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Clopen graphs

by

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Abstract. A graph G on a topological space X as its set of vertices is *clopen* if the edge relation of G is a clopen subset of X^2 without the diagonal.

We study clopen graphs on Polish spaces in terms of their finite induced subgraphs and obtain information about their cochromatic numbers. In this context we investigate modular profinite graphs, a class of graphs obtained from finite graphs by taking inverse limits. This continues the investigation of continuous colorings on Polish spaces and their homogeneity numbers started in [11] and [9].

We show that clopen graphs on compact spaces have no infinite induced subgraphs that are 4-saturated. In particular, there are countably infinite graphs such as Rado's random graph that do not embed into any clopen graph on a compact space. Using similar methods, we show that the quasi-orders of clopen graphs on compact zero-dimensional metric spaces with topological or combinatorial embeddability are Tukey equivalent to ω^ω with eventual domination. In particular, the dominating number $\mathfrak{d}$ is the least size of a family of clopen graphs on compact metric spaces such that every clopen graph on a compact zero-dimensional metric space embeds into a member of the family. We also show that there are $\aleph_0$-saturated clopen graphs on ω^ω, while no $\aleph_1$-saturated graph embeds into a clopen graph on a Polish space. There is, however, an $\aleph_1$-saturated F_σ graph on 2^ω.

1. Introduction and outline of the article. Some rather pathological uncountable graphs can be constructed using the Axiom of Choice. There is, for example, a graph on the real line $\mathbb{R}$ with the property that for each uncountable set $A \subseteq \mathbb{R}$, there are two vertices in A that form an edge and there are two vertices in A that do not form an edge in the graph. This graph shows that the infinite Ramsey theorem fails if "infinite" is replaced by "uncountable".

These pathologies vanish once we assume that the uncountable graphs under consideration are sufficiently definable. By theorems of Galvin and Mycielski, every graph G on an uncountable Polish space whose edge relation has the Baire property has a set H of size $2^{\aleph_0}$ of vertices such that either

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any two distinct vertices in H form an edge in G or no two vertices in H form an edge in G (see [14]).

Naturally, most uncountable graphs that occur in mathematics are definable in some way. And the study of Borel and analytic graphs on Polish spaces has been very successful. A landmark result that has found several interesting applications and generalizations is the $\mathcal{G}_0$-dichotomy of Kechris, Solecki, and Todorcevic [15] about Borel chromatic numbers of analytic graphs.

In this article, we carry out an in-depth study of definable graphs on Polish spaces of the least possible complexity: clopen graphs. In the guise of continuous colorings, clopen graphs on Polish spaces occurred in [11] in the context of planar convex geometry and were then further studied in [9]. However, the main objective in those two articles was to get information about a certain cardinal invariant of continuous colorings, the so called homogeneity number, which corresponds to the cochromatic number of graphs.

In Sections 3 and 4 we generalize results from [9] and study the class of *modular profinite graphs*, which can be represented in a natural way as limits of inverse systems of finite graphs. Even the class of modular profinite graphs of countable weight is rich in the sense that its embeddability relation is as complicated as set-theoretic inclusion on the power set of ω. There is a universal modular profinite graph of countable weight. The main reason we study modular profinite graphs is that they give canonical examples of clopen graphs on compact metric spaces with a prescribed family of finite induced subgraphs. This is used in the proof of the fact that the cochromatic number of a clopen graph on a Polish space is in many cases determined by the family of finite induced subgraphs of the graph under consideration. If the family of finite induced subgraphs is contained in the closure of a finite family of finite graphs under certain natural operations, then the cochromatic number of the clopen graph is either countable or the smallest possible uncountable cochromatic number. This is shown in Section 5.

We then turn to the structural properties of the class of clopen graphs on compact metric spaces. In Section 6 we show that there is a clopen graph on ω^ω such that every clopen graph on a compact zero-dimensional metric space embeds into it. It follows that there is a family of size $\mathfrak{d}$ of clopen graphs on the Cantor space such that every clopen graph on a compact zero-dimensional metric space embeds into a member of the family. Here $\mathfrak{d}$ denotes the *dominating number*, the least size of a family of compact sets that covers ω^ω.

Section 7 deals with saturation of induced subgraphs of clopen graphs on compact metric spaces and on Polish spaces. In the case of compact metric spaces there are no infinite 4-saturated induced subgraphs, and in the case of Polish spaces there are no uncountable $\aleph_1$-saturated induced subgraphs. There are, however, $\aleph_0$-saturated clopen graphs on ω^ω. Also, there is an $\aleph_1$-saturated F_σ graph on 2^ω.

The methods developed in Section 7 are then used in Section 8 to show that there is no universal clopen graph on a compact metric space. In fact, the smallest size of a family of clopen graphs on compact metric spaces such that every clopen graph on a zero-dimensional compact metric space embeds into a member of the family is at least $\mathfrak{d}$. Together with the results from Section 6 this shows that the least size of such a family is exactly $\mathfrak{d}$. This contrasts nicely with the fact that there are universal graphs on 2^ω at all $\mathbf{\Sigma}$- and $\mathbf{\Pi}$-levels of the Borel and projective hierarchies. Answering a question in a previous version of this article, Arnold Miller has shown that the least size of a family of clopen graphs on ω^ω such that every clopen graph on ω^ω embeds into a member of the family is $\aleph_1$ [21].

Finally, in Section 9, the result about the non-saturation of infinite induced subgraphs of clopen graphs on compact metric spaces is generalized to infinite induced subgraphs on clopen graphs of spaces that are just compact. In particular, there are countable graphs that do not embed into a clopen graph on a compact space.

2. Continuous colorings and clopen graphs

Definition 2.1. An *n-coloring with k colors* on a set X is a function $c : [X]^n \to k$. We are only interested in 2-colorings and among those we are mostly interested in colorings with 2 colors. Hence we call 2-colorings with 2 colors just colorings.

If X is a topological space and $k \in \omega$, then a coloring $c : [X]^2 \to k$ is *continuous* if for all $\{x, y\} \in [X]^2$ there are disjoint open sets $U, V \subseteq X$ with $x \in U$ and $y \in V$ such that for all $a \in U$ and all $b \in V$, $c(x, y) = c(a, b)$. This is just continuity with respect to the natural topology on $[X]^2$.

If c is a coloring on X and d is a coloring on Y, we write $c \leq d$ if there is a topological embedding $e : X \to Y$, i.e., a homeomorphism onto its image, that preserves colors in the sense that for all distinct $x_0, x_1 \in X$ we have

$$c(x_0, x_1) = d(e(x_0), e(x_1)).$$

A *graph* G is a set $V(G)$ of vertices together with a set $E(G) \subseteq [V(G)]^2$ of edges. G is a *graph on* X if $V(G) = X$. A graph G on a topological space is *open, closed, clopen, Borel,* or *analytic* if the *edge-relation*

$$\{(x, y) : \{x, y\} \in E(G)\}$$

of G has the respective property as a subset of $X^2 \setminus \{(x, x) : x \in X\}$.

In the case of graphs, we distinguish two different notions of embeddings:

A graph F embeds into a graph G *combinatorially* if there is an injective map from the vertices of F to the vertices of G that preserves both edge and non-edges. This is the usual notion embeddability between graphs. If the vertex sets of F and G carry a topology, we say that F embeds into G *topologically* if there is an embedding of F into G that is a homeomorphism onto its range.

A coloring c on a set X corresponds to the graph $G_c = (X, c^{-1}(1))$, and a graph $G = (V(G), E(G))$ corresponds to the coloring $c_G : [V(G)]^2 \to 2$ that is the characteristic function of the set $E(G)$ of edges.

Note that a coloring on a topological space X is continuous iff the corresponding graph is clopen. For two continuous colorings c and d we have $c \leq d$ iff the graph corresponding to c topologically embeds into the graph corresponding to d.

We are primarily interested in continuous colorings on Polish spaces and mostly in uncountable features of these colorings. Removing countably many points from an uncountable Polish space we obtain a perfect Polish space, i.e., a Polish space without isolated points. Every perfect Polish space is a continuous 1-1 image of the Baire space ω^ω (see [14]). Given a continuous coloring c on a perfect Polish space X, we can pull back c along a continuous injection from ω^ω onto X, obtaining a continuous coloring on ω^ω that, for our purposes, carries practically the same information as c itself. This is one of the reasons we can often restrict our attention to continuous colorings on ω^ω. A different reason why it is usually enough to consider continuous colorings on zero-dimensional spaces will be pointed out in Lemma 3.4.

Just as in the finite case, uncountable graphs can be studied using cardinal invariants. Popular cardinal invariants of graphs are the clique number and the chromatic number.

DEFINITION 2.2. Let $G = (V, E)$ be a graph. A set $C \subseteq V$ is a *clique* if any two distinct vertices in C form an edge of G. Furthermore, $C \subseteq V$ is *independent* if no two vertices in C form an edge in G. The *clique number* of G is the supremum of all sizes of cliques in G. The *chromatic number* of G is the least size of a family $\mathcal{F}$ of independent sets such that $\bigcup \mathcal{F} = V$.

Clopen graphs on Polish spaces are the simplest uncountable graphs in the sense of descriptive complexity. So simple, in fact, that both the clique number and the chromatic number are degenerate:

If a clopen graph on a Polish space has an uncountable clique, then it has a perfect clique and hence the clique number $2^{\aleph_0}$. If the chromatic number of a clopen graph on a Polish space is uncountable, then the graph has a perfect independent set and hence its chromatic number is $2^{\aleph_0}$.

Both of these facts follow from the definable instance of Todorcevic's Open Coloring Axiom, which can be proved in ZFC (see [5]):

THEOREM 2.3. *Let $G = (X, E)$ be an open graph on a Polish space X. Then either the chromatic number of G is countable or G has a perfect clique.*

One cardinal invariant of clopen graphs that has a more interesting behavior in the uncountable case is the *cochromatic number*, which translates into the *homogeneity number* of the associated continuous coloring.

DEFINITION 2.4. Let G be a graph. A set $H \subseteq V(G)$ is *homogeneous* if either any two distinct vertices of H form an edge or any two distinct vertices of H form a non-edge. The *cochromatic number* of G is the least size of a family of homogeneous sets that covers $V(G)$.

If $k \in \omega$ and $c : [X]^2 \to k$ is a coloring, then $H \subseteq X$ is *c-homogeneous* if c is constant on $[H]^2$. The homogeneity number $\mathfrak{hm}(c)$ is the least size of a family of c-homogeneous sets that covers X.

Homogeneity numbers of continuous colorings on Polish spaces came up in [11] in the context of planar convexity and were further studied in [9] and [7]. In [8] it was shown that the colorings that appear in the context of planar convexity are of a particularly simple form. Namely, the corresponding graphs have no induced paths of length 4. This result was the motivation for a part of the research presented in this article.

3. Modular maps and modular profinite graphs. Usually a map between the vertex sets of two graphs is called a *homomorphism* if it preserves edges. This definition favors edges over non-edges. We are, however, interested in graphs as an alternative description of 2-colorings with two colors. In the case of colorings neither of the two colors should be emphasized. Insisting on preservation of both edges and non-edges leaves us with graph embeddings only, which is too restrictive.

For our purposes a more suitable notion of structure preserving map is that of a modular map, which we present in Definition 3.3. We first introduce modules as they appear in Gallai's modular decomposition of finite graphs [6].

DEFINITION 3.1. Let $G = (V, E)$ be a graph. A set $M \subseteq V$ is a *module* if for all $v \in V \setminus M$ and all $u_0, u_1 \in M$ we have $\{u_0, v\} \in E$ iff $\{u_1, v\} \in E$. A module M is *trivial* if $|M| \leq 1$ or $M = V$.

REMARK 3.2. The following facts about modules are well-known and easily verified.

(1) The modules of a graph G are the same as the modules of the complement of G.
(2) Every connected component of a graph is a module.
(3) If M and N are both modules of G and $M \cap N \neq \emptyset$, then $M \cup N$ is a module.

(4) If M and N are disjoint modules of G, then either all vertices in M are connected to all vertices in N or no vertex in M is connected to any vertex in N.

Definition 3.3. A partition P of the vertex set $V(G)$ of a graph G into modules of G is a *modular partition* of G. By Remark 3.2(4), a modular partition P carries a natural graph structure by connecting two distinct modules $M, N \in P$ by an edge if every vertex in M is connected to every vertex in N. This graph is the *quotient* G/P.

Given two graphs G and H, a map $f : V(G) \to V(H)$ is *modular* if for all $u, v \in V(G)$ with $f(u) \neq f(v)$ we have $\{u, v\} \in E(G)$ iff $\{f(u), f(v)\} \in E(H)$. In case of a modular map, we will often write $f : G \to H$ instead of $f : V(G) \to V(H)$.

Clopen graphs on compact spaces have a natural modular partition into the connected components of the underlying space. The following is essentially Lemma 2.12 in [9].

Lemma 3.4. *Let $G = (X, E)$ be a clopen graph on a compact space X. Let* Comp(X) *denote the set of connected components of X in the topological sense. Then* Comp(X) *is a modular partition of G, each component $C \in$* Comp(X) *is homogeneous, the space* Comp(X) *with the quotient topology is compact and zero-dimensional, and the quotient graph $G/$*Comp(X) *is clopen.*

We now introduce inverse systems, in the particular case of graphs. Our notation for inverse systems follows [3].

Definition 3.5. Let $(I, \leq)$ be a directed set. A family $(G_i)_{i\in I}$ of graphs together with a family $(\pi_i^j)_{i,j\in I,\, i\leq j}$ maps is an *inverse system of graphs with modular bonding maps* if the following hold:

(1) For all $i, j \in I$ with $i \leq j$, $\pi_i^j : G_j \to G_i$ is modular.
(2) For all $i \in I$, π_i^i is the identity map on G_i.
(3) For all $i, j, k \in I$, if $i \leq j \leq k$, then $\pi_i^k = \pi_i^j \circ \pi_j^k$.

We do not require the bonding maps π_i^j to be onto.

