
$$\overline{h}^{\circ m}(\beta) = 1 - h^{\circ m}(\beta) \qquad \text{for } m \text{ odd.} \tag{2.4}$$

As mentioned before, the dynamical properties of distance doubling maps change at periodic angles. Therefore, we now characterize those angles for the two prototypes of distance doubling maps, h and $\overline{h}$:

Proposition 2.2 (Characterization of periodic angles). *An angle $\beta \in \mathbb{T}$ is periodic for $\overline{h}$* (*and h*) *iff it is of the form p/q with q odd. Moreover*, *the following statements are valid for $\beta \in \mathbb{T}$:*

(i) $h^{\circ m}(\beta) = \beta \Leftrightarrow \beta = p/(2^m - 1)$ *for some* $p \in \mathbb{N}$.
(ii) $\overline{h}^{\circ m}(\beta) = \beta$ *with m even* $\Leftrightarrow \beta = p/(2^m - 1)$ *for some* $p \in \mathbb{N}$.
(iii) $\overline{h}^{\circ m}(\beta) = \beta$ *with m odd* $\Leftrightarrow \beta = p/(2^m + 1)$ *for some* $p \in \mathbb{N}$.

Proof. Each h-iterate of a $\beta = p/q$ with q odd can be represented by a fraction with denominator q. So the orbit of β is finite, hence there is an n such that $h^{\circ n}(\beta)$ is periodic of some period m.

If $n > 0$, then either $h^{\circ n-1}(\beta) = h^{\circ n+m-1}(\beta) = r/q$ is periodic, or

$$h^{\circ n-1}(\beta) = \frac{r}{q} + \frac{1}{2} \bmod 1 = \frac{2r+q}{2q} \bmod 1$$

for some r. In the latter case $h^{\circ n-1}(\beta)$ cannot be represented by a fraction with denominator q, since $2r+q$ is odd. So $h^{\circ n-1}(\beta)$ is periodic. By induction one shows h-periodicity of β itself. Then (2.3) and (2.4) imply $\overline{h}$-periodicity. (i)–(iii) can be obtained by straightforward computations (for (ii) and (iii) see (2.3) and (2.4)). ∎

Finally, note that even if the sets of periodic angles of h and $\overline{h}$ are the same, the periods of angles can differ. For example, the angle $1/3$ has period 2 under h but period 1 under $\overline{h}$.

3. Chords under iteration. By the *length* $d(S)$ of a chord $S = \beta_1\beta_2$ we understand the distance $d(\beta_1, \beta_2)$ of its endpoints as given by (2.1). A subset of the unit disk is said to be *between* two disjoint chords if it has at least one point in the open component bounded by both chords, but none in the other two open parts of the disk. It is said to lie *behind* a chord of length less than $1/2$ if at least one of its points lies in the smaller component determined by the chord and none in the other component. For $\beta \in \mathbb{T}$ let $\beta' = \beta + \frac{1}{2} \bmod 1$, and for a chord $S = \beta_1\beta_2$ let $S' = \beta_1'\beta_2'$. Note that $f(\beta') = f(\beta)$ for all $\beta \in \mathbb{T}$.

Chord length under iteration. For each chord S we have

$$d(f(S)) = \begin{cases} 2d(S) & \text{for } d(S) \le 1/4, \\ 1 - 2d(S) & \text{for } d(S) > 1/4, \end{cases} \tag{3.1}$$

and

$$d(S) \in \left\{ \frac{d(f(S))}{2}, \frac{1 - d(f(S))}{2} \right\}, \tag{3.2}$$

which yields the following statement:

LEMMA 3.1. *For a chord S and $n \in \mathbb{N}$ assume that none of the chords $f(S), f^{\circ 2}(S), \ldots, f^{\circ n-1}(S)$ is longer than S. If $f^{\circ n}(S)$ lies between S and S', then $d(f^{\circ n}(S)) > d(S)$.*

Proof. For $d(S) < 1/3$ the statement is obvious, so assume that $d(S) \ge 1/3$. If $f^{\circ n}(S)$ lies between S and S', then either $d(f^{\circ n}(S)) > d(S)$ or $d(f^{\circ n}(S)) \le 1/2 - d(S)$. In the second case there would exist a least index $i \le n$ with $d(f^{\circ i}(S)) < 1 - 2d(S)$. Now apply formula (3.1). Since $d(f^{\circ i}(S)) = 2d(f^{\circ i-1}(S))$ would imply $d(f^{\circ i-1}(S)) < 1 - 2d(S)$, a contradiction to i being the least such index, we have $d(f^{\circ i}(S)) = 1 - 2d(f^{\circ i-1}(S))$. But then $f^{\circ i-1}(S)$ would be longer than S, contrary to our assumptions. ∎

Sets of angles with similar dynamics. For angles $\beta_1, \ldots, \beta_n \in \mathbb{T}$ we write $\beta_1 \curvearrowright \cdots \curvearrowright \beta_n$ (resp. $\beta_1 \curvearrowleft \cdots \curvearrowleft \beta_n$) if they lie in clockwise (counterclockwise) orientation, and we make the following simple but important observation: Provided $\beta_1, \ldots, \beta_n$ lie in an open half-circle, then

$$\beta_1 \curvearrowright \cdots \curvearrowright \beta_n \quad (\text{resp. } \beta_1 \curvearrowleft \cdots \curvearrowleft \beta_n)$$

implies

$$f(\beta_1) \curvearrowright \cdots \curvearrowright f(\beta_n) \quad (\text{resp. } f(\beta_1) \curvearrowleft \cdots \curvearrowleft f(\beta_n))$$

if f is orientation preserving, and

$$f(\beta_1) \curvearrowleft \cdots \curvearrowleft f(\beta_n) \quad (\text{resp. } f(\beta_1) \curvearrowright \cdots \curvearrowright f(\beta_n))$$

if f is orientation reversing. This has far-reaching consequences for chords as well as for finite sets of periodic angles, which we now present in a sequence of statements.

LEMMA 3.2. *If the iterates of a chord S do not cross each other*, *then they do not cross S'.*

Proof. If $f^{\circ i}(S)$ crossed S' but not S for some i, then the endpoints of $f^{\circ i}(S)$ and S' would lie in an open half-circle, hence $f^{\circ i+1}(S)$ would cross $f(S) = f(S')$, contradicting the assumption. ∎

The two preceding lemmata lead to the following observation for bifurcation chords, where, for $\alpha \in \mathrm{Per}^f$ of period m, we denote by $\dot{\alpha} = f^{\circ m-1}(\alpha)$ the periodic preimage of α and by $\ddot{\alpha} = \dot{\alpha}'$ the preperiodic one.

LEMMA 3.3. *The iterates of a bifurcation chord $\alpha\gamma$ do not meet the open part between $\dot{\alpha}\dot{\gamma}$ and $\ddot{\alpha}\ddot{\gamma}$, and $\dot{\alpha}\dot{\gamma}$ is the longest iterate of $\alpha\gamma$. Moreover*, *$\dot{\alpha}\dot{\gamma}$ is not shorter than* $1/3$, *and $\alpha\gamma$ is not longer than* $1/3$.

Proof. If S is the longest iterate of $\alpha\gamma$, then by Lemmata 3.1 and 3.2 no iterate of α or γ lies between S and S'. On the other hand, since $f^{-1}(\alpha)$ and $f^{-1}(\gamma)$ do not cross S, S', the endpoints of S and S' are $\dot{\alpha}, \dot{\gamma}, \ddot{\alpha}, \ddot{\gamma}$. For the chord lengths compare (3.1). ∎

Now we turn our attention to finite $f^{\circ m}$-invariant sets lying in an open half-circle and discuss a statement which will play a key role in the proof of Theorem 1. Note that the "orientation preserving" part of the following proposition is due to Thurston [16].

PROPOSITION 3.4. *Let $B = \{\beta_1, \ldots, \beta_n\} \subset \mathrm{Per}^f$ with $n \geq 3$ and $f^{\circ m}(B) = B$. Assume that the sets $B, f(B), f^{\circ 2}(B), \ldots, f^{\circ m-1}(B)$ are weakly unlinked and mutually disjoint*, *and each of them is contained in an open half-circle. Then exactly one of the following statements is valid*:

(i) $f^{\circ m}$ *is orientation preserving and* B *lies in a periodic orbit of period* nm.

(ii) $f^{\circ m}$ *is orientation reversing and* B *consists of three angles, one of period* m *and two of period* $2m$ *interchanged by* $f^{\circ m}$.

Proof. We can assume $\beta_1 \curvearrowright \cdots \curvearrowright \beta_n$. Let $\mathcal{C} = \{\beta_1\beta_2, \beta_2\beta_3, \ldots, \beta_{n-1}\beta_n, \beta_n\beta_1\}$. The conditions on the iterates of B imply that the chord sets $\mathcal{C}, f(\mathcal{C}), f^{\circ 2}(\mathcal{C}), \ldots, f^{\circ m-1}(\mathcal{C})$ define mutually disjoined "polygons" and that $f^{\circ m}$ permutes not only B but also $\mathcal{C}$. Let $\overline{S}$ be a longest chord in $\mathcal{Z} = \mathcal{C} \cup f(\mathcal{C}) \cup f^{\circ 2}(\mathcal{C}) \cup \cdots \cup f^{\circ m-1}(\mathcal{C})$. We can assume that $\overline{S}$ belongs to $\mathcal{C}$. Then besides $\overline{S}$ there is at most one chord $\overline{\overline{S}}$ in $\mathcal{C}$ which is not shorter than $1/3$.

For each chord $S \in \mathcal{Z}$ there exists a k with $d(f^{\circ k}(S)) \geq 1/3$. If $f^{\circ k}(S)$ itself does not coincide with $\overline{S}$ or $\overline{\overline{S}}$, then let $l > k$ be the least index with $f^{\circ l}(S) \in \mathcal{C}$. By changing k if necessary, we can assume that none of the chords $f^{\circ k+1}(S), f^{\circ k+2}(S), \ldots, f^{\circ l-1}(S)$ is longer than $f^{\circ k}(S)$. Observe that $\overline{S}$ (and thus the whole chord set $\mathcal{C}$) lies between $f^{\circ k}(S)$ and $f^{\circ k}(S)'$ (see Lemma 3.2), and by Lemma 3.1, $f^{\circ l}(S) = \overline{S}$ or $f^{\circ l}(S) = \overline{\overline{S}}$. Thus the chords in $\mathcal{Z}$ belong to at most two different orbits.

If $\overline{\overline{S}}$ exists, then again by Lemma 3.1 either $f^{\circ m}(\overline{\overline{S}}) = \overline{S}$ or $f^{\circ m}(\overline{\overline{S}}) = \overline{\overline{S}}$. In the first case (as well as in the case when $\overline{\overline{S}}$ does not exist) all chords in $\mathcal{Z}$ belong to the same orbit, since $\overline{S}$ is an iterate of each of them, whereas in the latter case $\overline{\overline{S}}$ lies in an orbit of period m so that $\mathcal{Z}$ splits into one orbit of period m and one of period $(n-1)m$.

In the orientation preserving case the latter is impossible. Otherwise, there would exist two chords in $\mathcal{C}$ with a common endpoint but different periods m and $(n-1)m$. One easily sees that $f^{\circ m}$ must fix the endpoints of the two chords, a contradiction.