A graph G together with a family $(\pi_i)_{i\in I}$ of maps is the *limit* of an inverse system $((G_i)_{i\in I}, (\pi_i^j)_{i,j\in I, i\leq j})$ of graphs with modular bonding maps if the following hold:

(4) For all $i \in I$, $\pi_i : G \to G_i$ is modular.
(5) For all $i, j \in I$ with $i \leq j$, $\pi_i = \pi_i^j \circ \pi_j$.
(6) Whenever F is a graph and $(\rho_i)_{i\in I}$ is a family of modular maps such that (4) and (5) hold for F and the ρ_i instead of G and the π_i, then there is a unique modular map $\rho : F \to G$ such that for all $i \in I$, $\rho_i = \pi_i \circ \rho$.

A graph G is *modular profinite* if it is the limit of an inverse system of finite graphs with modular bonding maps.

Observe that our modular profinite graphs are only vaguely related to the profinite graphs in the sense of Serre that occur for example as Cayley graphs of profinite groups.

Every modular profinite graph is the limit of an inverse system of finite sets and hence carries a compact, zero-dimensional topology. Namely, if G is the limit of an inverse system $((G_i)_{i\in I}, (\pi_i^j)_{i,j\in I, i\leq j})$ of finite graphs and this is witnessed by a family $(\pi_i)_{i\in I}$ of modular maps, then a subset A of $V(G)$ is clopen iff for some $i \in I$ there is a set $B \subseteq V(G_i)$ such that $A = \pi_i^{-1}[B]$. We refer to this topology as the *limit topology*.

It is well-known that every compact, zero-dimensional space is the limit of an inverse system of finite sets, and hence there are arbitrarily large, modular profinite graphs, for example all discrete and all complete graphs on compact, zero-dimensional spaces.

Definition 3.6. Let G be a graph. Two vertices v and w of G are *separated* by a modular partition P of G if v and w are members of distinct classes of the partition P.

Lemma 3.7. *Let G be a graph on a topological space $V(G)$. Then G is a modular profinite graph iff $V(G)$ is compact and for all distinct vertices v and w there is a finite, modular partition of G consisting of clopen sets that separates v and w. In particular, the space of vertices of a modular profinite graph is zero-dimensional and the graph is clopen.*

Proof. Suppose $V(G)$ is compact and any two vertices of G are separated by a finite modular partition into clopen sets.

If P and Q are modular partitions of G, we write $P \leq Q$ if Q refines P, i.e., if every $A \in Q$ is fully included in some $B \in P$. Given any two modular partitions P and Q of G into finitely many clopen sets, it is easily checked that

$$P \vee Q = \{A \cap B : A \in P,\ B \in Q, \text{ and } A \cap B \neq \emptyset\}$$

is again a modular partition of G into finitely many clopen sets. $P \vee Q$ is the smallest common refinement of P and Q. It follows that the set I of finite modular partitions of G into clopen sets ordered by $\leq$ is a directed set.

If P and Q are modular partitions of G and $P \leq Q$, then the natural map $\pi_P^Q : G/Q \to G/P$ is modular. It is easily checked that G together with the quotient maps $\pi_P : G \to G/P$, $P \in I$, is the inverse limit of the system consisting of the finite graphs G/P, $P \in I$, with the bonding maps π_P^Q, $P, Q \in I$, $P \leq Q$.

On the other hand, if I is a directed set ordered by $\leq$ and $(G_i)_{i\in \mathrm{I}}$ is a family of graphs that forms an inverse system with the modular bonding maps $\pi_i^j : G_i \to G_j$, $i \leq j$, and limit G together with modular maps $\pi_i : G \to G_i$, then for any two distinct vertices $v, w \in V(G)$ there is $i \in I$ with $\pi_i(v) \neq \pi_i(w)$. Now

$$P = \{\pi_i^{-1}(u) : u \in V(G_i)\}$$

is a finite modular partition of G into clopen sets that separates v and w.

As a limit of an inverse system of finite spaces, $V(G)$ is compact and zero-dimensional. Given two distinct vertices $v, w \in V(G)$ and a finite modular partition P of G into clopen sets separating them, let $A, B \in P$ be such that $v \in A$ and $w \in B$. By the modularity of P, for all $a \in A$ and all $b \in B$, $\{a, b\} \in E(G)$ iff $\{v, w\} \in E(G)$. This shows that G is clopen. ∎

We now describe the general form of a modular profinite graph of countable weight. Recall that a topological space is *of countable weight* if its topology has a countable basis.

Definition 3.8. Let $X \subseteq \omega^\omega$ be a closed set. Let

$$T = T(X) = \{t \in \omega^{<\omega} : \exists x \in X \ (t \subseteq x)\}.$$

For every $t \in T$ let $\mathrm{succ}_T(t)$ be the set of immediate successors of t in T. For $x, y \in \omega^\omega$ let $x \wedge y$ be the longest common initial segment of x and y. For $x \neq y$ let $\Delta(x, y)$ be the length of $x \wedge y$, i.e., $\Delta(x, y) = \min\{n \in \omega : x(n) \neq y(n)\}$.

A coloring $c : [X]^2 \to k$, $k \in \omega$, is a *node coloring* if there is a function $\overline{c} : T(X) \to k$ such that for all $\{x, y\} \in [X]^2$, $c(x, y) = \overline{c}(x \wedge y)$. In other words, c is a node coloring if $c(x, y)$ only depends on $x \wedge y$.

A coloring $c : [X]^2 \to k$ is an *almost node coloring* if $c(x, y)$ only depends on $\{x{\restriction}(\Delta(x, y)+1), y{\restriction}(\Delta(x, y)+1)\}$. That is, c is an almost node coloring if there is a family $(c_t)_{t \in T(X)}$ such that for all $t \in T(X)$, $c_t : [\mathrm{succ}_{T(X)}(t)]^2 \to k$ and for all $\{x, y\} \in [X]^2$,

$$c(x, y) = c_{x \wedge y}(x{\restriction}(\Delta(x, y) + 1), y{\restriction}(\Delta(x, y) + 1)).$$

Observe that a closed set $X \subseteq \omega^{<\omega}$ is compact iff for all $t \in T(X)$, $\mathrm{succ}_{T(X)}(t)$ is finite, i.e., if $T(X)$ is finitely branching.

Theorem 3.9. *A clopen graph G whose vertex space $V(G)$ has countable weight is modular profinite iff the corresponding coloring c_G is isomorphic to an almost node coloring c on a compact subspace of ω^ω.*

Proof. First let $X \subseteq \omega^\omega$ be compact and let c be an almost node coloring on X. For each $n \in \omega$ let

$$T_n = \{t \in T(X) : \mathrm{dom}(t) = n\} \quad \text{and} \quad P_n = \{\{x \in X : t \subseteq x\} : t \in T_n\}.$$

It is clear that each P_n is a finite partition of X into clopen sets and that the partitions P_n separate the points of X. We show that they are modular.

Let $n \in \omega$ and $t \in T_n$. If $x, y \in X$ are such that $t \subseteq x$ and $t \not\subseteq y$, then $c(x, y)$ only depends on $x{\restriction}(\Delta(x, y)+1)$ and $y{\restriction}(\Delta(x, y)+1)$. But $x{\restriction}(\Delta(x, y)+1)$

$\subseteq t$. It follows that $c(x,y)$ is already determined by t and $y\restriction(\Delta(x,y)+1)$ and in particular independent of the choice of x within the set $\{z \in X : t \subseteq z\}$. This shows that P_n is a modular partition of the graph G_c. Hence G_c and therefore G are modular profinite.

Now suppose that G is modular profinite. Since $V(G)$ is compact, zero-dimensional and of countable weight, there are only countably many clopen subsets of $V(G)$. It follows that there are only countably many finite modular partitions of G into clopen sets. These partitions form a directed set with respect to the ordering $\leq$. Since the directed set is countable, there is a sequence $(P_n)_{n\in\omega}$ of such partitions that is increasing with respect to $\leq$ such that for every finite modular partition Q of G into clopen sets there is $n \in \omega$ such that $Q \leq P_n$. Now the partitions P_n, $n \in \omega$, separate the vertices of G.

For each $n \in \omega$ let $G_n = G/P_n$. For $n \in \omega$ let $\pi_n : G \to G/P_n$ be the quotient map. Since each G_n is finite, we can identify $V(G_n)$ with the natural number $|V(G_n)|$. With this identification, the space $Y = \prod_{n\in\omega} G_n$ is a compact subspace of ω^ω. Let $\pi : V(G) \to Y$ be defined by $\pi(v) = (\pi_n(v))_{n\in\omega}$. This map is a homeomorphism onto its image since the P_n separate the vertices of G. Let $X = \pi[V(G)]$. On X we define a coloring c as follows:

Given two distinct vertices v and w of G let $c(\pi(v),\pi(w)) = 1$ if $\{v,w\} \in E(G)$, and $c(\pi(v),\pi(w)) = 0$ otherwise. Now given two distinct points x and y of X, let $v = \pi^{-1}(x)$ and $w = \pi^{-1}(y)$. $\Delta(x,y)$ is the minimal $n \in \omega$ with $\pi_n(v) \neq \pi_n(w)$ Since $\pi_n : G \to G_n$ is modular, $\{v,w\} \in E(G)$ iff $\{\pi_n(v),\pi_n(w)\} \in E(G_n)$. It follows that the color $c(x,y)$ is already determined by $x\restriction\Delta(x,y)+1$ and $y\restriction\Delta(x,y)+1$. This shows that c is an almost node coloring on a compact subspace on ω^ω. ∎

It is worth pointing out that not every continuous coloring on a compact metric space is isomorphic to an almost node coloring. Namely, for $\{x,y\} \in [2^\omega]^2$ let

$$c(x,y) = \begin{cases} 0 & \text{if } x(\Delta(x,y)+1) \neq y(\Delta(x,y)+1), \\ 1 & \text{if } x(\Delta(x,y)+1) = y(\Delta(x,y)+1). \end{cases}$$

A graph is *prime* if it has no nontrivial modules.

Lemma 3.10. *Let c be the coloring defined above. Then the corresponding graph G_c is prime. In particular, G_c has no nontrivial finite modular partition and is therefore not modular profinite.*

Proof. Let $M \subseteq 2^\omega$ be a module of G_c. Let $x_0, x_1 \in M$ and $y \in 2^\omega$. We say that x_0, x_1, y are in *critical configuration* if

$$\Delta(x_0,y) = \Delta(x_1,y) = \Delta(x_0,x_1) - 1.$$

Claim 3.11. *If x_0, x_1, y are in critical configuration, then $y \in M$.*

Let $n = \Delta(x_0, x_1)$. Without loss of generality we may assume that $x_0(n) = y(n)$ and $x_1(n) \neq y(n)$. Now $c(x_0, y) \neq c(x_1, y)$ and hence x_0 and y form an edge in G_c iff x_1 and y do not. Since M is a module, it follows that $y \in M$. This proves Claim 3.11.

Claim 3.12. *Let $n \in \omega$, $f : n \to 2$, and suppose there are two distinct vertices $x_0, x_1 \in M$ such that $f \subseteq x_0, x_1$. Then there are $y_0, y_1 \in M$ such that $y_0 \wedge y_1 = f$.*

Suppose the pair $\{x_0, x_1\} \in [M^2]$ is chosen so that $f \subseteq x_0, x_1$ and $\Delta(x_0, x_1)$ is minimal. We show that $\Delta(x_0, x_1) = n$.

If not, let $m = \Delta(x_0, x_1) - 1$. Since $\Delta(x_0, x_1) > n$, $m \geq n$. Let $y \in 2^\omega$ be such that $\Delta(x_0, y) = \Delta(x_1, y) = m$. Now $x_0(n) = y(n)$ iff $x_1(x) \neq y(n)$. It follows that x_0 and y form an edge in G_c iff x_1 and y do not. Since M is a module, this implies $y \in M$.

But now $f \subseteq x_0, y$ and $\Delta(x_0, y) \geq n$, contradicting the minimality of $\Delta(x_0, x_1)$. Hence $\Delta(x_0, x_1) = n$. This proves Claim 3.12.

Now suppose that M has two distinct elements x_0, x_1 with $\Delta(x_0, x_1) > 0$. Let $f = \{(0, x_0(0))\}$. By Claim 3.12, there are distinct $y_0, y_1 \in M$ such that $y_0 \wedge y_1 = f$. Let $y \in 2^\omega$ be such that $y(0) \neq f(0)$. The three vertices y_0, y_1, y are of critical configuration, and hence, by Claim 3.11, $y \in M$. In other words,

$$\{y \in 2^\omega : y(0) \neq f(0)\} \subseteq M.$$

In particular, there are distinct $z_0, z_1 \in M$ such that $\Delta(z_0, z_1) = 1$ and $z_0(0) = z_1(0) \neq f(0)$. For all $z \in 2^\omega$ with $z(0) = f(0)$, z_0, z_1, z are of critical configuration, and hence $z \in M$. This implies that $M = 2^\omega$.

It follows that if $M \neq 2^\omega$, then M contains at most two distinct elements and moreover, if $M \neq 2^\omega$ and M contains two distinct elements x_0, x_1, then $\Delta(x_0, x_1) = 0$.

Now suppose that $M = \{x_0, x_1\}$ and $\Delta(x_0, x_1) = 0$. We distinguish two cases. If $x_0(1) \neq x_1(1)$, let $y \in 2^\omega$ be such that $y(0) = x_0(0)$, $y(1) = x_1(1)$, and $y(2) \neq x_0(2)$. In this case $\Delta(x_0, y) = 1$, $c(x_0, y) = 0$, $\Delta(x_1, y) = 0$, and $c(x_1, y) = 1$. Since M is a module, $y \in M$, contradicting the assumption that $M = \{x_0, x_1\}$. If $x_0(1) = x_1(1)$, let $y \in 2^\omega$ be such that $y(0) = x_0(0)$, $y(1) \neq x_1(1)$, and $y(2) = x_0(2)$. In this case $\Delta(x_0, y) = 1$, $c(x_0, y) = 1$, $\Delta(x_1, y) = 0$, and $c(x_1, y) = 0$. Again, since M is a module, $y \in M$, a contradiction.