If $f^{\circ m}$ is orientation reversing, there must be an angle $\beta \in B$ which is fixed by $f^{\circ m}$. We can assume that $\beta = \beta_1$. Then $f^{\circ m}(\beta_2) = \beta_n$ and $f^{\circ m}(\beta_n) = \beta_2$, hence $f^{\circ m}(\beta_1\beta_2) = \beta_1\beta_n$ and $f^{\circ m}(\beta_1\beta_n) = \beta_1\beta_2$. Therefore $\beta_1\beta_2$ and $\beta_1\beta_n$ are on an orbit of period $2m < nm$, implying the existence of a second orbit of period m, and $n = 3$. Clearly, $\beta_2\beta_3 \in \mathcal{C}$. ∎

We will use Proposition 3.4 in the following version:

Corollary 3.5. *Let* $\delta \in \mathbb{T}$ *be nonperiodic and* $D = \{\beta_1, \ldots, \beta_k\} \subset \mathrm{Per}^f$ *with* $k \geq 3$ *such that for each* $i = 0, 1, 2, \ldots,$ *the set* $f^{\circ i}(D)$ *is contained in* $]\delta, \delta + 1/2[$ *or in* $]\delta + 1/2, \delta[$. *Then exactly one of the following statements is valid*:

(i) *The angles of* D *lie in a periodic orbit of some period* l *with* $f^{\circ l}$ *orientation preserving.*

(ii) $k = 3$, *one of the angles of* D *has some odd period* l *with* $f^{\circ l}$ *orientation reversing*, *and the other two have period* $2l$ *and are interchanged by* $f^{\circ l}$.

Proof. Let $B = \{\beta_1, \ldots, \beta_k, \ldots, \beta_n\}$ be the largest possible set containing D and further iterates of angles of D such that still for each $i = 0, 1, 2, \ldots$ the set $f^{\circ i}(B)$ is contained in $]\delta, \delta + 1/2[$ or in $]\delta + 1/2, \delta[$. Then nm is the number of all iterates of angles of D. Proposition 3.4 can be applied to B if we show that the iterates of B are weakly unlinked, i.e. that chords S_1 and S_2 with ends in $f^{\circ k_1}(B)$ and $f^{\circ k_2}(B)$, respectively, do not cross for $0 \leq k_1 < k_2 < m$. Assuming the contrary, $f^{\circ i}(S_1)$ and $f^{\circ i}(S_2)$ would cross each other for all i. Hence $f^{\circ k_1 + i}(B)$ and $f^{\circ k_2 + i}(B)$ would be contained in the same part $]\delta, \delta + 1/2[$ or $]\delta + 1/2, \delta[$ for each i, contradicting the assumed maximality of B. ■

4. Backward iteration of the critical chord. For the following two sections we fix α to be a periodic angle with respect to f and $m > 1$ to be its period. Recall from the previous section that $\dot{\alpha}$ denotes the periodic preimage of α and $\ddot{\alpha}$ the preperiodic one. The diameter $\dot{\alpha}\ddot{\alpha} = f^{-1}(\alpha)$ is critical in the sense that its image under f is not another chord but the single angle α. In this section we define a set $\mathcal{S}$ of backward iterates of $\dot{\alpha}\ddot{\alpha}$, investigate its properties and consider the important subset $\mathcal{S}_{\text{center}}$ of $\mathcal{S}$, which will be the base for constructing $\Theta^f(\alpha)$ as well as the $f^{\circ n}$-set corresponding to $\alpha\Theta^f(\alpha)$.

Recall that $\mathbb{T} \setminus \{\dot{\alpha}, \ddot{\alpha}\}$ consists of the two open half-circles $\mathbb{T}_0$, the one containing α, and $\mathbb{T}_1$. Clearly, the map f is invertible on both $\mathbb{T}_0$ and $\mathbb{T}_1$, with inverse maps called l_0 and l_1, defined on $\mathbb{T} \setminus \{\alpha\}$. More generally, for each finite 0-1-word $\mathbf{w} = w_1 \ldots w_n$, there is a map $l_{\mathbf{w}} = l_{w_1} \circ \cdots \circ l_{w_n}$ whose domain includes the set $\mathbb{T} \setminus \{\alpha, f(\alpha), \ldots f^{\circ n-1}(\alpha)\}$. For $\mathbf{w}$ being the empty word, let $l_{\mathbf{w}}$ be the identity.

Each of the maps l_s, $s = 0, 1$, has two injective extensions l_s^t, $t = 0, 1$, to the whole circle defined by

$$l_s^t(\beta) = \begin{cases} l_s(\beta) & \text{for } \beta \neq \alpha, \\ \dot{\alpha} & \text{for } \beta = \alpha \text{ and } s = t, \\ \ddot{\alpha} & \text{for } \beta = \alpha \text{ and } s \neq t. \end{cases}$$

Now we define the set $\mathcal{S}$ of backward iterates of $\dot{\alpha}\ddot{\alpha}$ as follows:

$$\mathcal{S} = \{l_{s_1}^{t_1} \circ l_{s_2}^{t_2} \circ \cdots \circ l_{s_n}^{t_n}(\dot{\alpha}\ddot{\alpha}) \mid s_1, \ldots, s_n, t_1, \ldots, t_n \in \{0, 1\},\ n \in \mathbb{N}\}.$$

Note that, by definition, no backward iterate can cross $\dot{\alpha}\ddot{\alpha}$. But the chords of $\mathcal{S}$ even satisfy a stronger property:

LEMMA 4.1. *The chords of* $\mathcal{S}$ *do not cross each other.*

Proof. Assume that $S = l_{s_1}^{t_1} \circ \cdots \circ l_{s_j}^{t_j}(\dot\alpha\ddot\alpha)$ and $\widetilde{S} = l_{u_1}^{v_1} \circ \cdots \circ l_{u_k}^{v_k}(\dot\alpha\ddot\alpha)$ cross each other for some $s_1, \ldots, s_j, t_1, \ldots, t_j, u_1, \ldots, u_k, v_1, \ldots, v_k \in \{0,1\}$ and $j, k \geq 0$. Then $s_1 = u_1$, since otherwise S and $\widetilde{S}$ would lie in different half-circles. This implies that $l_{s_2}^{t_2} \circ \cdots \circ l_{s_j}^{t_j}(\dot\alpha\ddot\alpha)$ and $l_{u_2}^{v_2} \circ \cdots \circ l_{u_k}^{v_k}(\dot\alpha\ddot\alpha)$ cross each other (cf. before Lemma 3.2). So by induction one easily shows that $j \neq k$ and that the jth iterate of $\widetilde{S}$ or the kth iterate of S crosses $\dot\alpha\ddot\alpha$, which is obviously false. ∎

A special system of chords. For the following we only need backward iterates of $\dot\alpha\ddot\alpha$ which "jump" between the two components $\mathbb{T}_0$ and $\mathbb{T}_1$ according to the kneading sequence of α. We denote the latter by $\overline{\mathbf{v}*}$, where $\mathbf{v} = v_1 \ldots v_{m-1}$ is a word in $\{0,1\}^{m-1}$. Looking at such backward iterates seems quite natural for finding $\Theta^f(\alpha)$, since $\Theta^f(\alpha)$ is supposed to exhibit the same dynamical behavior as α, i.e. to have the same kneading sequence and the same "jumps". Following the kneading sequence, we can trace the way of $\dot\alpha\ddot\alpha$ back. Whenever a $*$ occurs, we include both possibilities of iterating back further. This construction yields the subset $\mathcal{S}_{\mathrm{center}}$ of $\mathcal{S}$.

We define

$$\mathcal{S}_{\mathrm{center}} = \{S_{s_1 \ldots s_n}^t \mid t, s_1, \ldots, s_n \in \{0,1\}\},$$

where the chords $S_{s_1 \ldots s_n}^t$ are specified as follows.

$f^{\circ m}$ *is orientation preserving*: In this case let

$$S_{s_1 \ldots s_n}^t = l_{s_1}^t \circ l_{\mathbf{v}} \circ l_{s_2}^t \circ l_{\mathbf{v}} \circ \cdots \circ l_{s_{n-1}}^t \circ l_{\mathbf{v}} \circ l_{s_n}^t \circ l_{\mathbf{v}}(\dot\alpha\ddot\alpha).$$

$f^{\circ m}$ *is orientation reversing*: If n is even, let

$$S_{s_1 \ldots s_n}^t = l_{s_1}^{1-t} \circ l_{\mathbf{v}} \circ l_{s_2}^t \circ l_{\mathbf{v}} \circ l_{s_3}^{1-t} \circ l_{\mathbf{v}} \circ \cdots \circ l_{s_{n-1}}^{1-t} \circ l_{\mathbf{v}} \circ l_{s_n}^t \circ l_{\mathbf{v}}(\dot\alpha\ddot\alpha),$$

and if n is odd, let

$$S_{s_1 \ldots s_n}^t = l_{s_1}^t \circ l_{\mathbf{v}} \circ l_{s_2}^{1-t} \circ l_{\mathbf{v}} \circ l_{s_3}^t \circ l_{\mathbf{v}} \circ \cdots \circ l_{s_{n-1}}^{1-t} \circ l_{\mathbf{v}} \circ l_{s_n}^t \circ l_{\mathbf{v}}(\dot\alpha\ddot\alpha).$$

The set $\mathcal{S}_{\mathrm{center}}$ is $f^{\circ m}$-invariant by construction. Since $l_s^s \circ l_{\mathbf{v}}(\dot\alpha) = \dot\alpha$ and $l_s^{1-s} \circ l_{\mathbf{v}}(\dot\alpha) = \ddot\alpha$ for $s \in \{0,1\}$, each chord $S_{s_1 \ldots s_n}^t$ of $\mathcal{S}_{\mathrm{center}}$ has a common endpoint with either $S_{s_1 \ldots s_{n-1}}^t$ or $S_{s_1 \ldots s_{n-1}}^{1-t}$, which, by induction, implies that any two endpoints of chords in $\mathcal{S}_{\mathrm{center}}$ are connected by a finite sequence of chords in $\mathcal{S}_{\mathrm{center}}$. Hence the union of all chords in $\mathcal{S}_{\mathrm{center}}$ is a connected set in the unit disk. The same is true for the iterates $f^{\circ n}(\mathcal{S}_{\mathrm{center}})$ of $\mathcal{S}_{\mathrm{center}}$. We deduce from this that the corresponding sets of endpoints are unlinked.

PROPOSITION 4.2. *The sets of endpoints of* $f^{\circ n}(\mathcal{S}_{\mathrm{center}})$, $n = 0, \ldots, m-1$, *are unlinked. Consequently, their closures in* $\mathbb{T}$ *are weakly unlinked.*

Proof. We show that any two chords $S \in f^{\circ n_1}(\mathcal{S}_{\mathrm{center}})$ and $T \in f^{\circ n_2}(\mathcal{S}_{\mathrm{center}})$ with $0 \leq n_1 < n_2 < m$ are disjoint. This together with the above mentioned connectedness implies that the sets of endpoints are unlinked as stated. By Lemma 4.1, S and T do not cross. If they were not disjoint, they would have a common endpoint. Hence their f-iterates would have a common endpoint. After sufficiently long iteration some iterate of each of the two chords would be the critical chord $\dot{\alpha}\ddot{\alpha}$ and the further iterates would be orbit points of α. The assumed common endpoint would imply $\alpha = f^{\circ k}(\alpha)$ for some $0 < k < m$, a contradiction. ∎

Remark. For a moment assume $f = h$ or $\overline{h}$ and $\alpha = 0$. The case of α being a fixed point was excluded above, however, with $\mathbf{v}$ being the empty word, $\dot{\alpha} = 0$ and $\ddot{\alpha} = 1/2$, the definitions of $\mathcal{S}$ and $\mathcal{S}_{\text{center}}$ easily carry over and provide some insights for the non-fixed point case. For $f = h$ as well as for $f = \overline{h}$, the sets $\mathcal{S}$ and $\mathcal{S}_{\text{center}}$ coincide and consist of the chords $0\frac{1}{4}, \frac{1}{4}\frac{1}{2}, \frac{1}{2}\frac{3}{4}, \frac{3}{4}0, 0\frac{1}{8}, \frac{1}{8}\frac{1}{4}, \ldots$ (cf. Figure 5). (For $f = h$ the "symbolization" of

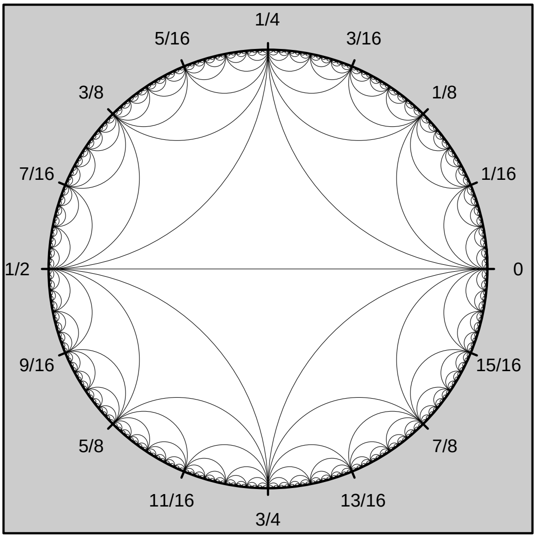

Fig. 5. $\mathcal{S} = \mathcal{S}_{\text{center}}$ in the fixed point case $\alpha = 0$

these chords is strongly related to the binary expansion of points in $[0, 1[$.) For the non-fixed point case, the incidence structure of $\mathcal{S}_{\text{center}}$, which is studied below, is the same as for $\alpha = 0$. The only difference is that the endpoints of chords of $\mathcal{S}_{\text{center}}$ are not dense in $\mathbb{T}$ any longer. Keeping this in mind might be useful for the comprehension of the next section.

5. Some important angles

Special triangle systems. Each chord of $\mathcal{S}_{\text{center}}$ is a common side of two triangles with sides in $\mathcal{S}_{\text{center}}$. This can be used for describing the mutual position of the chords in $\mathcal{S}_{\mathrm{center}}$ and their endpoints. We are especially interested in systems of triangles Δ_k^t, $k = 0, 1, 2, \ldots$, for $t = 0, 1$ defined as follows:

$f^{\circ m}$ is orientation preserving: Δ_k^t has angles $\dot\alpha$, $l_{(t\mathbf{v})^k}(\ddot\alpha)$, $l_{(t\mathbf{v})^{k+1}}(\ddot\alpha)$ and sides $S_{t^k}^t = \dot\alpha l_{(t\mathbf{v})^k}(\ddot\alpha)$, $S_{t^{k+1}}^{1-t} = l_{(t\mathbf{v})^k}(\ddot\alpha) l_{(t\mathbf{v})^{k+1}}(\ddot\alpha)$, $S_{t^{k+1}}^t = l_{(t\mathbf{v})^{k+1}}(\ddot\alpha)\dot\alpha$.

$f^{\circ m}$ is orientation reversing: Δ_k^t has angles $l_{(t\mathbf{v})^{k-1}}(\ddot\alpha), l_{(t\mathbf{v})^k}(\ddot\alpha), l_{(t\mathbf{v})^{k+1}}(\ddot\alpha)$ and sides $S_{t^k}^{1-t} = l_{(t\mathbf{v})^{k-1}}(\ddot\alpha) l_{(t\mathbf{v})^k}(\ddot\alpha)$, $S_{t^{k+1}}^{1-t} = l_{(t\mathbf{v})^k}(\ddot\alpha) l_{(t\mathbf{v})^{k+1}}(\ddot\alpha)$, $S_{t^{k+1}}^t = l_{(t\mathbf{v})^{k+1}}(\ddot\alpha) l_{(t\mathbf{v})^{k-1}}(\ddot\alpha)$ if $k > 0$, and Δ_0^t has sides $\dot\alpha\ddot\alpha, S_t^{1-t}, S_t^t$.

Considering the incidence structure of the triangle system Δ_k^t, $k = 1, 2, \ldots$, one easily sees that

$$(5.1)\qquad \begin{aligned} &\ddot\alpha \curvearrowright l_{t\mathbf{v}}(\ddot\alpha) \curvearrowright l_{(t\mathbf{v})^2}(\ddot\alpha) \curvearrowright l_{(t\mathbf{v})^3}(\ddot\alpha) \curvearrowright l_{(t\mathbf{v})^4}(\ddot\alpha) \curvearrowright \cdots \curvearrowright \dot\alpha \quad \text{or} \\ &\ddot\alpha \curvearrowleft l_{t\mathbf{v}}(\ddot\alpha) \curvearrowleft l_{(t\mathbf{v})^2}(\ddot\alpha) \curvearrowleft l_{(t\mathbf{v})^3}(\ddot\alpha) \curvearrowleft l_{(t\mathbf{v})^4}(\ddot\alpha) \curvearrowleft \cdots \curvearrowleft \dot\alpha \end{aligned}$$

if $f^{\circ m}$ is orientation preserving, and that

$$(5.2)\qquad \begin{aligned} &\ddot\alpha \curvearrowright l_{(t\mathbf{v})^2}(\ddot\alpha) \curvearrowright l_{(t\mathbf{v})^4}(\ddot\alpha) \curvearrowright \cdots \\ &\qquad \curvearrowright l_{(t\mathbf{v})^5}(\ddot\alpha) \curvearrowright l_{(t\mathbf{v})^3}(\ddot\alpha) \curvearrowright l_{t\mathbf{v}}(\ddot\alpha) \curvearrowright \dot\alpha \quad \text{or} \\ &\ddot\alpha \curvearrowleft l_{(t\mathbf{v})^2}(\ddot\alpha) \curvearrowleft l_{(t\mathbf{v})^4}(\ddot\alpha) \curvearrowleft \cdots \\ &\qquad \curvearrowleft l_{(t\mathbf{v})^5}(\ddot\alpha) \curvearrowleft l_{(t\mathbf{v})^3}(\ddot\alpha) \curvearrowleft l_{t\mathbf{v}}(\ddot\alpha) \curvearrowleft \dot\alpha \end{aligned}$$

if $f^{\circ m}$ is orientation reversing.

Length of chords in $\mathcal{S}$. There is a symbol $e = e^{f,\alpha} \in \{0, 1\}$ such that S_0^e (and S_1^e) are shorter than S_0^{1-e} (and S_1^{1-e}). Otherwise, both S_0^0 and S_0^1 would have length $1/4$, hence would be mapped to $\dot\alpha\ddot\alpha$, implying $f(\dot\alpha) = \dot\alpha$. This is impossible since the period of α was assumed to be greater than 1. Obviously,

$$(5.3)\qquad S_0^{1-e} \text{ and } S_1^{1-e} \text{ are the two longest chords in } \mathcal{S}.$$

S_0^e and S_1^e are not longer than $1/8$: Assume that $(1/4 >) d(S_0^e) > 1/8$. Then $d(S_0^{1-e}) < 3/8$ and $d(f(S_0^e)) > 1/4$. On the other hand, $f^{\circ m}(S_0^e) = \dot\alpha\ddot\alpha$ implies $d(f^{\circ m-1}(S_0^e)) = 1/4$. Let $i > 1$ be minimal with $d(f^{\circ i}(S_0^e)) \leq 1/4$. By (3.2), $d(f^{\circ i-1}(S_0^e)) \geq 3/8$, contradicting (5.3).

Again by (3.2) the preimage of a chord not longer than $d(S_0^e) = d(S_1^e)$ has either half its length or is longer than $d(S_0^{1-e}) = d(S_1^{1-e})$, implying the following:

$$(5.4)\qquad \text{If } f^{\circ n}(S) \in \{S_0^e, S_1^e\} \text{ for some } S \in \mathcal{S}, \text{ then } d(S) \leq 2^{-(n+3)}.$$

Important angles. In preparation for the definitions of $\Theta^f(\alpha)$ and $\Lambda^f(\alpha)$ we define some important angles as limits in $\mathcal{S}_{\mathrm{center}}$.

Proposition 5.1 (Important angles). *The angle*

$$\xi^f(\alpha) := \lim_{k\to\infty} l_{((1-e)\mathbf{v})^k}(\ddot{\alpha})$$

is well defined with period dividing m, *and* $d(\dot{\alpha}, \xi^f(\alpha)) \geq 1/3$. *Moreover* (*cf. Figure* 6):

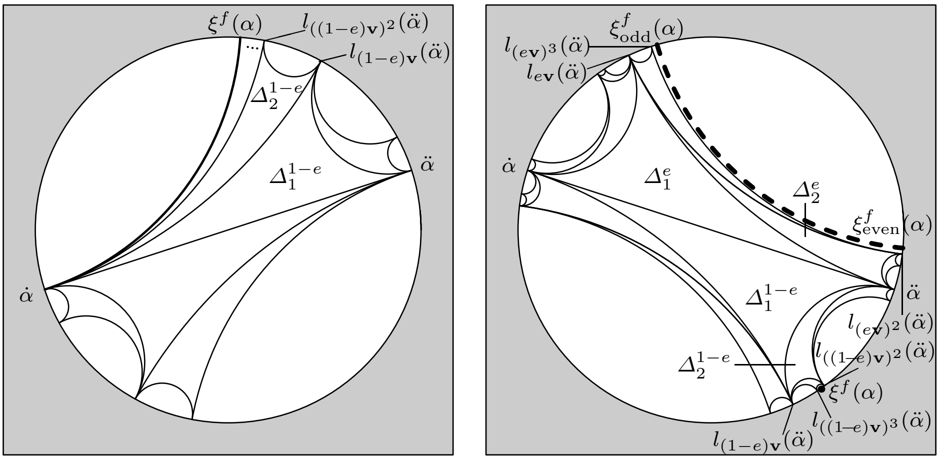

Fig. 6. Important angles: the orientation preserving case (left) and the orientation reversing case (right)

(i) *If* $f^{\circ m}$ *is orientation preserving* (*resp. orientation reversing*), *then the angle* $\lim_{k\to\infty} l_{(e\mathbf{v})^k}(\ddot{\alpha})$ (*resp.* $\lim_{k\to\infty} l_{((1-e)\mathbf{v}e\mathbf{v})^k}(\ddot{\alpha})$) *exists and equals* $\dot{\alpha}$.

(ii) *If* $f^{\circ m}$ *is orientation reversing*, *then*

$$\xi^f_{\mathrm{odd}}(\alpha) := \lim_{k\to\infty,\, k\,\mathrm{odd}} l_{(e\mathbf{v})^k}(\ddot{\alpha})$$

and

$$\xi^f_{\mathrm{even}}(\alpha) := \lim_{k\to\infty,\, k\,\mathrm{even}} l_{(e\mathbf{v})^k}(\ddot{\alpha})$$

are well defined, and separated from $\xi^f(\alpha)$ *by* $\dot{\alpha}\ddot{\alpha}$. *They satisfy the inequalities*

$$d(\xi^f_{\mathrm{odd}}(\alpha), \xi^f_{\mathrm{even}}(\alpha)) \geq 1/3,$$
$$d(\ddot{\alpha}, \xi^f_{\mathrm{even}}(\alpha)), d(\ddot{\alpha}, \xi^f(\alpha)) < d(\dot{\alpha}, \xi^f_{\mathrm{odd}}(\alpha)) < 1/6,$$

have period $2m$ *and are interchanged by* $f^{\circ m}$.

(iii) *If* f *is orientation reversing and* $m/2$ *is odd*, *then* $f^{\circ m/2}(\dot{\alpha}) \neq \xi^f(\alpha)$.

Proof. For $f^{\circ m}$ orientation reversing, $\lim_{k\to\infty} l_{((1-e)\mathbf{v})^k}(\ddot{\alpha})$ exists since, by (5.4), $d(l_{((1-e)\mathbf{v})^k}(\ddot{\alpha}), l_{((1-e)\mathbf{v})^{k+1}}(\ddot{\alpha})) = d(S^e_{(1-e)^k})$ exponentially converges to 0 for $k \to \infty$ (cf. (5.2) for $t = 1-e$). The limit $\lim_{k\to\infty} l_{((1-e)\mathbf{v}e\mathbf{v})^k}(\ddot{\alpha})$ is considered below when proving (i). The existence of all other limits is obvious (cf. (5.1) and (5.2)). So let us show that $d(\dot{\alpha},\xi) \geq 1/3$ for $\xi = \xi^f(\alpha)$.