It follows that G_c does not have any modules of size 2, showing that G_c is a prime graph. ■

Let us observe another interesting property of the coloring c defined above. The graph G_c contains an infinite path, which no modular profinite graph does.

EXAMPLE 3.13. For each $n \in \omega$ let $v_n \in 2^\omega$ be the sequence of 1's of length n followed by ω-many 0's. Then for all $n, m \in \omega$ with $n < m$, $c(v_n, v_m) = 1$ iff $m = n + 1$. In other words, $\{v_n : n \in \omega\}$ is the set of vertices of an infinite induced path in G_c.

LEMMA 3.14. *If G is a modular profinite graph, then G does not contain an infinite induced path.*

Proof. It is well-known that P_4, the path on four vertices, is prime. A simple induction shows that every path of length at least 4 and also every infinite path is prime.

If G is modular profinite, then the finite modular partitions separate the vertices of G. If P is an induced path in G of length at least 4, then there is some finite modular partition Q of G that separates at least two distinct vertices of P. But since P is prime, either it is completely contained in a single module of the partition Q, which does not happen since Q separates two distinct vertices of P, or each $M \in Q$ contains at most one vertex of P. Since Q is finite, it follows that P is finite. ∎

4. The finite induced subgraphs of a clopen graph. Sushanskiĭ [22] showed that there is a universal profinite group of countable weight. The analog is true for modular profinite graphs.

The continuous coloring corresponding to this universal modular profinite graph figured prominently in [9] and is called $c_{\max}$ in that article. We now put the construction of $c_{\max}$ and the corresponding graph into a more general framework. Clopen graphs, and hence continuous colorings, are studied in terms of their finite induced subgraphs.

DEFINITION 4.1. For any graph G let $\operatorname{age}(G)$ denote the class of finite graphs isomorphic to an induced subgraph of G. If $c : [X]^2 \to 2$ is a coloring, let $\operatorname{age}(c) = \operatorname{age}(X, c^{-1}(1))$. If X is a topological space and c a coloring on X, let the *hereditary age* of c be the class

$$\operatorname{hage}(c) = \bigcap \{\operatorname{age}(c \restriction O) : O \text{ is an nonempty open subset of } X\}.$$

The crucial observation in the proof of the existence of a universal modular profinite graph of countable weight is the following criterion for the embeddability of an almost node coloring into a continuous coloring.

LEMMA 4.2. *Let c be an almost node coloring on a compact subspace X of ω^ω. For each $t \in T(X)$ fix a coloring $c_t : [\operatorname{succ}_{T(X)}(t)]^2 \to 2$ witnessing the fact that c is an almost node coloring as in Definition* 3.8, *i.e., such that for all $\{x, y\} \in [X]^2$ we have*

$$c(x, y) = c_{x \wedge y}(x \restriction (\Delta(x, y) + 1), y \restriction (\Delta(x, y) + 1)).$$

If d is a continuous coloring on a Polish space Y with $G_{c_t} \in \mathrm{hage}(d)$ for all $t \in T(X)$, then $d \leq c$.

Proof. Fix a complete metric that induces the topology of the Polish space Y. We construct a scheme $(U_t)_{t\in T(X)}$ of nonempty open subsets of Y along with a family $(y_t)_{t\in T(X)}$ of points of Y such that for all $t \in T(X)$ the following hold:

(1) $y_t \in U_t$.
(2) If $t \in \omega^n$, then $\mathrm{diam}(U_t) < 2^{-n}$.
(3) For all $t_0 \in \mathrm{succ}_{T(X)}(t)$, $\mathrm{cl}(U_{t_0}) \subseteq U_t$.
(4) For all $t_0, t_1 \in \mathrm{succ}_{T(X)}(t)$ with $t_0 \neq t_1$, $\mathrm{cl}(U_{t_0}) \cap \mathrm{cl}(U_{t_1}) = \emptyset$.
(5) For all $t_0, t_1 \in \mathrm{succ}_{T(X)}(t)$ with $t_0 \neq t_1$, all $u_0 \in U_{t_0}$, and all $u_1 \in U_{t_1}$, $c_t(t_0, t_1) = d(u_0, u_1)$.

We start the recursive construction of $(U_t)_{t\in T(X)}$ and $(y_t)_{t\in T(X)}$ by choosing a nonempty open set $U_\emptyset \subseteq Y$ of diameter < 1 and a point $y_\emptyset \in U_\emptyset$. Suppose y_t and U_t have been chosen for some $t \in T(X)$. Let $(t_i)_{i<k}$ be a 1-1 enumeration of $\mathrm{succ}_{T(X)}(t)$.

Since $G_{c_t} \subseteq \mathrm{hage}(d)$, there are points y_{t_i}, $i < k$, in U_t such that the map sending each t_i to the corresponding y_{t_i} is an isomorphism between the graph G_{c_t} and the induced subgraph of G_d on the vertices y_{t_i}, $i < k$.

Since G is clopen, there are open neighborhoods U_{t_i}, $i < k$, of the points y_{t_i} such that for all $i, j < k$ with $i \neq j$, all $u_i \in U_{t_i}$, and all $u_j \in U_{t_j}$, $c(u_i, u_j) = c(y_{t_i}, y_{t_j})$. Choosing the U_{t_i} small enough, we can satisfy (2)–(4) for the given t. This finishes the recursive construction.

For each $x \in X$ the sequence $(y_{x\restriction n})_{n\in\omega}$ is Cauchy by (1) and (2) and therefore has a limit in Y. Let $e(x) = \lim_{n\to\infty} y_{x\restriction n}$. The limit $\lim_{n\to\infty} y_{x\restriction n}$ is the unique element of $\bigcap_{n\in\omega} U_{x\restriction n} = \bigcap_{n\in\omega} \mathrm{cl}(U_{x\restriction n})$. Condition (5) implies that $e : X \to Y$ witnesses $c \leq c_G$. ∎

This lemma allows us to characterize embeddability between certain kinds of clopen graphs on Polish spaces.

DEFINITION 4.3. A graph G on a topological space $V(G)$ is *self-similar* if $\mathrm{age}(G) = \mathrm{hage}(G)$.

COROLLARY 4.4. *If G is a modular profinite graph of countable weight and F is a self-similar clopen graph on a Polish space, then G embeds into F topologically iff $\mathrm{age}(G) \subseteq \mathrm{age}(F)$. In particular, G embeds into F combinatorially iff G embeds into F topologically.*

Let us have a look the properties of classes of finite graphs of the form $\mathrm{hage}(c)$.

DEFINITION 4.5. Let G and H be graphs and let v be a vertex of G. Let $F = G \otimes_v H$ be the graph obtained by replacing the vertex v with a copy of

H that is disjoint from G and connecting a vertex u in the copy of H with a vertex $w \in V(G) \setminus \{v\}$ by an edge iff $\{u, v\} \in E(G)$. Following [19], we call this operation *substitution of H for v in G*.

If $f : G \to H$ is modular, then $\{f^{-1}(v) : v \in V(H)\}$ is a modular partition of G. Furthermore, G is isomorphic to the graph obtained by substituting, for every vertex $v \in V(H)$, the induced subgraph of G on the set $f^{-1}(v)$.

LEMMA 4.6. *Let X be a topological space and let c be a continuous coloring on X. Then* $\mathrm{hage}(c)$ *is closed under substitution.*

Proof. Let $F \in \mathrm{hage}(c)$. We may assume that $V(F) \subseteq X$ and for all distinct $w_0, w_1 \in V(F)$, $\{w_0, w_1\} \in E(F)$ iff $c(w_0, w_1) = 1$. Let $v \in V(F)$ and $H \in \mathrm{hage}(c)$. For all $w \in V(F)$ fix pairwise disjoint open neighborhoods U_w such that for all distinct $w_0, w_1 \in V(F)$, all $u_0 \in U_{w_0}$, and all $u_1 \in U_{w_1}$, $c(u_0, u_1) = c(w_0, w_1)$. Since $H \in \mathrm{hage}(c)$, the graph $G_{c\restriction U_v}$ contains an induced copy of H. For simplicity assume that H is actually an induced subgraph of $G_{c\restriction U_v}$. Now the induced subgraph of G_c on the vertex set $(V(F) \setminus \{v\}) \cup V(H)$ is isomorphic to $F \otimes_v H$. ∎

DEFINITION 4.7. For a class $\mathcal{C}$ of finite graphs let $\mathrm{cl}(\mathcal{C})$ denote the closure of $\mathcal{C}$ under isomorphic copies, induced subgraphs, and substitution. A class $\mathcal{C}$ of finite graphs is *generated* by a set $\mathcal{G}$ of finite graphs if $\mathcal{C} = \mathrm{cl}(\mathcal{G})$.

It turns out that all classes of the form $\mathrm{cl}(\mathcal{G})$, $\mathcal{G}$ a nonempty set of finite graphs, are the hereditary age of some continuous coloring.

DEFINITION 4.8. Let $\overline{G} = (G_n)_{n\in\omega}$ be a sequence of (nonempty) finite graphs. We identify each G_n with a graph whose set of vertices is a natural number. We define a coloring $c_{\overline{G}}$ on the space $X = \prod_{n\in\omega} V(G_n)$ as follows:

Given two distinct points $x, y \in X$ and $n = \Delta(x, y)$, let

$$c_{\overline{G}}(x, y) = \begin{cases} 0 & \text{if } \{x(n), y(n)\} \notin E(G_n), \\ 1 & \text{if } \{x(n), y(n)\} \in E(G_n). \end{cases}$$

LEMMA 4.9. *For any sequence $\overline{G}$ of finite graphs the coloring $c_{\overline{G}}$ is an almost node coloring.*

LEMMA 4.10. *Let $\mathcal{G}$ be a nonempty class of finite graphs. Let $\overline{G} = (G_n)_{n\in\omega}$ be a sequence of finite graphs in $\mathcal{G}$ such that for all $G \in \mathcal{G}$ there are infinitely many $n \in \omega$ with $G_n \cong G$. Then* $\mathrm{age}(c_{\overline{G}}) = \mathrm{hage}(c_{\overline{G}}) = \mathrm{cl}(\mathcal{G})$.

Proof. Clearly, $\mathcal{G} \subseteq \mathrm{hage}(c_{\overline{G}})$. By Lemma 4.6, $\mathrm{hage}(c_{\overline{G}})$ is closed under substitution and hence $\mathrm{cl}(\mathcal{G}) \subseteq \mathrm{hage}(c_{\overline{G}}) \subseteq \mathrm{age}(c_{\overline{G}})$.

On the other hand, if F is a finite induced subgraph of the graph corresponding to $c_{\overline{G}}$, then there is some $n \in \omega$ such that the finite sequences $v\restriction n$, $v \in V(F)$, are pairwise distinct. It is clear that F is isomorphic to an induced subgraph of a graph that is obtained by iterated substitution of the graphs G_m, $m \leq n$. Hence $F \in \mathrm{cl}(\mathcal{G})$. It follows that $\mathrm{age}(c_{\overline{G}}) \subseteq \mathrm{cl}(\mathcal{G})$. ∎

Observe that by Corollary 4.4, for any two sequences $\overline{F} = (F_n)_{n\in\omega}$ and $\overline{G} = (G_n)_{n\in\omega}$ of graphs from $\mathcal{G}$ with each isomorphism type occurring infinitely often, $c_{\overline{F}} \leq c_{\overline{G}} \leq c_{\overline{F}}$. In other words, for each nonempty class $\mathcal{G}$ there is an almost node coloring $c_{\mathcal{G}}$ such that $\mathrm{hage}(c_{\mathcal{G}}) = \mathrm{age}(c_{\mathcal{G}}) = \mathrm{cl}(\mathcal{G})$ and $c_{\mathcal{G}}$ is unique up to bi-embeddability.

Corollary 4.11. *There is a universal almost node coloring $c_{\max}$ on a compact subspace of ω^ω, i.e., if $X \subseteq \omega^\omega$ is compact and c is an almost node coloring on X, then $c \leq c_{\max}$.*

In the language of graphs, there is a universal, modular profinite graph $G_{\max}$ of countable weight, i.e., every modular profinite graph of countable weight is isomorphic to an induced subgraph of $G_{\max}$.

Proof. Let $\mathcal{C}$ be the class of all finite graphs. By Lemma 4.10, there is an almost node coloring $c_{\max}$ on a compact subspace of ω^ω such that $\mathrm{hage}(c_{\max}) = \mathcal{C}$. Now by Lemma 4.2, $c_{\max}$ is a universal almost node coloring on a compact subset of ω^ω. ∎

The coloring $c_{\min}$ was introduced in [11] and it was shown to be $\leq$-minimal among all continuous colorings on Polish spaces that have an uncountable homogeneity number. We reprove this result using the theory developed so far.

Definition 4.12. For $\{x, y\} \in [\omega^\omega]^2$ let $c_{\mathrm{parity}}(x, y) = \Delta(x, y) \bmod 2$. Let $c_{\min} = c_{\mathrm{parity}} \restriction 2^\omega$.

It was shown in [9] that actually $c_{\mathrm{parity}} \leq c_{\min}$. This implies that $\mathfrak{hm}(c_{\min}) = \mathfrak{hm}(c_{\mathrm{parity}})$.

In our context, $c_{\min}$ can be described as follows: Let $\mathcal{G}$ be the class of graphs that consists of the complete graph on two vertices, the *edge*, and the discrete graph on two vertices, the *non-edge*. Then $c_{\min} = c_{\mathcal{G}}$.

Since no open set is $c_{\min}$-homogeneous and the closure of a $c_{\min}$-homogeneous set is again homogeneous, all $c_{\min}$-homogeneous sets are nowhere dense. Now by the Baire category theorem, $\mathfrak{hm}(c_{\min}) > \aleph_0$.