If $f^{\circ m}$ is orientation preserving, then $\ddot{\alpha}, l_{(1-e)\mathbf{v}}(\ddot{\alpha}), l_{((1-e)\mathbf{v})^2}(\ddot{\alpha}), \ldots, l_{((1-e)\mathbf{v})^k}(\ddot{\alpha}), \ldots$ are successively connected by the chords $S^e_{1-e}, S^e_{(1-e)^2}, \ldots, S^e_{(1-e)^k}, \ldots$ no longer than $1/8, 1/32, \ldots, 1/2^{2k+1}, \ldots$ (cf. (5.1) for $t = 1-e$, and (5.4)). Summing up these numbers provides $d(\ddot{\alpha},\xi) \leq 1/6$, hence $d(\dot{\alpha},\xi) \geq 1/2 - 1/6 = 1/3$. In the orientation reversing case $S^e_{1-e} = \ddot{\alpha} l_{(1-e)\mathbf{v}}(\ddot{\alpha})$ is not longer than $1/8$. For $t = 1-e$, (5.2) shows $d(\ddot{\alpha},\xi) \leq 1/8$, and thus $d(\dot{\alpha},\xi) \geq 1/3$.

Now let f be orientation reversing and $m/2$ odd. Then $f^{\circ m}$ is orientation preserving, and, by the above, the sequence of chords $S^{1-e}_{(1-e)^n}$ with endpoints $\dot{\alpha}$ and $l_{((1-e)\mathbf{v})^n}(\ddot{\alpha})$ converges to $\dot{\alpha}\xi$. The orientation of the convergence of $(l_{((1-e)\mathbf{v})^n}(\ddot{\alpha}))_{n=1}^{\infty}$ to ξ and of $(f^{\circ m/2}(l_{((1-e)\mathbf{v})^n}(\ddot{\alpha})))_{n=1}^{\infty}$ to $f^{\circ m/2}(\xi)$ is different, and, by definition, no chord $f^{\circ m/2}(S^{1-e}_{(1-e)^n})$ has endpoints in both $]\dot{\alpha},\ddot{\alpha}[$ and $]\ddot{\alpha},\dot{\alpha}[$. Thus $f^{\circ m/2}(\dot{\alpha}) = \xi$ is impossible, proving (iii).

(i) If $f^{\circ m}$ is orientation preserving, consider (5.1) for $t = e$. By (5.4), we obtain $\lim_{k\to\infty} d(l_{(e\mathbf{v})^k}(\ddot{\alpha}),\dot{\alpha}) = \lim_{k\to\infty} d(S^e_{e^k}) = 0$, implying $\lim_{k\to\infty} l_{(e\mathbf{v})^k}(\ddot{\alpha}) = \dot{\alpha}$.

In the orientation reversing case, we have

$$\begin{aligned}\dot{\alpha} l_{((1-e)\mathbf{v}e\mathbf{v})^k}(\ddot{\alpha}) &= S^{1-e}_{((1-e)e)^k} = (l^{1-e}_{(1-e)\mathbf{v}} \circ l^e_{e\mathbf{v}}))^{\circ k-1}(l^{1-e}_{(1-e)\mathbf{v}} \circ l^e_{e\mathbf{v}}(\dot{\alpha}\ddot{\alpha}))\\ &= (l^{1-e}_{(1-e)\mathbf{v}} \circ l^e_{e\mathbf{v}}))^{\circ k-1} \circ l^{1-e}_{(1-e)\mathbf{v}}(S^e_e)\end{aligned}$$

for $k = 1, 2, \ldots$, hence $\lim_{k\to\infty} d(\dot{\alpha}, l_{((1-e)\mathbf{v}e\mathbf{v})^k}(\ddot{\alpha})) = 0$ by (5.4).

(ii) Now set $t = e$ in (5.2). The angles $\dot{\alpha}, l_{e\mathbf{v}}(\ddot{\alpha})$ are connected by S^e_e, and the angles $l_{(e\mathbf{v})^{k-2}}(\ddot{\alpha}), l_{(e\mathbf{v})^k}(\ddot{\alpha})$ by $S^e_{e^k}$ for all $k = 2, 3, \ldots$. By (5.4), we have

$$d(S^e_{e^k}) \leq \frac{1}{2^{2k+1}} = \frac{1}{2}\left(\frac{1}{4}\right)^k, \qquad d(S^e_{e^k}) > d(S^e_{e^{k+1}}), \qquad k = 1, 2, \ldots.$$

The first inequality shows $\sum_{k=1}^{\infty} d(S^e_{e^k}) \leq 1/6$ and the second $\sum_{k \text{ odd}} d(S^e_{e^k}) > \sum_{k \text{ even}} d(S^e_{e^k})$. This implies $d(\xi_{\text{odd}}, \xi_{\text{even}}) = 1/2 - (d(\dot{\alpha},\xi_{\text{odd}}) + d(\ddot{\alpha},\xi_{\text{even}})) \geq 1/2 - 1/6 = 1/3$ and $d(\dot{\alpha},\xi_{\text{odd}}) > d(\ddot{\alpha},\xi_{\text{even}})$ for $\xi_{\text{even}} = \xi^f_{\text{even}}(\alpha)$ and $\xi_{\text{odd}} = \xi^f_{\text{odd}}(\alpha)$. By looking at the incidence structure of the system of triangles (cf. Figure 6, right), one easily sees that ξ lies between $\xi'_{\text{even}}\xi'_{\text{odd}}$ and $\dot{\alpha}\ddot{\alpha}$, hence $d(\dot{\alpha},\xi_{\text{odd}}(\alpha)) > d(\ddot{\alpha},\xi)$. The angles ξ_{even} and ξ_{odd} are interchanged by $f^{\circ m}$ since $f^{\circ m}(l_{(e\mathbf{v})^{k+1}}(\ddot{\alpha})) = l_{(e\mathbf{v})^k}(\ddot{\alpha})$ for $k = 1, 2, \ldots$ (cf. (5.2)), completing the proof. ∎

Remark. For the case $\alpha = 0$, described in the remark at the end of the previous section, the limit $\xi^f(\alpha)$ coincides with α. This corresponds to the fact that for fixed points $\mathcal{S}_{\text{center}}$ is dense in $\mathbb{T}$, in contrast to the non-fixed point case, where α and $\xi^f(\alpha)$ do not coincide any longer.

6. $f^{\circ n}$-sets

The morphology of $f^{\circ n}$-sets. The complement $\mathbb{T} \setminus C$ of a Cantor set C consists of countably many open intervals. All chords connecting the two endpoints of such an open interval form a set denoted by $\mathcal{B}_C$. We associate to C a topological circle $\mathbb{T}_C$ by contracting each chord of $\mathcal{B}_C$ to a point and letting the points in C not connected by a chord in $\mathcal{B}_C$ be points in $\mathbb{T}_C$. The elements of $\mathcal{B}_C$ can be considered both as chords with endpoints in C and as points of $\mathbb{T}_C$.

In case C is an $f^{\circ n}$-set, the construction of $\mathbb{T}_C$ is compatible with the dynamics of $f^{\circ n}$. This is what we want to discuss now. From the technical viewpoint it is also helpful to consider the $'$-symmetric iterate $\widetilde{C} = f^{\circ n-1}(C)$ of C and the corresponding chord set $\mathcal{B}_{\widetilde{C}}$. First we collect some properties of $f^{\circ n}$-sets.

LEMMA 6.1. *Let C be an $f^{\circ n}$-set.*

(i) *In $\mathcal{B}_C$ there is a unique longest chord $B_C = \beta_1\beta_2$. It is a bifurcation chord and satisfies $f^{\circ n}(B_C) = B_C$. Moreover, one of the following cases holds:*

- (a) *$f^{\circ n}$ is orientation preserving, and β_1, β_2 have period n.*
- (b) *f is orientation reversing, n is odd, β_1, β_2 have period $2n$, and $f^{\circ n}(\beta_1) = \beta_2$.*
- (c) *f is orientation reversing and β_1, β_2 have odd period $n/2$.*
- (d) *f is orientation reversing and β_1, β_2 have periods n and odd $n/2$.*

(ii) *$B_{\widetilde{C}} := f^{\circ n-1}(B_C)$ and $B'_{\widetilde{C}}$ are the longest chords of $\mathcal{B}_{\widetilde{C}}$, and for $i = 0, 1, \ldots, n-2$ the set $f^{\circ i}(C)$ is behind $B_{\widetilde{C}}$ or $B'_{\widetilde{C}}$.*

(iii) *For each chord $T \in \mathcal{B}_C$, there exists some $k \geq 0$ with $f^{\circ kn}(T) = B_C$.*

(iv) *$f^{\circ n}$ maps $\mathcal{B}_C$ onto $\mathcal{B}_C$ in a two-to-one way.*

Proof. For $0 \leq i \leq n$ write $C_i := f^{\circ i}(C)$ and $\mathcal{B}_i := \mathcal{B}_{f^{\circ i}(C)}$. Observe that by Definition 1.6(ii) for each chord $S \in \mathcal{B}_{n-1}$ the chord S' is also in $\mathcal{B}_{n-1}$ and has the same length. Since $f(\alpha) = f(\alpha')$, we only need to consider a semi-open half-circle to obtain the full image of C_{n-1} under f, i.e. $f(C_{n-1} \cap]\delta, \delta'[) = f(C_{n-1}) \subseteq C$ for δ in the complement of C_{n-1}.

Since the sets $C_0, C_1, \ldots, C_{n-2}, C_{n-1}$ are weakly unlinked, each of the sets $C_0, C_1, \ldots, C_{n-2}$ sits behind some chord of $\mathcal{B}_{n-1}$, hence is contained in an open half-circle. This ensures that for each $T \in \mathcal{B}_i$, $0 \leq i < n$, the chord

$f(T)$ is in $\mathcal{B}_{i+1}$, i.e. $f(\mathcal{B}_i) \subseteq \mathcal{B}_{i+1}$. The converse is also true, hence

$$f(\mathcal{B}_i) = \mathcal{B}_{i+1}. \tag{6.1}$$

From (6.1) we also have $f^{\circ i}(\mathcal{B}_0) = \mathcal{B}_i$, and the $f^{\circ n}$-invariance of C_0 now implies the $f^{\circ n}$-invariance of $\mathcal{B}_0$. (iv) is now easy to see. Another immediate consequence is the existence of a unique longest chord B_C of $\mathcal{B}_0$, separating C_0 and C_{n-1}.

Let S be a longest chord in $\mathcal{B}_{n-1}$. From formula (3.1) and the invariance of $\mathcal{B}_0 \cup \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_{n-1}$ under f, one sees that $d(S) \geq 1/3$. Moreover, S, S' is the unique pair of longest chords in $\mathcal{B}_{n-1}$ since the sum of lengths of all chords in $\mathcal{B}_{n-1}$ does not exceed 1. Clearly, S and S' are also the longest chords in $\mathcal{B}_0 \cup \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_{n-1}$.

Now observe that $d(f(R)) = 2d(R) < 2(1/2 - d(S)) = 1 - 2d(S) = d(f(S))$ for $R \in \mathcal{B}_{n-1} \setminus \{S, S'\}$. Thus $f(S) = f(S') = B_C$. The nth iterate $f^{\circ n}(S)$ of S is in $\mathcal{B}_{n-1}$ and thus it is either equal to S or S', or lies between these two chords. The latter is impossible since by Lemma 3.1, $f^{\circ n}(S)$ would be longer than S, a contradiction to S being a longest chord. Thus $f^{\circ n}(S) = S = B_{\widetilde{C}}$ or $f^{\circ n}(S') = S' = B_{\widetilde{C}}$ and $f^{\circ n}(B_C) = B_C$. Clearly, since $C_0, C_1, \ldots, C_{n-1}$ are weakly unlinked, B_C is a bifurcation chord.

Let γ, β be the endpoints of B_C, and let $(\gamma_i)_{i=1}^{\infty}$ be a sequence in C_0 converging to γ. It is important to note that, besides C_0, γ is in at most one further set C_k since the sets $C_0, C_1, \ldots, C_{n-1}$ are weakly unlinked. If γ lies in C_0 and C_k and $f^{\circ k}(\gamma) = \gamma$, then $(f^{\circ k}(\gamma_i))_{i=1}^{\infty} \subset C_k$ and $(\gamma_i)_{i=1}^{\infty} \subset C_0$ converge in different orientations to γ. This is the striking argument behind the following.

First of all, if $f^{\circ n}(\gamma) = \beta$ and $f^{\circ n}(\beta) = \gamma$, then γ and β have period $2n$. Moreover, $(f^{\circ n}(\gamma_i))_{i=1}^{\infty}$ lies in C and $\lim_{i\to\infty} f^{\circ n}(\gamma_i) = \beta$, implying that $f^{\circ n}$ is orientation reversing. This is case (b) in (i). If $f^{\circ n}(\gamma) = \gamma$ and $f^{\circ n}(\beta) = \beta$, denote the period of β by r. Here either $r = n$ and $f^{\circ n}$ is orientation preserving, or $r = n/2$ and $f^{\circ n/2}$ is orientation reversing. Since the same is obtained for the period r of β, one of cases (a), (c) or (d) is valid.

It remains to show (ii) and (iii). By Lemma 3.1, the iterates of $B_{\widetilde{C}}$, hence of B_C, cannot lie between $B_{\widetilde{C}}$ and $B'_{\widetilde{C}}$, which implies (ii). Let $B \in \mathcal{B}_C$. Since the number of chords in $\mathcal{B}_0 \cup \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_{n-1}$ not shorter than $1/3$ is finite and at least one iterate of B has length not less than $1/3$, there exists a longest iterate T of B not shorter than $1/3$. Let $\widetilde{T}$ be the first iterate of T belonging to $\mathcal{B}_{n-1}$. Then $T = B_{\widetilde{C}}$ or $T = B'_{\widetilde{C}}$ or, by Lemma 3.1, $\widetilde{T}$ is longer than T. Since the latter contradicts our assumption, some iterate of B is equal to B_C, showing (iii). ∎

π_C mapping C to the circle. Now we are going to define the map π_C of Theorem 2(iii). We start by constructing a map on $\mathbb{T}_{\widetilde{C}} := f^{\circ n-1}(\mathbb{T}_C)$,

the "preimage" of $\mathbb{T}_C$. B_C has two preimages in $\mathbb{T}_{\widetilde{C}}$, the periodic chord $B_{\widetilde{C}} = \gamma\delta$ and the preperiodic one $B'_{\widetilde{C}} = \gamma'\delta'$; these are the longest chords of $\mathcal{B}_{\widetilde{C}}$ (cf. Lemma 6.1(ii)). We can assume that $0 < \gamma < \delta' < \gamma' < \delta$ or $0 < \gamma' < \delta < \gamma < \delta'$. This is possible since the fixed point 0 lies behind $B_{\widetilde{C}}$ or behind $B'_{\widetilde{C}}$. (The intervals between $B_{\widetilde{C}}$ and $B'_{\widetilde{C}}$ with endpoints in $\{\gamma, \delta', \gamma', \delta\}$ are mapped to intervals behind $B_{\widetilde{C}}$ or $B'_{\widetilde{C}}$.)

For $B \in \mathcal{B}_{\widetilde{C}} \setminus \{B_{\widetilde{C}}, B'_{\widetilde{C}}\}$, let $L_0(B)$ and $L_1(B)$ be the unique chords in $\mathcal{B}_{\widetilde{C}}$ with nth f-iterate equal to B and with endpoints in $]\gamma, \delta'[$ and $]\gamma', \delta[$, respectively. Similarly, if $f^{\circ n}$ is orientation preserving (resp. orientation reversing), let $l_s^0(\beta)$, $\beta \neq 0$, be the h-preimage (resp. $\overline{h}$-preimage) of $\beta \in \mathbb{T}$ in $]0, 1/2[$ if $s = 0$ and in $]1/2, 0[$ if $s = 1$, respectively.

Now define $\pi_{\widetilde{C}}(B_{\widetilde{C}}) := 0$, $\pi_{\widetilde{C}}(B'_{\widetilde{C}}) := 1/2$, and

$$\pi_{\widetilde{C}}(L_{s_1} \circ \cdots \circ L_{s_k}(B'_{\widetilde{C}})) := l^0_{s_1} \circ \cdots \circ l^0_{s_k}(1/2)$$

for $s_1, \ldots, s_k$, $k \in \mathbb{N}$. By Lemma 6.1(iii), this provides an injective map from $\mathcal{B}_{\widetilde{C}}$ onto the set of all angles in $\mathbb{T}$ of the form $a/2^i$ with $a, i \in \mathbb{N} \cup \{0\}$, satisfying

$$\pi_{\widetilde{C}} \circ f^{\circ n} = h \circ \pi_{\widetilde{C}} \quad (\text{resp. } \pi_{\widetilde{C}} \circ f^{\circ n} = \overline{h} \circ \pi_{\widetilde{C}}) \tag{6.2}$$

in the orientation preserving (resp. orientation reversing) case. We show that

$$\pi_{\widetilde{C}} \text{ is orientation preserving on } \mathcal{B}_{\widetilde{C}}. \tag{6.3}$$

For $k \in \mathbb{N}$, let $\mathcal{B}^k = \{B_{\widetilde{C}}, B'_{\widetilde{C}}\} \cup \{L_{s_1} \circ \cdots \circ L_{s_i}(B'_{\widetilde{C}}) \mid s_1, \ldots, s_i \in \{0,1\}, i \leq k\}$. Clearly, $\pi_{\widetilde{C}}$ is orientation preserving on $\mathcal{B}^1$. Assuming that (6.3) is false, fix a minimal k such that $\pi_{\widetilde{C}}$ fails to be orientation preserving on $\mathcal{B}^k$. Then orientation preservation is violated on the set $\mathcal{B}^k_0 \subset \mathcal{B}^k$ of chords with endpoints in $]\gamma, \delta'[$ or the set $\mathcal{B}^k_1 \subset \mathcal{B}^k$ of chords with endpoints in $]\gamma', \delta[$, mapped into $]0, 1/2[$ and $]1/2, 0[$, respectively. Therefore, by (6.2), orientation preservation must be violated on $f^{\circ n}(\mathcal{B}^k_0) \subset \mathcal{B}^{k-1}$ or $f^{\circ n}(\mathcal{B}^k_1) \subset \mathcal{B}^{k-1}$, contradicting minimality of k.

It is clear from the definition that $\mathcal{B}_{\widetilde{C}}$ is dense in $\mathbb{T}_{\widetilde{C}}$. Equally, $\pi_{\widetilde{C}}(\mathcal{B}_{\widetilde{C}})$ is dense in $\mathbb{T}$. Thus the continuous extension of $\pi_{\widetilde{C}}$—also called $\pi_{\widetilde{C}}$—is well defined and is a homeomorphism satisfying (6.2). Finally, let $\pi_C = \pi_{\widetilde{C}} \circ f^{\circ n-1}$. Then the following is easily seen:

Lemma 6.2. *π_C (resp. $\pi_{\widetilde{C}}$) conjugates $f^{\circ n}$ restricted to $\mathbb{T}_C$ (resp. $\mathbb{T}_{\widetilde{C}}$) to h if f is orientation preserving or n is even, and to $\overline{h}$ otherwise.* ■

As a consequence of Lemmata 6.2 and 6.1 we have the following

Corollary 6.3. *In each $f^{\circ n}$-set C there are exactly two angles α, γ of period not greater than n. Their periods belong to $\{n, n/2\}$, and $\alpha\gamma$ is a bifurcation chord. If f is orientation reversing and n is odd, C contains* *exactly two angles ξ_1, ξ_2 of period $2n$, which are the ends of B_C, and α, γ have period n. Otherwise, $B_C = \alpha\gamma$.*

Proof. The main point to notice is that, according to the conjugacy π_C, points in $\mathbb{T}_C$ of period dividing n correspond to fixed points of h (or $\overline{h}$) in $\mathbb{T}$. Moreover, by Lemma 6.1(iii),(iv) (cf. also construction of π_C) all chords in $\mathcal{B}_C \setminus \{B_C\}$ are preperiodic, and by Lemma 6.1(ii) the period of periodic angles in C not being an endpoint of B_C is a multiple of n.

If $f^{\circ n}$ is orientation preserving, 0 is the unique fixed point of h and by definition $\pi_C^{-1}(0) = B_C$. So according to Lemma 6.1(i) (see (a), (c), (d)), the endpoints α, γ of B_C are the only two angles of period dividing n in C, and $\alpha\gamma$ is a bifurcation chord.

In the orientation reversing case π_C conjugates $f^{\circ n}$ to $\overline{h}$, which has the three fixed points $0, 1/3, 2/3$ and no angles of period 2. Here by Lemma 6.1(i) (see (b)), $\pi_C^{-1}(0) = B_C$ has endpoints of period $2n$. On the other hand, according to the considerations at the beginning of the proof, $1/3$ and $2/3$ are π_C-images of single angles α and γ in C of period n. Thus we have exactly two angles of period n and two of period $2n$ in C. To complete the proof, observe that $\alpha\gamma$ is a bifurcation chord (since the sets $C, C_1, \ldots, C_{n-1}$ are weakly unlinked). ■

Up to this point we have collected some properties of $f^{\circ n}$-sets, in particular in the above corollary we proved that for each $f^{\circ n}$-set C there is a unique corresponding bifurcation chord with endpoints of period n in C, but so far we have not asked if $f^{\circ n}$-sets exist. This is now answered in the affirmative by constructing a class of $f^{\circ n}$-sets starting from periodic angles.

The $f^{\circ n}$-set $C^f(\alpha)$. For $\alpha \in \mathrm{Per}^f$ define the set $\widetilde{C}^f(\alpha)$ to consist of the accumulation points of sequences of endpoints of chords in $\mathcal{S}_{\mathrm{center}}$ described in Section 4. Let $C^f(\alpha) := f(\widetilde{C}^f(\alpha))$.

LEMMA 6.4. *For $\alpha \in \mathrm{Per}^f$ of period n, $C^f(\alpha)$ is an $f^{\circ n}$-set. $\widetilde{C}^f(\alpha)$ is the closure of the set of endpoints of chords in $\mathcal{S}_{\mathrm{center}}$.*

Proof. Recall that $f^{\circ n}(\mathcal{S}_{\mathrm{center}} \setminus \{\dot{\alpha}\ddot{\alpha}\}) = \mathcal{S}_{\mathrm{center}}$, which implies $f^{\circ n}(\widetilde{C}^f(\alpha)) = \widetilde{C}^f(\alpha)$. Moreover, the invariance of $\mathcal{S}_{\mathrm{center}}$ under $\beta \mapsto \beta + \frac{1}{2} \bmod 1$ carries over to $\widetilde{C}^f(\alpha)$. This shows that $\widetilde{C}^f(\alpha)$ is $f^{\circ n}$-invariant and that (ii) of Definition 1.6 is satisfied. Moreover, by Proposition 4.2, also condition (i) of Definition 1.6 is satisfied.