On the other hand, if c is a continuous coloring on a Polish space X with $\mathfrak{hm}(c) > \aleph_0$, we can iteratively remove open c-homogeneous sets from X, obtaining a nonempty Polish space $Y \subseteq X$ with the property that no open subset of Y is c-homogeneous. Hence $\mathcal{G} \subseteq \mathrm{hage}(c \restriction Y)$. Now by Lemma 4.2, $c_{\min} \leq c$. This shows

Corollary 4.13. *For every continuous coloring c on a Polish space, $\mathfrak{hm}(c) > \aleph_0$ iff $c_{\min} \leq c$.*

Because of the significance of $\mathfrak{hm}(c_{\min})$ as the smallest uncountable homogeneity number of a continuous coloring on a Polish space, we write just $\mathfrak{hm}$ for $\mathfrak{hm}(c_{\min})$.

By Lemma 4.10, $\operatorname{age}(c_{\min}) = \operatorname{hage}(c_{\min}) = \operatorname{cl}(\mathcal{G})$. It is well-known that $\operatorname{cl}(\mathcal{G})$ is the class of finite P_4-free graphs, i.e., the class of finite graphs that do not contain an induced path of length 4 (see [1]). That $\operatorname{age}(c_{\min})$ is exactly the class of finite P_4-free graphs was already pointed out in [8] and this fact is one of the motivations for the theory presented here.

The following theorem was proved in [9], but was not stated explicitly in that paper. An explicit statement and survey of the proof can be found in [8]. The theorem clarifies why we are interested in almost node colorings.

THEOREM 4.14. *For every continuous coloring $c : [X]^2 \to 2$ on a Polish space X, X is the union of $\mathfrak{hm}$-many sets $Y \subseteq X$ such that $c{\restriction}Y$ is isomorphic to an almost node coloring on a compact subset of ω^ω. If $\mathfrak{hm}(c)$ is uncountable, then $\mathfrak{hm}(c) = \mathfrak{hm}(d)$ for some almost node coloring d on a compact subset of ω^ω such that $d \leq c$.*

COROLLARY 4.15. *Let c be a continuous coloring on a Polish space. Let $\mathcal{G}$ be a class of finite graphs such that* $\operatorname{age}(c) \subseteq \operatorname{cl}(\mathcal{G})$. *Then $\mathfrak{hm}(c) \leq \mathfrak{hm}(c_{\mathcal{G}})$.*

Proof. By Theorem 4.14, there is an almost node coloring d on a compact subset of ω^ω such that $d \leq c$ and $\mathfrak{hm}(d) = \mathfrak{hm}(c)$. Since $d \leq c$, $\operatorname{age}(d) \subseteq \operatorname{age}(c)$. By Lemma 4.10, $\operatorname{hage}(c_{\mathcal{G}}) = \operatorname{cl}(\mathcal{G})$. Hence $\operatorname{age}(d) \subseteq \operatorname{hage}(c_{\mathcal{G}})$. By Lemma 4.2, $d \leq c_{\mathcal{G}}$ and thus, in particular, $\mathfrak{hm}(c) = \mathfrak{hm}(d) \leq \mathfrak{hm}(c_{\mathcal{G}})$. ∎

The following was shown in [9]:

COROLLARY 4.16. *For every continuous coloring c on a Polish space, $\mathfrak{hm}(c) \leq \mathfrak{hm}(c_{\max})$.*

Proof. By Corollary 4.15, $\mathfrak{hm}(c) \leq \mathfrak{hm}(c_{\operatorname{age}(c)})$. Since $c_{\max}$ is a universal almost node coloring on a compact subset of ω^ω, $c_{\operatorname{age}(c)} \leq c_{\max}$. It follows that $\mathfrak{hm}(c) \leq \mathfrak{hm}(c_{\max})$. ∎

We finish this section by pointing out that the embeddability ordering on modular profinite graphs of countable weight is complicated. Recall from Corollary 4.4 that for self-similar, modular profinite graphs F and G of countable weight, F embeds into G iff $\operatorname{age}(G) \subseteq \operatorname{age}(F)$.

Since there are only countably many isomorphism types of finite graphs, this observation shows that embeddability between self-similar, modular profinite graphs is not more complicated than the subset relation between subsets of ω. The next result shows that the embeddability relation between self-similar, modular profinite graphs is also not less complicated than the subset relation between subsets of ω.

Definition 4.17. For $A \subseteq \omega \setminus 5$ let

$$\mathcal{C}(A) = \mathrm{cl}(\{C_n : n \in A\}),$$

where C_n is the cycle of length n. Let $G(A) = G_{\mathcal{C}(A)}$.

Lemma 4.18. *For $A, B \subseteq \omega \setminus 5$, $G(A)$ embeds into $G(B)$ iff $A \subseteq B$.*

Proof. It follows from Corollary 4.4 that $G(A)$ embeds into $G(B)$ if $A \subseteq B$. On the other hand, if $G(A)$ embeds into $G(B)$, then

$$\mathcal{C}(A) = \mathrm{age}(G(A)) \subseteq \mathcal{C}(B) = \mathrm{age}(G(B)).$$

Claim 4.19. *Suppose $n \in A \setminus B$. Then $G(B)$ has no induced cycle of length n and therefore $G(A)$ does not embed into $G(B)$.*

It is clear that the lemma follows from the claim. To show the claim, observe that every cycle of length at least 5 is prime, i.e., has no nontrivial modules. If G is a finite induced subgraph of $G(B)$, then it is isomorphic to an induced subgraph of a graph that is obtained by iterated substitution from cycles of length different from n. It follows that every prime subgraph of G is a subgraph of a cycle of length different from n, i.e., it is itself a cycle of length different from n or a path. This proves that G is not isomorphic to C_n, finishing the proof of the claim. ∎

Lemma 4.18 shows for example that there is an uncountable family of pairwise nonembeddable, self-similar, modular profinite graphs of countable weight.

5. Colorings of finite depth

Definition 5.1. Let $X \subseteq \omega^\omega$ be a closed set. Given $n \in \omega$, a continuous coloring $c : [X]^2 \to k$ is of *depth n* if for all $\{x, y\} \in [X]^2$, $c(x, y)$ only depends on $x \restriction (\Delta(x, y) + n)$ and $y \restriction (\Delta(x, y) + n)$. A graph $G = (X, E)$ is of *depth n* if the corresponding coloring is.

A coloring $c : [X]^2 \to k$ is an almost node coloring iff it is of depth 1, and a node coloring iff it is of depth 0.

Lemma 5.2. *Let $X \subseteq [\omega^\omega]^2$ be a closed set and let $c : [X]^2 \to 2$ be a node coloring. If $\mathfrak{hm}(c)$ is uncountable, then $\mathfrak{hm}(c) = \mathfrak{hm}$.*

Proof. Every node coloring is continuous. Hence, if $\mathfrak{hm}(c)$ is uncountable, then, by Corollary 4.13, $\mathfrak{hm} \leq \mathfrak{hm}(c)$.

Let $\overline{c} : T(X) \to 2$ witness the fact that c is a node coloring. We define a graph $G = (T(X), E)$ as follows: for any two distinct elements s and t of $T(X)$ let $\{s, t\} \in E$ if one of the two is an immediate successor of the other in $T(X)$ and, moreover, $\overline{c}(s) = \overline{c}(t)$.

By the definition of E, $\overline{c}$ is constant on every connected component of G.

CLAIM 5.3. *Let $C \subseteq T(X)$ be a connected component of G. Then C has a smallest element with respect to $\subseteq$.*

Since $\subseteq$ is well-founded on $T(X)$, it is enough to show that any two elements of C have a common lower bound in C. Let $s, t \in C$ be distinct. Let

$$P = \{r \in T(X) : s \wedge t \subseteq r \text{ and } (r \subseteq s \text{ or } r \subseteq t)\}.$$

Since $T(X)$ is a tree and two elements of $T(X)$ form an edge of G only if one is an immediate successor of the other in the tree-order, every path connecting s and t has to contain all the vertices in P. It follows that $P \subseteq C$ and thus $s \wedge t \in C$. This proves the claim.

Let T_c be the subset of $T(X)$ consisting of the minimal elements of all connected components of G. Then T_c is isomorphic to a subtree of $\omega^{<\omega}$. The map $\overline{c}\restriction T_c$ induces a node coloring d on the space $[T_c]$ of infinite branches of T_c. Every branch b of T_c generates a branch of $T(X)$ and thus corresponds to an element $x(b)$ of X.

If $x \in X$ is not of the form $x(b)$, then there are some $n_0 \in \omega$ and a connected component C of G such that for all $n \geq n_0$, $x\restriction n \in C$. We say that x is *eventually in* C. Given a connected component C of G, the set

$$\{x \in X : \exists n_0\ \forall n \geq n_0\ (x\restriction n \in C)\}$$

of all $x \in X$ that are eventually in C is a closed c-homogeneous subset of X.

If $H \subseteq [T_c]$ is d-homogeneous, then $\{x(b) : b \in H\}$ is a c-homogeneous subset of X. Since G has at most countably many connected components and each $x \in X$ is either of the form $x(b)$ for some $b \in [T_c]$ or eventually in some connected component of G, $\mathfrak{hm}(c) \leq \mathfrak{hm}(d) + \aleph_0$.

If t is an immediate successor of s in T_c, then $\overline{c}(s) = 1 - \overline{c}(t)$. It follows that $([T_c], d)$ is isomorphic to a closed subspace of ω^ω equipped with the coloring c_{parity} as defined in Definition 4.12. Hence $\mathfrak{hm}(c) \leq \mathfrak{hm}(c_{\text{parity}})$. In [9] it was shown that $c_{\text{parity}} \leq c_{\min}$ and thus $\mathfrak{hm}(c_{\text{parity}}) = \mathfrak{hm}$. It follows that $\mathfrak{hm}(c) \leq \mathfrak{hm}$. ∎

LEMMA 5.4. *For all $n > 1$ and all $\{x, y\} \in [\omega^\omega]^2$ let $c_n(x, y) = \Delta(x, y)$ mod n. Then $\mathfrak{hm}(c_n) = \mathfrak{hm}$.*

Proof. It is easily checked that for all $n, m \in \omega$ with $n < m$ there is an embedding $e : \omega^\omega \to \omega^\omega$ such that for all $\{x, y\} \in [\omega^\omega]$, $c_n(x, y) = c_m(e(x), e(y))$. It follows that $\mathfrak{hm}(c_n) \leq \mathfrak{hm}(c_m)$. In particular, for all $n \in \omega$ with $n > 1$, $\mathfrak{hm} \leq \mathfrak{hm}(c_n)$.

To finish the proof of the lemma, it is enough to show that for all $n > 0$, $\mathfrak{hm}(c_{2^n}) \leq \mathfrak{hm}$. We have $c_{\text{parity}} = c_2$ and thus $\mathfrak{hm}(c_2) = \mathfrak{hm}(c_{\text{parity}}) = \mathfrak{hm}$. Now suppose we have already shown that $\mathfrak{hm}(c_{2^n}) \leq \mathfrak{hm}$. If $H \subseteq \omega^\omega$ is

$c_{\rm parity}$-homogeneous of color 0, then

$$f : H \to \omega^\omega : (x_0, x_1, x_2, \ldots) \mapsto (x_0, x_2, x_4, \ldots)$$

is 1-1. If $G \subseteq \omega^\omega$ is c_{2^n}-homogeneous, then $f^{-1}(G)$ is $c_{2^{n+1}}$-homogeneous. It follows that H can be covered by $\mathfrak{hm}(c_{2^n})$ and hence by $\mathfrak{hm}$-many $c_{2^{n+1}}$-homogeneous sets.

Similarly, every $c_{\rm parity}$-homogeneous set of color 1 can be covered by $\mathfrak{hm}$-many $c_{2^{n+1}}$-homogeneous sets. Since ω^ω is the union of $\mathfrak{hm}$-many $c_{\rm parity}$-homogeneous sets, ω^ω can be covered by $\mathfrak{hm} \cdot \mathfrak{hm} = \mathfrak{hm}$ sets that are $c_{2^{n+1}}$-homogeneous. It follows that for all $n > 0$, $\mathfrak{hm}(c_{2^n}) \le \mathfrak{hm}$. This finishes the proof of the lemma. ∎

DEFINITION 5.5. A set $X \subseteq \omega^\omega$ is *n-ary* if for every $t \in T(X)$ we have

$$|\mathrm{succ}_{T(X)}(t)| \le n.$$

THEOREM 5.6. *Let $X \subseteq \omega^\omega$ be n-ary for some $n \in \omega$. If $c : [X]^2 \to 2$ is a coloring of depth n and $\mathfrak{hm}(c)$ is uncountable, then $\mathfrak{hm}(c) = \mathfrak{hm}$.*

Proof. Again it follows from Corollary 4.13 that $\mathfrak{hm}(c)$ is at least $\mathfrak{hm}$, provided $\mathfrak{hm}(c)$ is uncountable. By Lemma 5.4, X is the union of not more than $\mathfrak{hm}$ c_{2n}-homogeneous sets. If $H \subseteq X$ is c_{2n}-homogeneous, then $c \restriction H$ is an almost node coloring. The set H is n-ary. In [10] it was shown that every n-ary subset of ω^ω is the union of not more than $\mathfrak{hm}$ binary sets.

If $B \subseteq H$ is binary, the restricted coloring $c \restriction B$ is actually a node coloring. Hence, by Lemma 5.2, for every binary c_{2n}-homogeneous set $B \subseteq X$, $\mathfrak{hm}(c \restriction B) \le \mathfrak{hm}$. Since X is the union of not more than $\mathfrak{hm}$ such sets, $\mathfrak{hm}(c) \le \mathfrak{hm}$. ∎

We now combine Theorem 5.6 with the results of Section 4.

COROLLARY 5.7. *Let c be a continuous coloring on a Polish space. If there is a finite set $\mathcal{G}$ of finite graphs such that* $\mathrm{age}(c) \subseteq \mathrm{cl}(\mathcal{G})$, *then $\mathfrak{hm}(c) \le \mathfrak{hm}$.*

Proof. By Corollary 4.15 we have $\mathfrak{hm}(c) \le \mathfrak{hm}(c_{\mathcal{G}})$. Since $\mathcal{G}$ is finite, $c_{\mathcal{G}}$ is of finite width. Also $c_{\mathcal{G}}$ is an almost node coloring and hence of depth 1. Now by Theorem 5.6, $\mathfrak{hm}(c) \le \mathfrak{hm}(c_{\mathcal{G}}) \le \mathfrak{hm}$. ∎

6. A universal coloring. We continue to study the embeddability relation between continuous colorings on compact metric spaces. In Section 8 we will show that the class of continuous colorings on a compact metric space does not have a largest element with respect to combinatorial or topological embeddability. However, there is a continuous coloring on ω^ω such that all continuous colorings on a compact, zero-dimensional, metric space embed into it topologically.

Theorem 6.1. *There is a continuous coloring* $c_{\text{universal}} : [\omega^\omega]^2 \to 2$ *such that for every continuous coloring c on a compact, zero-dimensional space X of countable weight there is a topological embedding* $e : X \to \omega^\omega$ *such that for all* $\{x, y\} \in [X]^2$, $c(x, y) = c_{\text{universal}}(e(x), e(y))$. *In other words,* $c \leq c_{\text{universal}}$.

Before giving the proof of this theorem, let us derive a corollary from it. Recall that $\mathfrak{d}$ is the least size of a family of compact sets that covers ω^ω. It is well-known that $\mathfrak{d}$ is also the least size of a family $\mathcal{C}$ of compact sets such that for every compact set $K \subseteq \omega^\omega$, there is $C \in \mathcal{C}$ such that $C \subseteq K$. It is consistent that $\mathfrak{d}$ is strictly smaller than $2^{\aleph_0}$.

Corollary 6.2. *There is a family* $\mathcal{D}$ *of size* $\mathfrak{d}$ *of continuous colorings on* 2^ω *such that for every continuous coloring c on a separable, zero-dimensional, compact space X there is* $d \in \mathcal{D}$ *such that* $c \leq d$.

Proof. Let $c_{\text{universal}}$ be as in Theorem 6.1. Let $\mathcal{C}$ be a family of size $\mathfrak{d}$ of compact subsets of ω^ω such that every compact subset of ω^ω is contained in some member of $\mathcal{C}$. Let $\mathcal{D} = \{c_{\text{universal}} \restriction C : C \in \mathcal{C}\}$.

If c is a continuous coloring on a compact, zero-dimensional space X of countable weight, then $c \leq c_{\text{universal}}$. Let $e : X \to \omega^\omega$ be a topological embedding witnessing this. By the continuity of e, $e[X]$ is compact and hence there is $C \in \mathcal{C}$ such that $e[X] \subseteq C$. Now, clearly, $c \leq c_{\text{universal}} \restriction C$ and $c_{\text{universal}} \restriction C \in D$. ∎

We now turn to the proof of Theorem 6.1, which requires a couple of lemmas. One crucial ingredient, Lemma 6.6, was essentially shown in [9]. But since the lemma is not explicitly stated in that paper, we provide a proof. We use this opportunity to extract some additional information from the argument.

Let us first define a property of continuous colorings that is weaker than being of depth n for some $n \in \omega$.

Definition 6.3. A continuous coloring $c : [X]^2 \to 2$ on a closed subset of ω^ω is *uniformly continuous* if there is a function $f : \omega \to \omega$ such that for all $\{x, y\} \in [X]^2$ the color $c(x, y)$ only depends on $x \restriction f(\Delta(x, y))$ and $y \restriction f(\Delta(x, y))$.

Lemma 6.4. *Let X be a compact subspace of* ω^ω *and let* $c : [X]^2 \to 2$ *be a continuous coloring. Then c is uniformly continuous.*

Proof. Let $T = T(X)$ be the tree of finite initial segments of elements of X. Since X is compact, T is finitely branching. Let $n \in \omega$ and let $s, t \in \omega^{n+1} \cap T$ be such that $s \wedge t \in \omega^n$. Let $[s]$ and $[t]$ denote the basic open subsets of ω^ω determined by s and t, respectively.

Since c is continuous, for all $(x, y) \in ([s] \times [t]) \cap (X \times X)$ there is $m_{x,y} \in \omega$ such that $c(x, y)$ only depends on $x \restriction m_{x,y}$ and $y \restriction m_{x,y}$. Since X is compact, so is $([s] \times [t]) \cap (X \times X)$. It follows that there are $k \in \omega$ and

$$(x_0, y_0), \dots, (x_{k-1}, y_{k-1}) \in ([s] \times [t]) \cap (X \times X)$$

such that for all $(x, y) \in ([s] \times [t]) \cap (X \times X)$ there is $j < k$ such that $x_j \restriction m_{x_j,y_j} \subseteq x$ and $y_j \restriction m_{x_j,y_j} \subseteq y$. Let $m_{s,t} \in \omega$ be such that $n < m_{s,t}$ and for all $j < k$, $m_{x_j,y_j} \leq m_{s,t}$. Now for all $(x, y) \in ([s] \times [t]) \cap (X \times X)$, $c(x, y)$ only depends on $x \restriction m_{s,t}$ and $y \restriction m_{s,t}$.

Since for each $n \in \omega$, $\omega^n \cap T$ is finite, we can define

$$f(n) = \max\{m_{s,t} : s, t \in T \cap \omega^{n+1} \text{ and } s \wedge t \in \omega^n\}$$

and obtain a function such that for all $\{x, y\} \in [X]^2$, $c(x, y)$ only depends on $x \restriction f(\Delta(x, y))$ and $y \restriction f(\Delta(x, y))$. The function f witnesses the uniform continuity of c. ∎

Lemma 6.5. *Let $c : [X]^2 \to 2$ be a uniformly continuous coloring on a closed subset of ω^ω. Then there are a closed set $Y \subseteq \omega^\omega$ and a homeomorphism $h : X \to Y$ such that the coloring $d : [Y]^2 \to 2$ defined by $d(x, y) = c(h^{-1}[\{x, y\}])$ is of depth 2.*

Proof. Let $f : \omega \to \omega$ witness the uniform continuity of c. We can choose f strictly increasing and such that $f(0) \geq 1$. For $n \in \omega$ let $g(n) := f^n(0)$.

Identifying $\omega^{<\omega}$ and ω, we define the required embedding $h : X \to \omega^\omega$ by letting $h(x) := (x \restriction g(0), x \restriction g(1), \dots)$. Let $Y := e[X]$. The coloring c induces a continuous pair-coloring d on Y such that for all $\{x, y\} \in [X]^2$, $d(h(x), h(y)) = c(x, y)$. By the choice of f, for $\{u, v\} \in [Y]^2$, $d(u, v)$ only depends on $u \restriction (\Delta(u, v) + 2)$ and $v \restriction (\Delta(u, v) + 2)$. This can be seen as follows:

If $n = \Delta(u, v)$ and $x, y \in X$ are such that $h(x) = u$ and $h(y) = v$, then $\Delta(x, y) < g(n)$ and $c(x, y)$ only depends on $x \restriction f(\Delta(x, y))$ and $y \restriction f(\Delta(x, y))$. Since f is strictly increasing, $f(\Delta(x, y)) < f(g(n)) = g(n+1)$. It follows that $c(x, y)$ only depends on $x \restriction g(n+1)$ and $y \restriction g(n+1)$. But $u(n+1) = x \restriction g(n+1)$ and $v(n + 1) = y \restriction g(n + 1)$. Hence $d(u, v)$ only depends on $u \restriction n + 2$ and $v \restriction n + 2$. ∎

Lemma 6.6. *Let X be a compact, zero-dimensional space of countable weight and let $c : [X]^2 \to 2$ be continuous. Then there are a compact set $Y \subseteq \omega^\omega$ and a homeomorphism $h : X \to Y$ such that the coloring $d : [Y]^2 \to 2$ defined by $d(x, y) = c(h^{-1}[\{x, y\}])$ is of depth 2.*

Proof. Every compact, zero-dimensional space X of countable weight is homeomorphic to a subspace of 2^ω (see [3, Theorem 6.2.16]). Hence we may assume that X is a compact subspace of ω^ω. By Lemma 6.4, c is uniformly continuous. By Lemma 6.5, there are a closed set $Y \subseteq \omega^\omega$ and a homeomorphism $h : X \to Y$ such that the coloring $d : [Y]^2 \to 2$ defined by $d(x, y) = c(h^{-1}[\{x, y\}])$ is of depth 2. Since Y is homeomorphic to X, Y is compact. ∎

The next lemma is the combinatorial core of the proof of Theorem 6.1. The lemma establishes the existence of something like a two-dimensional analog of the random graph.

LEMMA 6.7. *There is a graph $G^2_{\mathrm{random}} = (\omega^2, E^2_{\mathrm{random}})$ such that for all $n \in \omega$ and all finite disjoint sets $A, B \subseteq (\omega \setminus \{n\}) \times \omega$ there is $m \in \omega$ such that for all $s \in A$, $\{s, (n,m)\} \in E^2_{\mathrm{random}}$, and for all $s \in B$, $\{s, (n,m)\} \notin E^2_{\mathrm{random}}$.*

Proof. Let $\mathbb{P}$ be the collection of all graphs $G = (V(G), E(G))$ such that $V(G)$ is a finite subset of ω^2. Given $G, F \in \mathbb{P}$, we say that G *extends* F ($G \geq F$) if F is an induced subgraph of G. For all $n \in \omega$ and all finite disjoint sets $A, B \subseteq (\omega \setminus \{n\}) \times \omega$ let

$$\begin{aligned} D^n_{A,B} = \{G \in \mathbb{P} : A \cup B \subseteq V(G) \wedge \exists m \in \omega\ \forall s \in V(G) \\ ((s \in A \Rightarrow \{s, (n,m)\} \in E(G)) \wedge (s \in B \Rightarrow \{s, (n,m)\} \notin E(G)))\}. \end{aligned}$$

It is easily checked that for all $n \in \omega$ and all finite disjoint sets $A, B \subseteq (\omega \setminus \{n\}) \times \omega$, every $G \in \mathbb{P}$ has an extension F in $D^n_{A,B}$. Moreover, if a graph G is an element of $D^n_{A,B}$, then so is every extension of G in $\mathbb{P}$.

By recursion using some suitable book-keeping we choose a sequence

$$G_0 \leq G_1 \leq \ldots$$

of graphs in $\mathbb{P}$ such that for all finite disjoint sets $A, B \subseteq \omega^2$ and every $n \in \omega$, there is $k \in \omega$ such that $G_k \in D^n_{A,B}$. Let G^2_{random} be the direct limit of the graphs G_k, $k \in \omega$, i.e., let

$$G^2_{\mathrm{random}} = \Big(\bigcup_{n\in\omega} V(G_n), \bigcup_{n\in\omega} E(G_n)\Big).$$

It is not difficult to verify that this graph has the desired properties. ∎

We are now ready to construct the coloring that witnesses Theorem 6.1.

DEFINITION 6.8. Let $\{x, y\} \in [\omega^\omega]^2$ and $n = \Delta(x, y)$. Let $s, t \in \omega^2$ be such that $x\restriction(n+2) = (x\restriction n)^\frown s$ and $y\restriction(n+2) = (y\restriction n)^\frown t$. Now put

$$c_{\mathrm{universal}}(x, y) = \begin{cases} 0, & \{s, t\} \notin E^2_{\mathrm{random}}, \\ 1, & \{s, t\} \in E^2_{\mathrm{random}}. \end{cases}$$

Clearly, $c_{\mathrm{universal}}$ is a coloring of depth 2. Whenever $s, t \in \omega^{<\omega}$ are such that $\Delta(s, t)$ is defined and $\Delta(s, t) + 1 \in \mathrm{dom}(s) \cap \mathrm{dom}(t)$, let $\bar{c}_{\mathrm{universal}}(s, t)$ be the unique color in 2 of the form $c_{\mathrm{universal}}(x, y)$ where $s \subseteq x$ and $t \subseteq y$.

LEMMA 6.9. *For every continuous coloring $c : [X]^2 \to 2$ of depth 2 on a closed subset of ω^ω, $c \leq c_{\mathrm{universal}}$. In other words, $c_{\mathrm{universal}}$ is universal for continuous colorings of depth 2 on closed subsets of ω^ω.*

Proof. Let X and c be as in the statement of the lemma and let $T = T(X)$. Since c is of depth 2, there is a map $\bar{c}$ from a subset of $[T]^2$ to 2 such that for all $\{x,y\} \in [X]^2$ and all $n \geq 2$, $\bar{c}(x{\restriction}(\Delta(x,y)+n), y{\restriction}(\Delta(x,y)+n))$ is defined and equal to $c(x,y)$.

We define a level preserving, monotone injection $i : T \to \omega^{<\omega}$ that induces a topological embedding from X into ω^ω witnessing $c \leq c_{\mathrm{universal}}$. Here by monotone we mean that if $s \subseteq t$, then $i(s) \subseteq i(t)$.

Since i is supposed to be level preserving, it has to map the empty sequence to the empty sequence. For $i{\restriction}(X \cap \omega^1)$ we choose any 1-1 map into ω^1. Now suppose that for some $n > 1$, $i(s)$ has been defined for all $s \in T \cap \omega^{<n}$. Let $\ell = |T \cap \omega^n| \leq \aleph_0$ and let $(s_k)_{k<\ell}$ be an enumeration of $T \cap \omega^n$.

Suppose for some $k_0 < \ell$, $i(s_k)$ has been chosen for all $k < k_0$. Let

$$A = \{s_k : k < k_0 \wedge \Delta(s_k, s_{k_0}) < n-1 \wedge \bar{c}(s_k, s_{k_0}) = 1\},$$
$$B = \{s_k : k < k_0 \wedge \Delta(s_k, s_{k_0}) < n-1 \wedge \bar{c}(s_k, s_{k_0}) = 0\}.$$

Let $t = i(s_{k_0}{\restriction}(n-1))$. Observe that since i is level preserving, 1-1, and monotone, for all $s \in A \cup B$, $\Delta(i(s), t) < n-1$. By the definition of $c_{\mathrm{universal}}$ and by the properties of G^2_{random}, there is some $m \in \omega$ such that for all $s \in A$, $\bar{c}_{\mathrm{universal}}(i(s_k), t^\frown m) = 1$, and for all $s \in B$, $\bar{c}_{\mathrm{universal}}(i(s_k), t^\frown m) = 0$. Let $i(s_{k_0}) = t^\frown m$.