We show that $C^f(\alpha)$ is a Cantor set, i.e. that it is closed, totally disconnected and perfect. Obviously $C^f(\alpha)$ is closed. Assume $C^f(\alpha)$ is not totally disconnected. Then it would contain some interval I of $\mathbb{T}$. The iteration of I would produce larger and larger intervals that would finally cover $\mathbb{T}$. By the $f^{\circ n}$-invariance of $C^f(\alpha)$ this would impliy $\widetilde{C}^f(\alpha) = \mathbb{T}$ in contradiction to its definition as the limit set of endpoints of chords in $\mathcal{S}_{\text{center}}$. Thus $C^f(\alpha)$ is totally disconnected. Finally, by Proposition 5.1(i), $\dot{\alpha}$ belongs to $\widetilde{C}^f(\alpha)$. By backward iteration of $\dot{\alpha}$ one sees that all endpoints of chords in $\mathcal{S}_{\text{center}}$ belong to $\widetilde{C}^f(\alpha)$, implying that $\widetilde{C}^f(\alpha)$ is perfect. Hence $C^f(\alpha)$ is perfect, which completes the proof that $C^f(\alpha)$ is an $f^{\circ n}$-set. ∎

For $f = h$ the periodic angles $1/3$ and $2/3$ are those of minimal period greater than 1. In some sense, they provide the simplest $h^{\circ n}$-set (with $n = 2$). It is interesting that this set is also an $\overline{h}^{\circ 2}$-set, as shown below. Moreover, it will serve as a kind of prototype for the construction of more $f^{\circ n}$-sets in Section 8.

LEMMA 6.5. *The sets $C^h(1/3)$ and $C^h(2/3)$ coincide. They are both $h^{\circ 2}$-sets and $\overline{h}^{\circ 2}$-sets, and are invariant with respect to $\beta \mapsto 1 - \beta \bmod 1$.*

Proof. By Lemma 6.4, $C_1 = C^h(1/3)$ and $C_2 = C^h(2/3)$ are $h^{\circ 2}$-sets, and by Lemma 6.1(i) (see (a)), $B_{C_1} = B_{C_2} = \frac{1}{3}\frac{2}{3}$. Moreover, Lemma 6.1 implies $B_{\widetilde{C}_1} = B_{\widetilde{C}_2} = \frac{1}{3}\frac{2}{3}$ and $B'_{\widetilde{C}_1} = B'_{\widetilde{C}_2} = \frac{1}{6}\frac{5}{6}$. Note that for $T \in \mathcal{B}_{\widetilde{C}_1} \cap \mathcal{B}_{\widetilde{C}_2}$ there exist exactly two chords between the chords $\frac{1}{3}\frac{2}{3}$ and $\frac{1}{6}\frac{5}{6}$ not separating them and with second iterate equal to T. So induction and Lemma 6.1(iii),(iv) provide $\mathcal{B}_{\widetilde{C}_1} = \mathcal{B}_{\widetilde{C}_2}$, hence $\widetilde{C}_1 = \widetilde{C}_2$ and $C_1 = C_2$. One easily shows that $\widetilde{C}_2 = \{1 - \beta \bmod 1 \mid \beta \in \widetilde{C}_1\}$, implying that $\widetilde{C}_1$ and C_1 are invariant under $\beta \mapsto 1 - \beta \bmod 1$. Now the rest is obvious. ∎

7. Proof of the Bifurcation Theorem. This section is devoted to the proof of Theorem 1. We call a bifurcation chord *symmetric* if both endpoints have the same period, and otherwise *nonsymmetric*. It will turn out later that one of the endpoints of a nonsymmetric bifurcation chord always has odd period, while the other has the double period.

The angles $\Theta^f(\alpha)$ and $\Lambda^f(\alpha)$. For f a distance doubling map let $\alpha \in \mathrm{Per}^f$ be an angle of period m (i.e., in particular, $m > 1$). First note that according to Proposition 5.1 each of the angles $\xi = \xi^f(\alpha)$, $\xi_{\text{even}} = \xi^f_{\text{even}}(\alpha)$ and $\xi_{\text{odd}} = \xi^f_{\text{odd}}(\alpha)$, if defined for α, belongs to $\widetilde{C}^f(\alpha)$, and that the period of ξ divides m. According to Corollary 6.3, α and $f(\xi)$ are the only periodic angles in the set $C^f(\alpha)$ of period less than or equal to m and

$$\alpha f(\xi) \text{ is a bifurcation chord.} \tag{7.1}$$

By Lemma 6.1(i), the period of ξ is m or $m/2$, and period $m/2$ is only possible if f is orientation reversing and $m/2$ is odd. Similarly Corollary 6.3 implies that ξ_{even} and ξ_{odd} are the only angles of period $2m$ in $\widetilde{C}^f(\alpha)$ and that

$$f(\xi_{\text{even}})f(\xi_{\text{odd}}) \text{ forms a symmetric bifurcation chord.} \tag{7.2}$$

Moreover,

(7.3) $\alpha f(\xi_{\mathrm{even}})$ and $f(\xi)f(\xi_{\mathrm{odd}})$ are nonsymmetric bifurcation chords

since on the one hand $C^f(\alpha)$ and its first $m-1$ iterates are weakly unlinked and any two iterates of $\alpha f(\xi_{\mathrm{even}})$ ($f(\xi)f(\xi_{\mathrm{odd}})$, respectively), being in the same iterate of $C^f(\alpha)$, have one endpoint in common. On the other hand, the diameters $\dot\alpha\ddot\alpha$ and $\xi_{\mathrm{even}}\xi'_{\mathrm{even}}$ ($\xi\xi'$ and $\xi_{\mathrm{odd}}\xi'_{\mathrm{odd}}$) do not cross any iterate since, by the second inequality in Proposition 5.1(ii), $f^{\circ m}(\dot\alpha\xi_{\mathrm{even}}) = \dot\alpha\xi_{\mathrm{odd}}$ is behind $\dot\alpha\xi_{\mathrm{even}}$ (and $f^{\circ m}(\xi\xi_{\mathrm{odd}}) = \xi\xi_{\mathrm{even}}$ is behind $\xi\xi_{\mathrm{odd}}$). Also compare Figure 6. These observations justify the following definition.

Definition 7.1 (Angles $\Theta^f(\alpha)$ and $\Lambda^f(\alpha)$). If f is orientation reversing, $m/2$ is odd, and ξ has period $m/2$, then let $\Theta^f(\alpha) := f(\xi^f_{\mathrm{odd}}(f(\xi)))$ and $\Lambda^f(\alpha) := f(\xi)$. Otherwise, let $\Theta^f(\alpha) := f(\xi)$. For f orientation reversing and m odd, define $\Lambda^f(\alpha) := f(\xi_{\mathrm{even}})$.

It is clear from (7.1)–(7.3) that $\alpha\Theta^f(\alpha)$ is a symmetric bifurcation chord, while $\alpha\Lambda^f(\alpha)$, if defined, is a nonsymmetric bifurcation chord. The following lemma shows that apart from $\alpha\Theta^f(\alpha)$ and $\alpha\Lambda^f(\alpha)$ there are no other bifurcation chords with endpoint α.

Lemma 7.2.

(i) *For each $\alpha \in \mathrm{Per}^f$, the chord $\alpha\Theta^f(\alpha)$ is the only symmetric bifurcation chord with endpoint α, and $\Theta^f(\Theta^f(\alpha)) = \alpha$.*

(ii) *Whenever $\Lambda^f(\alpha)$ is defined for $\alpha \in \mathrm{Per}^f$, the chord $\alpha\Lambda^f(\alpha)$ is the only nonsymmetric bifurcation chord with endpoint α, and $\Lambda^f(\Lambda^f(\alpha)) = \alpha$. Otherwise there is no nonsymmetric bifurcation chord with endpoint α.*

Proof. Assume that $\alpha\gamma$ is a bifurcation chord with $\gamma \neq \Theta^f(\alpha)$. Let $\theta = \Theta^f(\alpha)$. For the periodic preimage $\dot\gamma$ of γ there are two possibilities. Either (I) $\dot\gamma$ and $\dot\theta$ are separated by $\dot\alpha\ddot\alpha$, or (II) $\dot\gamma$ and $\dot\theta$ are not separated by $\dot\alpha\ddot\alpha$.

(I) Suppose $\dot\gamma$ and $\dot\theta$ are separated by $\dot\alpha\ddot\alpha$. First we assume that $f^{\circ m}$ is orientation preserving and prove that in this case γ does not exist. By Lemma 3.3, $\dot\alpha\dot\gamma$ and $\dot\alpha\dot\theta$ are not shorter than $1/3$, hence $\dot\gamma$ must lie behind $\ddot\alpha\ddot\theta$. Since $\theta\alpha$ and $\gamma\alpha$ are bifurcation chords, for i not a multiple of m the angles of $f^{\circ i}(\{\ddot\alpha, \ddot\theta, \dot\gamma\})$ are not separated by $\dot\alpha\ddot\alpha$, implying that f does not change the orientation on $f^{\circ i}(\{\ddot\alpha, \ddot\theta, \dot\gamma\})$. So $\dot\gamma$ behind $\ddot\alpha\ddot\theta$ implies that $f^{\circ m}(\dot\gamma)$ lies behind $\dot\alpha\dot\theta$ because f is also orientation preserving on $\{\ddot\alpha, \ddot\theta, \dot\gamma\}$. More generally, induction shows that $f^{\circ km}(\dot\gamma)$ lies behind $\dot\alpha\dot\theta$ for $k = 1, 2, \ldots$. For some k we have $f^{\circ km}(\dot\gamma) = \dot\gamma$, which contradicts $\dot\gamma$ lying behind $\ddot\alpha\ddot\theta$. Hence such a γ cannot exist.

Now assume that $f^{\circ m}$ is orientation reversing, i.e. f is orientation reversing and m is odd. Then, according to Definition 7.1, $\dot{\theta} = \xi^f(\alpha)$. Moreover, $\xi_{\mathrm{even}} = \xi^f_{\mathrm{even}}(\alpha)$ and $\xi_{\mathrm{odd}} = \xi_{\mathrm{odd}}(\alpha)$ are well defined and separated from $\xi^f(\alpha)$ by $\dot{\alpha}\ddot{\alpha}$. Hence $\dot{\gamma}$, ξ_{even} and ξ_{odd} lie on the same side of $\dot{\alpha}\ddot{\alpha}$. Let $D = \{\dot{\alpha}, \dot{\gamma}, \xi_{\mathrm{odd}}, \xi_{\mathrm{even}}\}$. For some nonperiodic δ sufficiently near to $\dot{\alpha}$ and $n = 0, 1, \ldots$, we have $f^{\circ n}(D) \subset]\delta, \delta + 1/2[$ or $f^{\circ n}(D) \subset]\delta + 1/2, \delta[$. The angles in D do not have the same period, hence Corollary 3.5 implies $\dot{\gamma} = \xi_{\mathrm{odd}}$ or $\dot{\gamma} = \xi_{\mathrm{even}}$. Since $\xi_{\mathrm{odd}}\xi'_{\mathrm{odd}}$ separates ξ_{even} and $\dot{\alpha}$ (compare (5.2)) and since, by Proposition 5.1, $\xi_{\mathrm{even}}\dot{\alpha}$ is the mth iterate of the chord $\xi_{\mathrm{odd}}\dot{\alpha}$, the chord $f(\xi_{\mathrm{odd}})\alpha$ is not a bifurcation chord. So $\dot{\gamma} = \xi_{\mathrm{even}}$ and, by Definition 7.1, $\gamma = \Lambda^f(\alpha)$.