It is clear that i is level preserving, monotone, and 1-1. Moreover, by the recursive construction, whenever $s, t \in T$ are such that $\bar{c}(s,t)$ is defined, then $\bar{c}_{\mathrm{universal}}(i(s), i(t))$ is defined and equal to $c(s,t)$.

We now define $e : X \to \omega^\omega$ by letting $e(x) = \bigcup\{i(x{\restriction}n) : n \in \omega\}$. Since i is level preserving, 1-1, and monotone, e is well-defined and a homeomorphism onto its image. Let $\{x,y\} \in [X]^2$ and $n = \Delta(x,y)+2$. Then

$$c(x,y) = \bar{c}(x{\restriction}n, y{\restriction}n) = \bar{c}_{\mathrm{universal}}(i(x{\restriction}n), i(y{\restriction}n)) = c_{\mathrm{universal}}(e(x), e(y)).$$

This shows that e witnesses $c \leq c_{\mathrm{universal}}$. ∎

Proof of Theorem 6.1. Let X be a compact, zero-dimensional space of countable weight and let $c : [X]^2 \to 2$ be continuous. By Lemma 6.6, we may assume that X is a compact subset of ω^ω and c is of depth 2. By Lemma 6.9, $c \leq c_{\mathrm{universal}}$. ∎

7. Subgraphs of clopen graphs and saturation. In Lemma 3.14 we observed that the infinite path does not embed into any modular profinite graph. At this point it is not clear whether there is any countable graph that does not embed into a clopen graph on a compact metric space. It turns out that sufficiently saturated infinite graphs do not embed into any clopen graph on a compact metric space.

DEFINITION 7.1. Let $G = (V, E)$ be a graph, $A \subseteq V$. A *type* over A is a function $f : A \to 2$. A vertex $v \in V \setminus A$ *realizes* a type f over A if for all $a \in A$, a and v are connected by an edge in G iff $f(a) = 1$.

Similarly, if $c : [X]^2 \to 2$ is a coloring and $A \subseteq X$, then a point $x \in X$ realizes a type f over A if $x \notin A$ and for all $a \in A$, $c(a, x) = f(a)$.

For a cardinal κ, a graph $G = (V, E)$, respectively a coloring $c : [V]^2 \to 2$, is *κ-saturated* if it is nonempty and every type over every subset A of V of size $< \kappa$ is realized. A graph of size κ is *saturated* if it is κ-saturated.

In the graph corresponding to the coloring $c_{\min}$ introduced in Definition 4.12, every vertex has neighbors and non-neighbors. This shows that this graph is 2-saturated.

LEMMA 7.2. *The graph G in Lemma* 3.10 *is* 3*-saturated.*

Proof. Let $x, y \in 2^\omega$ be distinct and let $n = \Delta(x, y)$. Let $z \in 2^\omega$ extend $y{\restriction}(n+1)$. Now the value of $z(n+1)$ determines whether or not x and z form an edge in G, and the value of $z(n+2)$ determines whether or not y and z form an edge in G. It follows that every type over $\{x, y\}$ is realized in G. ■

It turns out that there are uncountable modular profinite graphs of countable weight that are 3-saturated. The graph corresponding to $c_{\min}$ is actually almost 3-saturated: only if $x, y \in 2^\omega$ have $\Delta(x, y) = 0$, then there are types over $\{x, y\}$ that are not realized.

LEMMA 7.3. *There is a modular profinite graph of countable weight that is* 3*-saturated.*

Proof. By results of Glebskii, Kogan, Liogon'kii, and Talanov [12] and Fagin [4], for every $n \in \omega$ there are finite graphs that are n-saturated, and in fact, in a precise sense, almost all finite graphs are n-saturated. So, let $m \in \omega$ be such that there is a 3-saturated graph on the set m of vertices. Let $c : [m]^2 \to 2$ be the corresponding coloring. Let $d : [m^\omega]^2 \to 2$ be defined by as follows: for distinct $x, y \in m^\omega$ let $n = \Delta(x, y)$ and $d(x, y) = c(x(n), y(n))$.

Clearly, the graph G_d corresponding to d is modular profinite and of countable weight. Whenever $x, y \in m^\omega$ are distinct and f is a type over $\{x, y\}$, then f is realized by some $z \in m^\omega$ with $\Delta(x, y) = \Delta(x, z) = \Delta(y, z)$. The vertex z can be chosen so that $z{\restriction}\Delta(x, y) = x{\restriction}\Delta(x, y)$ and $z(\Delta(x, y))$ realizes the obvious type over $\{x(\Delta(x, y)), y(\Delta(x, y))\}$ with respect to the coloring c. ■

In the following we will be specifically interested in 4-saturated graphs. Note that every 4-saturated graph has more than three elements.

LEMMA 7.4. *Every* 4*-saturated graph is prime.*

Proof. Let $G = (V, E)$ be 4-saturated and let $A \subseteq V$ be a module of G containing at least two distinct vertices x and y. Suppose that there is a vertex $z \in V$ outside A. Since every type over a 3-element set is realized in G, there are $a, b \in V$ such that a is connected to x and z but not to y, and b is connected to y but not to x or z. Since A is a module and a and b

are each connected to only one of x and y, we have $a, b \in A$. But now z is connected to a and not to b. It follows that $z \in A$, a contradiction. ■

For a set $A \subseteq \omega^\omega$ and $n \in \omega$ let $A{\restriction}n$ denote the set $\{a{\restriction}n : a \in A\}$.

LEMMA 7.5. *Let $c : [F]^2 \to 2$ be a coloring of depth 2 on a closed subset of ω^ω. Let $A \subseteq F$ be of size at least 3 and suppose that every type over every 3-element subset of A is realized in F. Let $m = \min\{\Delta(x, y) : \{x, y\} \in [A]^2\}$. Then for any two distinct points $a, b, \in A$, $\Delta(a, b) \leq m + 2$. In particular, $|A{\restriction}(m + 3)| = |A|$.*

Proof. Note that $m = \min\{k \in \omega : |A{\restriction}(k + 1)| > 1\}$. Let $a, b \in A$ be distinct and assume that $\Delta(a, b) > m$. Then there is $z \in A$ such that $\Delta(a, z) = \Delta(b, z) = m$. If $c(a, z) \neq c(b, z)$, then, since c is of depth 2, $a{\restriction}(m + 2) \neq b{\restriction}(m + 2)$.

Now assume that $i = c(a, z) = c(b, z)$. Since every type over $\{a, b, z\}$ is realized in F, there is $x \in F$ such that $c(x, z) = 1 - i$ and $c(a, x) \neq c(b, x)$. Since $c(x, z)$ is different from $c(a, z)$ and since $\Delta(a, z) = m$, $x{\restriction}(m + 2) \neq a{\restriction}(m+2)$. In other words, $\Delta(a, x) \leq m+1$. Since $c(a, x) \neq c(b, x)$, $a{\restriction}(m+3) \neq b{\restriction}(m + 3)$. This implies that $\Delta(a, b) \leq m + 2$. ■

We are now ready to show that infinite 4-saturated graphs such as the random graph, the unique countably infinite saturated graph, do not embed into any clopen graph on a compact metric space.

THEOREM 7.6. *No infinite 4-saturated graph embeds into a clopen graph on a compact metric space.*

Proof. Let $G = (X, E)$ be a clopen graph on a compact metric space. Let $A \subseteq X$ be such that the induced subgraph on A is 4-saturated. We will show that A is finite.

Recall the modular partition $\mathrm{Comp}(X)$ of G into the (topologically) connected components of X in Lemma 3.4. Since the induced subgraph on A is prime by Lemma 7.4, either A is contained in a single connected component of X or no two vertices of A are in the same connected component of X. Since A is 4-saturated, it has both edges and non-edges. Hence, again by Lemma 3.4, no connected component of X contains more than two vertices from A. It follows that A embeds into the quotient $G/\mathrm{Comp}(X)$. The underlying space $\mathrm{Comp}(X)$ of $G/\mathrm{Comp}(X)$ is compact and zero-dimensional.

Since X is of countable weight, so is $\mathrm{Comp}(X)$. Hence $\mathrm{Comp}(X)$ embeds into 2^ω. By Lemma 6.6, $G/\mathrm{Comp}(X)$ is isomorphic to a clopen graph of depth 2 on a compact subset of ω^ω. Hence we can assume that X is a compact subset of ω^ω and G is of depth 2.

Now let F be the closure of A in ω^ω. Let

$$m = \min\{\Delta(x, y) : \{x, y\} \in [A]^2\}.$$

By Lemma 7.5, $|A\restriction(m+3)| = |A|$. But since F is compact, for all $n \in \omega$, $\{a\restriction n : a \in F\}$ is finite. It follows that $A\restriction(n+3)$ and hence also A are finite. ∎

From Lemma 7.5 it follows immediately that no clopen graph of depth 2 on a closed subset of ω^ω has an uncountable induced subgraph that is 4-saturated. By Lemma 6.5, every uniformly continuous coloring on a closed subset of ω^ω is isomorphic to a coloring of depth 2. It follows that no graph G on a closed subset of ω^ω with c_G uniformly continuous has an uncountable 4-saturated induced subgraph.

However, there are clopen graphs on ω^ω that are even $\aleph_0$-saturated. This together with the previous remark shows that there are clopen graphs on ω^ω that do not embed into any clopen graph whose corresponding coloring is uniformly continuous. This is stated in Corollary 7.8 below. In particular, $c_{\text{universal}}$ is not universal for continuous colorings on ω^ω.

Let us call a graph G on a topological space X *locally κ-saturated* if for every open set $O \subseteq X$ and every $x \in O$ there is an open set U such that $x \in U \subseteq O$ and the induced subgraph of G on U is κ-saturated.

Lemma 7.7.

(a) *There is a clopen graph on ω^ω that is $\aleph_0$-saturated and has chromatic and cochromatic number $\aleph_0$.*
(b) *There is a clopen graph on ω^ω that is locally $\aleph_0$-saturated. The cochromatic number of this graph is $\mathfrak{hm}(c_{\max})$.*

Proof. (a) Let $\mathcal{S}$ be the collection of all finite sets $S \subseteq \omega^{<\omega}$ with all $s \in S$ of the same length > 0. Let $\mathcal{F}$ be the collection of all functions f with values in 2 and $\mathrm{dom}(f) \in \mathcal{S}$. Let $(f_n)_{n\in\omega\setminus\{0\}}$ be an enumeration of $\mathcal{F}$ such that for all $n \in \omega\setminus\{0\}$ and all $s \in \mathrm{dom}(f_n)$, $s(0) \le n$. For each $n \in \omega\setminus\{0\}$ let m_n denote the common length of all $s \in \mathrm{dom}(f_n)$.

We now define the set E of edges of the clopen graph G as follows: E contains no edge $\{x, y\}$ with $x(0) = y(0)$. For $\{x, y\} \in [\omega^\omega]^2$ with $x(0) \neq y(0)$ assume without loss of generality that $x(0) < y(0)$ and let $\{x, y\} \in E$ iff for $n = y(0)$ we have $x\restriction m_n \in \mathrm{dom}(f_n)$ and $f_n(x\restriction m_n) = 1$.

It is immediate from the definition of E that $G = (\omega^\omega, E)$ is clopen. Now, whenever S is a finite subset of ω^ω and $f : S \to 2$ is a type over S, then there is some m such that $|S\restriction m| = |S|$. Moreover, there is $n \in \omega\setminus\{0\}$ such that $m_n = m$, $\mathrm{dom}(f_n) = S$, and for all $x \in S$, $f_n(x\restriction m) = f(x)$. Any $y \in \omega^\omega$ with $y(0) = n$ realizes the type f over S. This shows that G is $\aleph_0$-saturated.

The sets $B_n = \{x \in \omega^\omega : x(0) = n\}$, $n \in \omega$, are independent and cover all of ω^ω. Therefore the chromatic number of G is countable. Since G is $\aleph_0$-saturated, G does not have a finite cochromatic number. It follows that both the chromatic and the cochromatic number of G are $\aleph_0$.

(b) Let G be the clopen graph constructed in the proof of (a). Recall that G has no edge $\{x, y\}$ with $\Delta(x, y) > 0$. We will add more edges to G, obtaining an enlarged set E^* of edges. The graph (ω^ω, E^*) will then witness (b).

For each $s \in \omega^{<\omega}$, the basic open subset $[s]$ of ω^ω of all functions extending s is homeomorphic to ω^ω by the homeomorphism $h_s : [s] \to \omega^\omega$ that maps $x \in [s]$ to the function $h_s(x) : \omega \to \omega$ defined by $h_s(x)(n) = x(n + |s|)$. For $\{x, y\} \in [\omega^\omega]^2$ let $\{x, y\} \in E^*$ iff for the longest common initial segment s of x and y we have $\{h_s(x), h_s(y)\} \in E$.

It is easily checked that $G^* = (\omega^\omega, E^*)$ is clopen and locally $\aleph_0$-saturated. Since every finite graph embeds into every open subset of G^* by the local $\aleph_0$-saturation, the cochromatic number of G^* is the maximal possible cochromatic number $\mathfrak{hm}(c_{\max})$. ∎

Corollary 7.8. *No clopen graph G on a closed subset of ω^ω with c_G uniformly continuous is universal for all clopen graphs on closed subsets of ω^ω.*

While there are $\aleph_0$-saturated clopen graphs on ω^ω, there is no $\aleph_1$-saturated clopen graph on any Polish space. This was shown in a previous version of this article. Then Arnold Miller observed that there is not even an $\aleph_1$-saturated $\mathbf{\Delta}^0_2$ graph on a Polish space. His argument gives slightly more.

Theorem 7.9. *No $\aleph_1$-saturated graph embeds into a $\mathbf{\Delta}^0_2$ graph on a Polish space.*

Proof. Let G be a $\mathbf{\Delta}^0_2$ graph on a Polish space X and suppose F is an $\aleph_1$-saturated induced subgraph of G. Let $Y \subseteq X$ be the set of vertices of F. Since F is $\aleph_1$-saturated, Y is an uncountable subset of X.