(II) Now suppose that $\dot{\gamma}$ and $\dot{\theta}$ are not separated by $f^{-1}(\alpha) = \dot{\alpha}\ddot{\alpha}$. In this case let $\eta \neq \alpha, \theta$ be a (possibly second) angle with the property that $\alpha\eta$ forms a bifurcation chord and that $\dot{\eta}$ and $\dot{\theta}$ are not separated by $f^{-1}(\alpha)$. Let $D = \{\dot{\alpha}, \dot{\gamma}, \dot{\eta}, \dot{\theta}\}$. Then for some δ sufficiently near to $\dot{\alpha}$ and $n = 0, 1, \ldots$, we have $f^{\circ n}(D) \subset]\delta, \delta + 1/2[$ or $f^{\circ n}(D) \subset]\delta + 1/2, \delta[$ since $\alpha\gamma$ and $\alpha\eta$ were assumed to be bifurcation chords. By Lemma 3.3, $\dot{\gamma}$ lies between $\dot{\alpha}\dot{\theta}$ and $\ddot{\alpha}\ddot{\theta}$ or $\dot{\theta}$ between $\dot{\alpha}\dot{\gamma}$ and $\ddot{\alpha}\ddot{\gamma}$ and therefore the set D cannot be contained in a periodic orbit. Hence (ii) of Corollary 3.5 applies. f must be orientation reversing, γ and η must coincide, and, since α and θ have period m, γ is an angle of period $m/2$ with $m/2$ odd. Moreover, $\theta = f^{\circ m/2}(\alpha)$. By Proposition 5.1(iii), this implies that $\theta \neq f(\xi^f(\alpha))$, in contrast to the case above. Therefore we are in the situation where $\Lambda^f(\alpha) = f(\xi^f(\alpha))$ and $\theta = f(\xi^f_{\mathrm{odd}}(\xi^f(\alpha)))$. Since γ is unique with the above properties, we have $\gamma = \Lambda^f(\alpha)$.

We summarize that all bifurcation chords are either of the form $\alpha\Theta^f(\alpha)$ or $\alpha\Lambda^f(\alpha)$. For each α, the chord $\alpha\Theta^f(\alpha)$ is the unique symmetric bifurcation chord with endpoint α, which also implies $\Theta^f(\Theta^f(\alpha)) = \alpha$. Similarly, the uniqueness of the nonsymmetric bifurcation chord, if defined, yields $\Lambda^f(\Lambda^f(\alpha)) = \alpha$. Finally, nonsymmetric bifurcation chords with endpoint α do not exist, if $\Lambda^f(\alpha)$ is not defined, completing the proof. ∎

Remark. Note that, for f orientation reversing, $\Lambda^f(\alpha)$ exists for all α of odd period, while for α of even period m, $\Lambda^f(\alpha)$ exists if and only if $m/2$ is odd and $\Theta^f(\alpha) = f^{\circ m/2}(\alpha)$.

Lemma 7.2 states that all bifurcation chords described in Theorem 1 exist—the existence of the three bifurcation chords connecting the fixed points of an orientation reversing map f is trivial—and that the described sets Bif^f are complete, i.e. there are no other bifurcation chords. In particular, this shows (ii) and (iv). To complete the proof of Theorem 1 it remains to show the following three things: firstly, the relation between the kneading sequences of the endpoints of symmetric bifurcation chords as stated in (i), secondly, the disjointness of symmetric bifurcation chords and finally, the existence and disjointness of the rectangles described in (iii).

Bifurcation chords and kneading sequences. Recall that an angle β is said to be *behind* a chord $\alpha\gamma$ if $d(\alpha,\gamma) < 1/2$ and β lies in the smaller open arc with endpoints α and γ.

LEMMA 7.3. *Let $\alpha\gamma$ be a symmetric bifurcation chord. Then the kneading sequences of α and γ coincide and are different from the kneading sequence of any* (*periodic*) *β behind $\alpha\gamma$.*

Proof. Equality of the kneading sequences of α and γ can easily be seen from Lemma 3.3. So assume that some β behind $\alpha\gamma$ has the same kneading sequence as α (and γ). Let $D = \{\dot{\alpha}, \dot{\gamma}, \dot{\beta}\}$. Applying Lemma 3.3 to α and γ, one easily sees that for some nonperiodic δ sufficiently close to $\dot{\alpha}$ and all $n = 0, 1, 2, \ldots$ we have $f^{\circ n}(D) \subset\,]\delta, \delta + 1/2[$ or $f^{\circ n}(D) \subset\,]\delta + 1/2, \delta[$. By Corollary 3.5, $\dot{\beta}$ would lie on the orbit of α but between the chords $\dot{\alpha}\dot{\gamma}$ and $\ddot{\alpha}\ddot{\gamma}$. This is impossible by Lemma 3.3. ■

Lemma 7.3 and the existence and uniqueness of a symmetric bifurcation chord for each periodic $\alpha \in \mathrm{Per}^f$ (compare Lemma 7.2) provide the following obvious but important statement: The number of periodic angles of period $m > 1$ with a given kneading sequence is even.

Up to the end of this section it is convenient to assume that $f = h$ or $f = \overline{h}$. Conjugacy of f to h or $\overline{h}$ guarantees that all statements presented below remain true in the general case (Lemma 7.4 only with the obvious change in the order of the α_i with respect to a fixed point of f).

For $f = h$ or $f = \overline{h}$ the angle 0 does not lie behind a bifurcation chord. Otherwise, the chord would cross $f^{-1}(\delta)$ where δ denotes its nearest endpoint to 0. Therefore, for general f we have:

(7.4) Behind a bifurcation chord there is no fixed point.

Moreover, the following is valid:

LEMMA 7.4. *For $f = h$ or $f = \overline{h}$, let $\{\alpha_1, \ldots, \alpha_k\} \subset \mathrm{Per}^f$ be the set of angles of period greater than* 1 *with a given kneading sequence. Then k is even, and if $\alpha_1 < \cdots < \alpha_k$ then $\Theta^f(\alpha_l) = \alpha_{l+1}$ for all odd $l < k$.* ■

We will make use of the following fact.

LEMMA 7.5. *For a bifurcation chord $\alpha_1\alpha_2$ and $n \in \mathbb{N}$, if β_1 and β_2 are angles and $\alpha_1\alpha_2$ and sufficiently close to α_1 and α_2, respectively, then the nth symbols of their kneading sequences coincide.*

Proof. We assume that n is a multiple of (at least) one of the periods m_i of α_i. Otherwise the statement is obvious. For $i = 1, 2$ choose β_i close enough to α_i to ensure $d(f^{\circ l}(\beta_i), f^{\circ l}(\alpha_i)) < \frac{1}{2}d(\beta_1,\beta_2)$ for $l = 1,\dots,n$, and let $\widetilde{\beta}_i$ be the endpoint of $f^{-1}(\beta_i)$ closer to $\dot{\alpha}_i$, and $\widetilde{\widetilde{\beta}}_i$ be the other one.

Assume $m_1 \geq m_2$. If n is a multiple of both periods, i.e. $n = km$ for some $k \in \mathbb{N}$ and either $m = m_1 = m_2$ or $m = m_1 = 2m_2$, we have $f^{\circ km}(\dot{\alpha}_i) = \dot{\alpha}_i$ for $i = 1,2$. Then $d(f^{\circ km}(\widetilde{\beta}_i), \dot{\alpha}_i) > d(\widetilde{\beta}_i, \dot{\alpha}_i)$. So depending on the orientation of $f^{\circ km}$, the chord $f^{\circ km-1}(\beta_1\beta_2) = f^{\circ km}(\widetilde{\beta}_1\widetilde{\beta}_2)$ is either between $\widetilde{\beta}_1\widetilde{\beta}_2$ and $\widetilde{\widetilde{\beta}}_1\widetilde{\widetilde{\beta}}_2$ or behind $\dot{\alpha}_1\dot{\alpha}_2$, implying that the nth symbols of the kneading sequences of β_1 and β_2 coincide. In the remaining case, n is a multiple of only one period, i.e. $n = km$ for some odd k and $m = m_2 = \frac{1}{2}m_1$.

Here f is orientation reversing, m is odd, and $\alpha_1 = \Lambda^f(\alpha_2) = f(\xi^f_{\mathrm{even}}(\alpha_2))$ (see Definition 7.1). Moreover, $f^{\circ n}(\dot{\alpha}_2) = \dot{\alpha}_2$ while $f^{\circ n}(\dot{\alpha}_1) = f^{\circ m}(\dot{\alpha}_1) = f^{\circ m}(\xi^f_{\mathrm{even}}(\alpha_2)) = \xi^f_{\mathrm{odd}}(\alpha_2) \neq \dot{\alpha}_1$. By Proposition 5.1(ii) (cf. Figure 6, right), the angle $\xi^f_{\mathrm{odd}}(\alpha_2)$—and thus $f^{\circ n-1}(\beta_1)$ for β_1 sufficiently near to α_2—is behind the chord $\dot{\alpha}_1\dot{\alpha}_2 = \xi^f_{\mathrm{odd}}(\alpha_2)\dot{\alpha}_2$. Also $f^{\circ n-1}(\beta_2)$ is behind $\dot{\alpha}_1\dot{\alpha}_2$, since $f^{\circ n}$ is orientation reversing, implying again coincidence of the nth symbols of the kneading sequences. ∎

On the base of (7.4) and Lemma 7.5, we get the following statement:

Corollary 7.6. *Behind each bifurcation chord the number of periodic angles of a given period is even.*

Proof. According to Lemma 1.3, the nth symbol of the kneading sequence switches from 0 to 1 exactly in nonfixed periodic angles for which n is a multiple of their period. If there existed a bifurcation chord B with an odd number of periodic angles of some period n behind B, one could assume n to be minimal with this property. Then the number of periodic angles behind B of period properly dividing n would be even. This and (7.4) would imply that the number of switches of the nth symbol of the kneading sequence is odd, contradicting Lemma 7.5. ∎

Disjointness of symmetric bifurcation chords. In order to show the disjointness of all symmetric bifurcation chords for given f, we construct a system of mutually disjoint chords and show that this system coincides with the system of all chords $\beta\Theta^f(\beta)$ with $\beta \in \mathrm{Per}^f$. This generalizes an idea of Lavaurs (see [9]).

Call an angle $\alpha_1 \in \mathrm{Per}^f$ *combinatorially smaller than* $\alpha_2 \in \mathrm{Per}^f$ if the period of α_1 is less than the period of α_2 or the periods coincide and $\alpha_1 < \alpha_2$ (in $]0,1[$). Construct chords $B_1, B_2, \dots$ with endpoints in Per^f as follows:

Start with an empty set of chords and with $n = 0$. If $B_1,\dots,B_n$ are already constructed, let γ be the combinatorially smallest angle in Per^f different from the endpoints of $B_1,\dots,B_n$, and let δ be the combinatorially smallest angle different from γ and the endpoints of $B_1, \dots, B_n$ and such that $\gamma\delta$ does not cross any of the chords $B_1, \dots, B_n$. Set $B_{n+1} = \gamma\delta$.

It remains to prove that each B_i is of the form $\beta\Theta^f(\beta)$. So we assume that this is true for $i = 1, \dots, n$. Let $B_{n+1} = \gamma\delta$ be given as above and let p (> 1) be the period of γ. Since by Corollary 7.6 behind each B_i ($i = 1, \dots, n$) the number of periodic angles of period p is even, at least one of them is different from γ and not separated from γ by one of the chords $B_1, \dots, B_n$. This implies that δ has period p.

By Lemma 7.4, $\delta = \Theta^f(\gamma)$ if for all $\eta \in \mathrm{Per}^f$ of period p with $\gamma < \eta < \delta$ the kneading sequence is different from that of γ. Indeed, such η is separated from γ by a chord $B \in \{B_1, \dots, B_n\}$. Assume that B has minimal possible period, say q. Then by the construction of the B_i this period is less than p and B is unique. (Two such chords would be separated by one of a smaller period.) Consequently, the number of angles in Per^f with period dividing q lying in the interval $]\gamma, \eta[$ is odd. Since the qth symbol of the kneading sequence changes exactly at such angles, the kneading sequences of γ and η have different qth symbols.

We have shown that all symmetric bifurcation chords are mutually disjoint. This completes the proof of Theorem 1 for the orientation preserving case. For f orientation reversing, note that, by (7.4), the three chords connecting the fixed points of f do not cross any other bifurcation chord. Moreover, we have to discuss the nonsymmetric bifurcation chords in this case.

Existence and disjointness of rectangles. Let $\alpha \in \mathrm{Per}^f$ be of odd period, $\lambda = \Lambda^f(\alpha) = f(\xi^f_{\mathrm{even}}(\alpha))$, $\Theta^f(\lambda) = f(\xi^f_{\mathrm{odd}}(\alpha))$ and $\Theta^f(\alpha) = f(\xi^f(\alpha))$. Since, by (7.3), $\Theta^f(\alpha)\Theta^f(\lambda)$ is a bifurcation chord, the four angles form a rectangle with all sides being bifurcation chords as described in Theorem 1. To show the disjointness of this rectangle from any other bifurcation chord it remains to prove that there is no symmetric chord separating the chords $\alpha\Theta^f(\alpha)$ and $\lambda\Theta^f(\lambda)$. All other possibilities of chords crossing the rectangle would either violate the disjointness of symmetric chords or contradict the fact that $\lambda\Theta^f(\lambda)$ is always the longest chord of a rectangle (cf. Lemma 6.1(ii)).

Assuming that there exist symmetric chords separating $\alpha\Theta^f(\alpha)$ and $\lambda\Theta^f(\lambda)$, let k be the minimal possible period of the endpoints of such a separating chord. Since each angle in Per^f is the endpoint of a unique symmetric bifurcation chord, and symmetric bifurcation chords do not cross each other, Corollary 7.6 implies that $\alpha\Theta^f(\alpha)$ and $\lambda\Theta^f(\lambda)$ are separated by at least two symmetric bifurcation chords with endpoints of period k. The latter can be chosen so that no symmetric chord of period not greater than k separates them. Then their endpoints must have the same kneading sequences, which immediately follows from Lemmata 1.3 and 7.5. This contradicts Lemma 7.3.

So the two symmetric bounding chords of a rectangle are not separated by a symmetric bifurcation chord. Therefore, rectangles are mutually disjoint. Since each nonsymmetric bifurcation chord belongs to some rectangle, this completes the proof of Theorem 1.

8. Proof of the Similarity Theorem. According to Lemma 6.1, there is a unique longest chord B_C in each $f^{\circ n}$-set C, which is a bifurcation chord and satisfies one of statements (a)–(d) in assertion (i) of that lemma. In view of Theorem 1 and the paragraph following it, it is clear that B_C satisfies (a) if and only if it is a free bifurcation chord. Similarly, (b) corresponds to B_C being regular, and irregular chords B_C satisfy either '(c) or (d), implying the classification of bifurcation chords B_C stated in Theorem 2(ii).

To show the one-to-one correspondence between $f^{\circ n}$-sets and bifurcation chords via the map $C \mapsto B = B_C$, we first prove this assertion for $f^{\circ n}$-sets with free and regular longest chords B_C. If B is a free bifurcation chord with endpoints of period n, denote these endpoints by α and γ. If $B = \beta_1\beta_2$ is a regular bifurcation chord with endpoints of period $2n$ (n odd), let $\alpha = \Lambda^f(\beta_1)$ and $\gamma = \Lambda^f(\beta_2)$ be the other two vertices of the corresponding rectangle. In both cases $\alpha\gamma$ is a bifurcation chord with endpoints of period n. Moreover, by Lemma 6.4, $C^f(\alpha)$ (as well as $C^f(\gamma)$) is an $f^{\circ n}$-set and, by Corollary 6.3, its longest chord is $B_{C^f(\alpha)} = B$. Thus there is at least one $f^{\circ n}$-set with longest chord B for each free or regular B, i.e. the map $C \mapsto B = B_C$ is surjective. We show injectivity by proving that given B all $f^{\circ n}$-sets C with longest chord B coincide.

Proposition 8.1. *Let C be an $f^{\circ n}$-set with free or regular B_C and let α and $\gamma = \Theta^f(\alpha)$ be its unique angles of period n. Then $C = C^f(\alpha) = C^f(\gamma)$.*

Proof. We show that $\widetilde{C}^f(\alpha) \subseteq \widetilde{C}$. For α let $\mathcal{S}_{\text{center}}$ be as in Section 4. Since $\alpha \in C$, we have $\dot\alpha = f^{\circ n-1}(\alpha) \in \widetilde{C}$ and thus $\ddot\alpha \in \widetilde{C}$ because of the $'$-symmetry of $\widetilde{C}$. Moreover, since for $i = 1, \dots, n-1$ the set $f^{\circ i}(\widetilde{C})$ lies behind $B_{\widetilde{C}}$ or $B'_{\widetilde{C}}$ (see Lemma 6.1(ii)), and $\dot\alpha\ddot\alpha$ between $B_{\widetilde{C}}$ and $B'_{\widetilde{C}}$, one shows by induction that the endpoints of chords in $\mathcal{S}_{\text{center}}$ are contained in $\widetilde{C}$. This yields $\widetilde{C}^f(\alpha) \subseteq \widetilde{C}$, implying $C^f(\alpha) \subseteq C$. Now it is an immediate consequence of Corollary 6.3 that B_C and $B_{C^f(\alpha)}$ coincide. So the endpoints of B_C are elements of $C^f(\alpha)$. Using Lemma 6.1(iii),(iv), one shows inductively that all endpoints of chords in $\mathcal{B}_C$ belong to $C^f(\alpha)$. Since $C^f(\alpha)$ is closed, it also contains all accumulation points of such endpoints, hence C. Therefore, $C = C^f(\alpha)$. In the proof we can replace α by γ and show in the same way that $C = C^f(\gamma)$. ■

According to Proposition 8.1, we have bijectivity of the map $C \mapsto B = B_C$ if the chord $\alpha\gamma$ (and thus the set $C^f(\alpha) = C^f(\gamma)$) is uniquely determined for each free or regular bifurcation chord B. For free chords B this is obviously true, since $B = \alpha\gamma$. For regular chords B, note that for sets $C^f(\alpha)$ with longest chord B, by (7.3), $\alpha\beta$ is a (nonsymmetric) bifurcation chord, where β is one of the endpoints of B. So, by Lemma 7.2, $\alpha\gamma$ is uniquely determined. Summarizing the above we arrive at the following:

COROLLARY 8.2. *$C \mapsto B_C$ bijectively maps the family of all $f^{\circ n}$-sets with B_C free or regular onto the set of all free or regular bifurcation chords.* ■

Note that assertion (iii) of Theorem 2 is a direct consequence of Lemma 6.2. For uniqueness of the map π_C with the required properties see its construction.

We complete the proof of bijectivity of the map $C \mapsto B_C$ by treating the case of irregular bifurcation chords B_C. First recall that, according to Lemma 6.5, $C_0 = C^h(1/3)$ is an $\overline{h}^{\circ 2}$-set with $B_{C_0} = \frac{1}{3}\frac{2}{3}$. So C_1 and C_2, obtained by rotating C_0 by $1/3$ and $2/3$, respectively, are $\overline{h}^{\circ 2}$-sets as well. Their longest chords are $B_{C_1} = \frac{2}{3}0$ and $B_{C_2} = 0\frac{1}{3}$, respectively.

For f orientation reversing and $\alpha \in \mathrm{Per}^f$ of odd period n, consider the three irregular sides $\alpha\Theta^f(\alpha)$, $\alpha\Lambda^f(\alpha)$ and $\Theta^f(\alpha)\Lambda^f(\Theta^f(\alpha))$ of the corresponding rectangle. Their endpoints belong to the $f^{\circ n}$-set $C = C^f(\alpha)$. Now $\pi_C : \mathbb{T}_C \to \mathbb{T}$ conjugates $f^{\circ n}$ to $\overline{h}$ and maps the chord $\alpha\Theta^f(\alpha)$ to the chord $\frac{1}{3}\frac{2}{3}$ and the chord $\Lambda^f(\alpha)\Lambda^f(\Theta^f(\alpha))$ to 0. Therefore $\pi_C^{-1}(C_0)$, $\pi_C^{-1}(C_1)$ and $\pi_C^{-1}(C_2)$ define three $f^{\circ 2n}$-sets, the longest chords of which are the three irregular chords. Thus for each irregular bifurcation chord B there is an $f^{\circ n}$-set with longest chord B, implying surjectivity.

For injectivity of the map $C \mapsto B_C$ for $f^{\circ n}$-sets with irregular B_C we can argue as at the end of the proof of Proposition 8.1 (also cf. Lemma 6.1): The forward orbit of B_C determines whether $f^{\circ i}(C)$ for $i = 0, 1, \ldots$ is behind the chord $B_{\widetilde{C}} = f^{\circ n-1}(B_C)$, behind $B'_{\widetilde{C}}$, or between these two chords. From this one shows that $\mathcal{B}_C$, hence C, is determined by B_C.

In order to show (iv) of Theorem 2, let C be an $f^{\circ n}$-set and g be h or $\overline{h}$ as in Theorem 2. By Lemma 6.1(i), B_C has period dividing n and all other chords in $\mathcal{B}_C$ are preperiodic. Therefore, Per^f_C is contained in the set $\mathbb{T}_C \setminus \mathcal{B}_C$ (here regard $\mathcal{B}_C$ as a set of points in $\mathbb{T}_C$). Clearly, π_C maps Per^f_C bijectively onto Per^g, where angles mapped to angles of period k have period kn.

Let $\gamma \in \mathrm{Per}^f_C$ be of period kn and $\overline{\gamma} = \pi_C^{-1}(\Theta^g(\pi_C(\gamma)))$. We show that $\gamma\overline{\gamma}$ is a symmetric bifurcation chord. Clearly, $\overline{\gamma}$ has the same period as γ, and by definition of Θ^g, the iterates of $\pi_C(\gamma)\Theta^g(\pi_C(\gamma))$ do not cross each other and do not cross the diameters $g^{-1}(\{\pi_C(\gamma)\})$ and $g^{-1}(\{\Theta^g(\pi_C(\gamma))\})$. Therefore, the $f^{\circ n}$-iterates of $\gamma\overline{\gamma}$ (with endpoints in C) do not cross each other, and thus the same holds for the $f^{\circ n}$-iterates of $f^{\circ i}(\gamma\overline{\gamma})$ (with endpoints in $f^{\circ i}(C)$) for $i = 1, \dots, n-1$. Since, moreover, the f-iterates of C are weakly unlinked, we conclude that the f-iterates of $\gamma\overline{\gamma}$ do not cross each other. Furthermore, since $\pi_{\widetilde{C}}$ conjugates $f^{\circ n}$ restricted to $\mathbb{T}_{\widetilde{C}}$ to g, the $f^{\circ n}$-iterates of the chord $f^{\circ n-1}(\gamma\overline{\gamma})$ (with endpoints in $\widetilde{C}$) do not cross the chords $f^{-1}(\{\gamma\})$ and $f^{-1}(\{\overline{\gamma}\})$. Here note that $\pi_{\widetilde{C}}$ maps $g^{-1}(\{\pi_C(\gamma)\})$ to $f^{-1}(\{\gamma\})$ and $g^{-1}(\{\pi_C(\overline{\gamma})\})$ to $f^{-1}(\{\overline{\gamma}\})$. Hence $\gamma\overline{\gamma}$ is indeed a symmetric bifurcation chord, and, by Lemma 7.2, $\overline{\gamma} = \Theta^f(\gamma)$.

In a similar way one can show that the images of nonsymmetric bifurcation chords under π_C^{-1} are again nonsymmetric bifurcation chords, i.e. for f orientation reversing and $\gamma \in \mathrm{Per}_C^f$ of odd period, the chord $\gamma\widehat{\gamma}$ is a nonsymmetric bifurcation chord, where $\widehat{\gamma} = \pi_C^{-1}(\Lambda^{\bar{h}}(\pi_C(\gamma)))$. Now (iv) of Theorem 2 can easily be seen, and so the proof of Theorem 2 is complete.

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