Let Z be the set of points $x \in X$ such that every open neighborhood of x has an uncountable intersection with Y. Then Z is a closed subset of X. Since X is Polish and Y is uncountable, Z is nonempty and has no isolated points. Also $Y \cap Z$ is dense in Z. Choose a sequence $(a_n)_{n\in\omega}$ in Y without repetition such that both $\{a_{2n} : n \in \omega\}$ and $\{a_{2n+1} : n \in \omega\}$ are dense in Z. Since the induced subgraph on Y is $\aleph_1$-saturated, there is $a \in Y$ such that a forms an edge with each a_{2n}, $n \in \omega$, and no edge with any a_{2n+1}, $n \in \omega$.

But since G is $\mathbf{\Delta}^0_2$, both G and its complement are G_δ. It follows that the set A of neighbors of a in G and also the set B of non-neighbors of a in G are G_δ subsets of X. Hence $A \cap Z$ and $B \cap Z$ are G_δ subsets of Z. Since $\{a_{2n} : n \in \omega\} \subseteq A$ and $\{a_{2n+1} : n \in \omega\} \subseteq B$, both A and B are dense G_δ subsets of Z. But dense G_δ subsets of a complete metric space are comeager. This contradicts the fact that A and B are disjoint. ∎

Theorem 7.9 is optimal. As Clinton Conley pointed out, there is an $\aleph_1$-saturated F_σ graph on 2^ω. The graph itself is rather well-known since it is an example of an F_σ graph that has an uncountable clique but no perfect clique. Graphs of this kind are mentioned in [17], [20], and in [18]. Wiesław Kubiś has shown that every G_δ graph with an uncountable clique has a perfect clique [16].

DEFINITION 7.10. Fix a homeomorphism $h : 2^\omega \to (2^\omega)^\omega$. For each $n \in \omega$ let $\pi_n : (2^\omega)^\omega \to 2^\omega$ be the projection to the nth coordinate and $f_n = \pi_n \circ h$. Let $\{x, y\} \in E$ if there is $n \in \omega$ such that $f_n(x) = y$ or $f_n(y) = x$.

Since graphs of continuous functions are closed subsets of the plane, the graph $G = (2^\omega, E)$ is F_σ.

LEMMA 7.11. *G is $\aleph_1$-saturated.*

Proof. Let $A \subseteq 2^\omega$ be a countable set and let $f : A \to 2$ be a type over A. After enlarging A if necessary, we may assume that there are infinitely many $a \in A$ such that $f(a) = 1$. Fix a 1-1-enumeration $(a_n)_{1 \le n < \omega}$ of $f^{-1}(1)$. There are only countably many $b \in 2^\omega$ that are of the form $f_m(a)$ for some $m \in \omega$ and $a \in A$. It follows that there is some $a_0 \in 2^\omega \setminus A$ such that for all $m \in \omega$ and all $a \in A$, $h^{-1}((a_n)_{n\in\omega})$ is different from all $f_m(a)$. Let $b = h^{-1}((a_n)_{n\in\omega})$. Now for each $n \in \omega$, $a_n = f_n(b)$. It follows that for all $n \in \omega$, $\{a_n, b\} \in E$. On the other hand, for no $a \in A$ and no $n \in \omega$ do we have $f_n(a) = b$. It follows that for $a \in A$ we have $\{a, b\} \in E$ iff $a \in \{a_n : n \in \omega\}$. This shows that b realizes the type f over A. ∎

COROLLARY 7.12. *Every graph of size $\aleph_1$ embeds into an F_σ graph on 2^ω.*

Note that if a graph is κ-saturated then its complement is also κ-saturated. Hence there is an $\aleph_1$-saturated G_δ graph. Under CH the $\aleph_1$-saturated graph G from Lemma 7.11 and its complement are combinatorially isomorphic, but they are not topologically isomorphic since one has a perfect clique and the other does not. In fact, neither of the two graphs can be embedded into the other by a continuous map since such an embedding preserves perfect cliques and perfect independent sets.

8. Nonexistence of universal clopen graphs on compact metric spaces. Let us first observe that there are universal graphs on 2^ω and on ω^ω of higher complexity than clopen. This was pointed out by Ben Miller.

THEOREM 8.1. *Let X be either the Cantor space 2^ω or the Baire space ω^ω. Let $\alpha \in \omega_1 \setminus \{0\}$ and $n \in \omega \setminus \{0\}$. Let Γ be one of the following classes of subsets of $X^2 \setminus \{(x, x) : x \in X\}$: $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^1_n$, and $\mathbf{\Pi}^1_n$. Then there is a graph G on X in the class Γ such that every graph on X in Γ embeds into G by a topological embedding.*

Proof. Since X is homeomorphic to $X \times X$, we may use $X \times X$ as the space of vertices of G. Let $C \subseteq X \times (X \times X)$ be a set in the class Γ that is universal in the sense that for each set $A \subseteq X \times X$ in Γ there is $x \in X$ such that

$$A = C_x = \{y \in X \times X : (x, y) \in C\}$$

(see [13] for the existence of universal sets of class Γ). We define the set E of edges of G as follows:

For all $(x_0, y_0), (x_1, y_1) \in 2^\omega \times 2^\omega$ with $(x_0, y_0) \neq (x_1, y_1)$ let

$$\{(x_0, y_0), (x_1, y_1)\} \in E$$

iff $(x_0, (y_0, y_1))$, $(x_0, (y_1, y_0))$, $(x_1, (y_0, y_1))$, and $(x_1, (y_1, y_0))$ are all elements of C. Since Γ is closed under finite intersections and under permutation of coordinates, $G = (X \times X, E)$ is a graph of class Γ.

If H is a graph of class Γ on X, let $x \in X$ be such that $C_x \setminus \{(x, x) : x \in X\}$ is the edge relation of H. Now H is isomorphic to the induced subgraph of G on the set $\{x\} \times X$. ∎

Some minimal care was necessary in the proof of Theorem 8.1 to capture the case of open graphs. A slightly more straightforward argument works for closed graphs and up. Observe that our construction of universal graphs does not immediately show that there are, for example, Borel graphs that do not embed into Borel graphs of lower complexity, topologically or just combinatorially. In fact, under CH every graph on a Polish space, independently of its definability, combinatorially embeds into a single $\aleph_1$-saturated F_σ graph on 2^ω.

Our next goal is to show that there is no universal clopen graph on a compact metric space. In fact, we can compute precisely how many clopen graphs on a compact metric space are needed so that every clopen graph on a compact zero-dimensional metric space embeds into one of them.

THEOREM 8.2. *For every family $\mathcal{G}$ of size less than $\mathfrak{d}$ of clopen graphs on compact metric spaces, there is a clopen graph on 2^ω that does not embed into any member of $\mathcal{G}$.*

COROLLARY 8.3. *The least size of a family $\mathcal{G}$ of clopen graphs on compact, zero-dimensional, metric spaces such that every clopen graph on a compact, zero-dimensional, metric space embeds into a member of $\mathcal{G}$ is $\mathfrak{d}$.*

Proof. By Corollary 6.2, there is a family $\mathcal{F}$ of clopen graphs on 2^ω such that every clopen graph on a compact, zero-dimensional, metric space embeds into a member of $\mathcal{F}$.

By Theorem 8.2, there is no family $\mathcal{F}$ of size $< \mathfrak{d}$ such that every clopen graph on a compact, zero-dimensional, metric space embeds into a member of $\mathcal{F}$. ∎

Lemma 7.5 plays a crucial role in the proof of Theorem 8.2. We first assign a graph G_f to every nondecreasing function $f : \omega \to \omega \setminus 3$.

DEFINITION 8.4. Let $f : \omega \to \omega \setminus 3$ be nondecreasing. We define a graph G_f of depth 2 on a compact subset V_f of ω^ω as follows:

First let

$$F_f = \{x \in \omega^\omega : \forall n \in \omega\ (x(n) < f(n))\},$$
$$T_f = T(F_f) = \{x \upharpoonright n : x \in F_f \wedge n \in \omega\}.$$

Fix $t \in T_f$ and let $n = |t|$ and

$$A_t = \{s \in T_f : |s| = n + 2 \wedge t \subseteq s\}.$$

Let $(g_{t,i})_{i<k_t}$ be an enumeration of all types over A_t. For each $i < k_t$ and all $m \in \omega$ let

$$x_{t,i}(m) = \begin{cases} t(m) & \text{if } m < n, \\ f(n) + i & \text{if } m = n, \\ 0 & \text{if } m > n. \end{cases}$$

Now let $X_f = \{x_{t,i} : t \in T_f \wedge i < k_t\}$ and $V_f = F_f \cup X_f$. We define the set E_f of edges in the graph G_f as follows:

All edges in G_f go between the sets F_f and X_f. Let $y \in F_f$ and $x \in X_f$. Then $\{x, y\} \in E_f$ iff for $n = \Delta(x, y)$ and $t = x \upharpoonright n$ there is some $i < k_t$ such that $x = x_{t,i}$ and $g_i(y \upharpoonright (n + 2)) = 1$.

Note that all vertices in X_f are isolated points of V_f. Every accumulation point of X_f is in F_f. Furthermore, F_f is compact. It follows that V_f is a closed subset of ω^ω. Clearly, the tree of finite initial segments of V_f is finitely branching. Hence V_f is compact. From the definition of E_f it follows that G_f is of depth 2.

The reason for the particular definition of G_f is the following: Let $A \subseteq F_f$ and $t \in T_f$ be such that the map $a \mapsto a \upharpoonright (|t| + 2)$ is a bijection between A and A_t. If g is a type over A, then there is a unique $i < k_t$ such that for all $a \in A$, $g_i(a \upharpoonright (n + 2)) = g(a)$. The vertex $x_{t,i}$ now realizes the type g over the set A.

This argument proves the next lemma:

LEMMA 8.5. *Let $A \subseteq F_f$ be such that for some $n \in \omega$ and all distinct $a, b \in A$, either $\Delta(a, b) = n$ or $\Delta(a, b) = n + 1$. Then every type over A is realized in G_f.*

For two functions $f, g : \omega \to \omega$ we write $f \leq^* g$ if for all but finitely many $n \in \omega$, $f(n) \leq g(n)$.

LEMMA 8.6. *For each $g : \omega \to \omega$ there is a function $f : \omega \to \omega$ such that whenever G_f embeds into a clopen graph G of depth 2 on a compact subset V of ω^ω, then for all but finitely many $n \in \omega$, $|V \upharpoonright n| > g(n)$.*

Proof. We may assume that g is nondecreasing. Choose a nondecreasing function $f : \omega \to \omega \setminus 4$ such that for all $k \in \omega$ and all but finitely many $n \in \omega$,

$f(n) > g(k+2n)$. This is possible since every countable set of functions from ω to ω is bounded with respect to $\leq^*$.

Let e be an embedding of G_f into a clopen graph G of depth 2 on a compact subset V of ω^ω. Let $a, b \in F_f$ be distinct and $\ell = \Delta(e(a), e(b)) + 2$. Let t be the longest common initial segment of a and b and let $t_a = a\restriction(|t|+1)$ and $t_b = b\restriction(|t|+1)$.

Claim 8.7. *Suppose $x, y \in F_f$ are distinct, extend t, and do not extend t_a or t_b. Then*

$$\Delta(e(x), e(y)) \leq \ell + 2(\Delta(x, y) - |t|).$$

We show the claim by induction on $n = \Delta(x, y)$, starting with the minimal possible value, namely $n = |t|$. Let $x, y \in F_f$ be such that $t \subseteq x, y$, $\Delta(x, y) = n$, and both x and y do not extend t_a or t_b. Then for all $c, d \in \{a, b, x, y\}$, $\Delta(c, d) = n$. Hence, by Lemma 8.5, all types over the set $\{a, b, x, y\}$ are realized in G_f. Hence all types over $e[\{a, b, x, y\}]$ are realized in G. By Lemma 7.5,

$$|e[\{a, b, x, y\}]\restriction(\Delta(e(a), e(b)) + 3)| = 4.$$

It follows that

$$\Delta(e(x), e(y)) \leq \Delta(e(a), e(b)) + 2 = \ell = \ell + 2(\Delta(x, y) - |t|).$$

Now let $n \geq |t|$ and suppose that for all $x, y \in F_f$ with $\Delta(x, y) = n$ that extend t, but not t_a or t_b, we have $\Delta(e(x), e(y)) \leq \ell + 2(\Delta(x, y) - |t|)$. Let $x, y \in F_f$ be such that they extend t, but not t_a or t_b, and satisfy $\Delta(x, y) = n + 1$. Choose $z \in F_f$ extending t, but not t_a or t_b, such that $\Delta(x, z) = n$. By the inductive hypothesis, $\Delta(x, z) \leq \ell + 2(\Delta(x, y) - |t|)$. By Lemma 8.5, all types over $\{x, y, z\}$ are realized in G_f. It follows that all types over $e[\{x, y, z\}]$ are realized in G. Hence, by Lemma 7.5,

$$\begin{aligned} \Delta(e(x), e(y)) &\leq \Delta(e(x), e(z)) + 2 \\ &\leq \ell + 2(\Delta(x, z) - |t|) + 2 = \ell + 2(\Delta(x, y) - |t|). \end{aligned}$$

This finishes the proof of the claim.

Now let $n_0 = |t|$ and $k = \ell + 1$. From the claim it follows that for every $n > n_0$,

$$|V\restriction(\ell + 2(n - |t|) + 1)| \geq |\{x\restriction(n+1) : x \in F_f \wedge t \subseteq x\}| \geq f(n).$$

Since f is nondecreasing, for all $n > n_0$, $|V\restriction(k + 2n)| \geq f(n)$. By the choice of f, for all but finitely many $n \in \omega$ we have $f(n) > g(k + 2 + 2n)$. It follows that for all but finitely many $n \in \omega$, $|V\restriction(k + 2n)| > g(k + 2 + 2n)$. Since the functions $n \mapsto |V\restriction(k + 2n)|$ and g are nondecreasing, this implies that for all but finitely many $n \in \omega$, $|V\restriction(k + 2n + 1)| > g(k + 2n + 1)$ and $|V\restriction(k + 2n)| > g(k + 2n)$. Hence for almost all $n \in \omega$, $|V\restriction(n)| > g(n)$. ∎

We are now ready to prove Theorem 8.2.

Proof of Theorem 8.2. Let $\mathcal{G}$ be a family of clopen graphs on compact metric spaces and suppose that $|\mathcal{G}| < \mathfrak{d}$. For each $G \in \mathcal{G}$ let $G/\mathrm{Comp}(V(G))$ be the quotient of G by the modular partition consisting of the (topologically) connected components of the space $V(G)$ of vertices of G. The space of connected components of $V(G)$ is compact, metric, and zero-dimensional. By Lemma 6.6, $G/\mathrm{Comp}(V(G))$ is isomorphic to a graph C_G of depth 2 on a compact subset of ω^ω. Let $f_G : \omega \to \omega$ be such that for all $n \in \omega$,

$$f_G(n) \geq |V(C_G){\restriction}n|.$$

Since $|\mathcal{G}| < \mathfrak{d}$, there is a function $g : \omega \to \omega$ such that for all $G \in \mathcal{G}$ there are infinitely many $n \in \omega$ with $g(n) > f_G(n)$. By Lemma 8.6 there is a function $f : \omega \to \omega$ such that whenever G_f embeds into a clopen graph G of depth 2 on a compact subset V of ω^ω, then for all but finitely many $n \in \omega$, $|V{\restriction}n| > g(n)$. The theorem follows immediately from the following claim.

Claim 8.8. *G_f does not embed into any $G \in \mathcal{G}$.*

Let $G \in \mathcal{G}$ and assume that there is an embedding e of G_f into G. For any two distinct members x and y of F_f there is $z \in X_f$ such that x and z form an edge in G_f but y and z do not. Since $\mathrm{Comp}(V(G))$ is a modular partition of G and each connected component of G is homogeneous, it follows that any two distinct vertices of G_f are mapped by e into distinct connected components of G. Hence G_f embeds into $G/\mathrm{Comp}(V(G))$.

The graph embedding of G_f into $G/\mathrm{Comp}(V(G))$ induces an embedding of G_f into C_G. By the properties of f, for almost all $n \in \omega$,

$$|V(C_G){\restriction}n| > g(n).$$

However, by the choice of g, there are infinitely many $n \in \omega$ such that $g(n) > f_G(n)$. But by the choice of f_G, for almost all $n \in \omega$,

$$f_G(n) \geq |V(C_G){\restriction}n| > g(n),$$

a contradiction. This finishes the proof of the claim and hence of the theorem. ∎

We finish this section with a discussion of the quasi-orders of clopen graphs on 2^ω ordered by combinatorial, respectively topological embeddability.

Definition 8.9. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be directed sets. A map $\varphi : P \to Q$ is *Tukey* if for all $q \in Q$ there is $p \in P$ such that for all $x \in P$, $\varphi(x) \leq_Q q$ implies $x \leq_P p$. In other words, a map is Tukey if preimages of bounded sets are bounded.

If there is a Tukey map from P to Q we say that P is *Tukey reducible* to Q and write $P \leq_T Q$. If P is Tukey reducible to Q and Q is Tukey reducible to P, then P and Q are *Tukey equivalent* and we write $P \equiv_T Q$.

The *bounding number* $\mathfrak{b}(P)$ of a directed set $(P, \leq_P)$ is the least size of an unbounded subset of P. The *dominating number* $\mathfrak{d}(P)$ of P is the least size of a subset C of P such that for all $p \in P$ there is $c \in C$ such that $p \leq_P c$. The usual bounding number $\mathfrak{b}$ and dominating number $\mathfrak{d}$ are just the bounding and dominating number of $(\omega^\omega, \leq^*)$.

Note that if $P \leq_T Q$, then $\mathfrak{b}(P) \geq \mathfrak{b}(Q)$ and $\mathfrak{d}(P) \leq \mathfrak{d}(Q)$.

Let $\mathcal{C}$ denote the class of clopen graphs on closed subspaces of 2^ω. Recall that every clopen graph on a compact zero-dimensional metric space is topologically isomorphic to a member of $\mathcal{C}$. Let $\leq_c$ denote combinatorial embeddability and let $\leq_t$ denote topological embeddability between members of $\mathcal{C}$. Then the identity map on $\mathcal{C}$ shows that $(\mathcal{C}, \leq_c) \leq_T (\mathcal{C}, \leq_t)$.

LEMMA 8.10. $(\omega^\omega, \leq^*) \leq_T (\mathcal{C}, \leq_c)$.

Proof. We define a Tukey map φ from $(\omega^\omega, \leq^*)$ to $(\mathcal{C}, \leq_c)$. For $g : \omega \to \omega$ let $f : \omega \to \omega$ be the function given by Lemma 8.6. Let $\varphi(g)$ be an isomorphic copy of G_f that lives on a closed subset of 2^ω. Given a graph $G \in \mathcal{C}$, choose an isomorphic copy of G that lives on a compact subset X of ω^ω and is of depth 2. For each $n \in \omega$, let $f(n) = |X{\restriction}n|$.

Now, if for some $g : \omega \to \omega$, $\varphi(g) \leq_c G$, then by the choice of $\varphi(g)$, for all but finitely many $n \in \omega$, $f(n) = |X{\restriction}n| > g(n)$. It follows that $g \leq^* f$, showing that φ is a Tukey map. ∎

LEMMA 8.11. $(\mathcal{C}, \leq_t) \leq_T (\omega^\omega, \leq^*)$.

Proof. In the following proof all embeddings of graphs are topological.

Each $G \in \mathcal{G}$ embeds into the graph $G_{\text{universal}}$ corresponding to the universal coloring of depth 2. For $f : \omega \to \omega$ let

$$H(f) = G_{\text{universal}}{\restriction}\{x \in \omega^\omega : \forall n \in \omega(x(n) < f(n))\}.$$

For each $G \in \mathcal{G}$ choose a strictly increasing function $f_G : \omega \to \omega$ such that G already embeds into $H(f_G)$. We show that $\psi : \mathcal{C} \to \omega^\omega : G \mapsto f_G$ is a Tukey map.

Let $f \in \omega^\omega$. Suppose that for some $F \in \mathcal{C}$, $\psi(F) \leq^* f$. Since $\psi(F) \leq^* f$ and $\psi(F)$ is strictly increasing, there is $k \in \omega$ such that for all $n \in \omega$ we have $\psi(F)(n) \leq f(n+k)$. For each $n \in \omega$ let $g(n) = f(n+k)$. Then $H(\psi(F))$ embeds into $H(g)$.

Since for distinct $x, y \in \omega^\omega$, $c_{\text{universal}}$ only depends on the values of x and y at $\Delta(x,y)$ and $\Delta(x,y)+1$, whenever for some graph G, $e : G \to G_{\text{universal}}$ is an embedding, then for all $s \in \omega^\omega$ the map $e_s : G \to G_{\text{universal}} : v \mapsto s^\frown e(v)$ is an embedding as well.

This shows that the graph $H(g)$ embeds into $H(f)$. Hence F embeds into $H(f)$. This shows that ψ is indeed a Tukey map. ∎

Since Tukey reducibility is transitive, from Lemmas 8.10, 8.11, and the remark before Lemma 8.10 we get the following corollary:

COROLLARY 8.12. *The quasi-orders $(\omega^\omega, \leq^*)$, $(\mathcal{C}, \leq_t)$, and $(\mathcal{C}, \leq_c)$ are Tukey equivalent. In particular,*

$$\mathfrak{d}(\mathcal{C}, \leq_t) = \mathfrak{d}(\mathcal{C}, \leq_c) = \mathfrak{d} \quad \textit{and} \quad \mathfrak{b}(\mathcal{C}, \leq_t) = \mathfrak{b}(\mathcal{C}, \leq_c) = \mathfrak{b}.$$

9. Clopen graphs on large compact spaces. It is natural to ask whether every countable graph embeds into a clopen graph on some large compact space such as $\beta\omega$, the Stone–Čech compactification of the countably infinite discrete space. This is however not the case. In general, there is no such thing as a compactification of graphs.

THEOREM 9.1. *No infinite, 4-saturated graph embeds into a clopen graph on a compact space.*

It turns out that the proof of Theorem 7.6 generalizes to arbitrary compact spaces, but we have to give up the very convenient use of concepts like depth that are related to the tree representation of closed subsets of ω^ω.

We will instead use the fact that every compact zero-dimensional space X is homeomorphic to a closed subspace of the generalized Cantor cube 2^κ where κ is the weight of X.

Proof of Theorem 9.1. Exactly as in the proof of Theorem 7.6 we see that if a 4-saturated graph embeds into any clopen graph on a compact space, then it embeds into a clopen graph on a compact zero-dimensional space.

So let $G = (X, E)$ be a clopen graph on the compact zero-dimensional space X and let A be a subset of X of size at least 3 such that every type over a 3-element subset of A is realized in G. We show that A is finite.

Since X is compact and zero-dimensional, it is homeomorphic to a closed subspace of 2^κ, where κ is the weight of X. Hence we may assume that X is actually a closed subset of 2^κ.

Let $c : [X]^2 \to 2$ denote the continuous coloring associated with G. For $S \subseteq T \subseteq \kappa$ we say that T *determines* S if for all $x, y \in X$ with $x{\restriction}S \neq y{\restriction}S$ and all $x', y' \in X$ with $x{\restriction}T = x'{\restriction}T$ and $y{\restriction}T = y'{\restriction}T$ we have $c(x, y) = c(x', y')$. In other words, T determines S if for all $x, y \in X$ with $x{\restriction}S \neq y{\restriction}S$ the color $c(x, y)$ is already determined by the restrictions of x and y to T.

CLAIM 9.2. *If $S \subseteq \kappa$ is finite, then there is a finite set $T \subseteq \kappa$ such that $S \subseteq T$ and T determines S.*

For a finite set $F \subseteq \kappa$ and a function $f : F \to 2$ let $[f]$ denote the set of all $x \in 2^\kappa$ with $f \subseteq x$. Let $a, b : S \to 2$ be distinct. By the continuity of c, for all $x \in [a] \cap X$ and $y \in [b] \cap X$ there are disjoint open sets $U_{x,y} \subseteq 2^\kappa$ and $V_{x,y} \subseteq 2^\kappa$ such that for all $x' \in U_{x,y} \cap X$ and all $y' \in V_{x,y} \cap X$, $c(x, y) = c(x', y')$. The sets $U_{x,y}$ and $V_{x,y}$ can be chosen of the form $U_{x,y} = [f_{x,y}]$ and $V_{x,y} = [g_{x,y}]$ for some functions $f_{x,y}, g_{x,y} : F_{x,y} \to 2$ with $F_{x,y} \subseteq \kappa$ finite.

The family

$$\{U_{x,y} \times V_{x,y} : x \in [a] \cap X \wedge y \in [b] \cap X\}$$

is an open cover of the compact set $([a] \times [b]) \cap (X \times X)$ and hence has a finite subcover. Let $x_1, \dots, x_n$ and $y_1, \dots, y_n$ be such that

$$([a] \times [b]) \cap (X \times X) \subseteq \bigcup_{i=1}^{n} ([f_{x_i,y_i}] \times [g_{x_i,y_i}]).$$

Let $F = \bigcup_{i=1}^{n} F_{x_i,y_i}$. Now for all $x, y \in X$ with $x{\restriction}S = a$ and $y{\restriction}S = b$, $c(x,y)$ is determined by the restrictions of x and y to F.

Since S is finite, 2^S is finite. For all $\{a,b\} \in [2^S]^2$ choose a finite set $F_{a,b} \subseteq \kappa$ such that for all $x, y \in X$ with $x{\restriction}S = a$ and $y{\restriction}S = b$, $c(x,y)$ is determined by the restrictions of x and y to $F_{a,b}$. Let

$$T = S \cup \bigcup \{F_{a,b} : \{a,b\} \in [2^S]^2\}.$$

Then T is a finite superset of S that determines S. This finishes the proof of the claim.

Now let $S_0 \subseteq \kappa$ be finite and such that $A{\restriction}S_0 = \{a{\restriction}S_0 : a \in A\}$ has at least two elements. By the claim, there are finite sets $S_1, S_2 \subseteq \kappa$ such that $S_0 \subseteq S_1 \subseteq S_2$, S_1 determines S_0, and S_2 determines S_1.

Claim 9.3. *For all distinct $a, b \in A$, $a{\restriction}S_2 \neq b{\restriction}S_2$.*

Let $a, b \in A$ be distinct. If $a{\restriction}S_0 \neq b{\restriction}S_0$, then certainly $a{\restriction}S_2 \neq b{\restriction}S_2$. Hence we can assume that $a{\restriction}S_0 = b{\restriction}S_0$. Since $A{\restriction}S_0$ has at least two elements, there is $z \in A$ such that $a{\restriction}S_0 = b{\restriction}S_0 \neq z{\restriction}S_0$. If $c(a,z) \neq c(b,z)$, then, since S_1 determines S_0, $a{\restriction}S_1 \neq b{\restriction}S_1$ and thus $a{\restriction}S_2 \neq b{\restriction}S_2$.

Hence we can assume that $c(a,z) = c(b,z)$. Since every type over $\{a,b,z\}$ is realized in G, there is $x \in X$ such that $c(a,x) \neq c(b,x)$ and $c(x,z) \neq c(a,z)$. Since $c(x,z) \neq c(a,z)$ and $a{\restriction}S_0 \neq z{\restriction}S_0$, $x{\restriction}S_1 \neq a{\restriction}S_1$. Since $c(a,x) \neq c(b,x)$ and since S_2 determines S_1, $a{\restriction}S_2 \neq b{\restriction}S_2$, showing the claim.

From the claim it follows immediately that $|A{\restriction}S_2| = |A|$. But $A{\restriction}S_2 \subseteq 2^{S_2}$ and 2^{S_2} is finite. Hence A is finite. This finishes the proof of Theorem 9.1. ∎

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