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The virtual and universal braids

by

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Abstract. We study the structure of the virtual braid group. It is shown that the virtual braid group is a semi-direct product of the virtual pure braid group and the symmetric group. Also, it is shown that the virtual pure braid group is a semi-direct product of free groups. From these results we obtain a normal form of words in the virtual braid group. We introduce the concept of a universal braid group. This group contains the classical braid group and has as quotients the singular braid group, virtual braid group, welded braid group, and classical braid group.

Recently some generalizations of classical knots and links were defined and studied: singular links [20, 5], virtual links [15, 12] and welded links [10].

One of the ways to study classical links is to study the braid group. Singular braids [1, 5], virtual braids [15, 21], welded braids [10] were defined similarly to the classical braid group. A theorem of A. A. Markov [4, Ch. 2.2] reduces the problem of classification of links to some algebraic problems of the theory of braid groups. These problems include the word problem and the conjugacy problem. There are generalizations of Markov's theorem to singular links [11], virtual links, and welded links [14].

There are some different ways to solve the word problem for the singular braid monoid and singular braid group [8, 7, 22]. The solution of the word problem for the welded braid group follows from the fact that this group is a subgroup of the automorphism group of the free group [10]. A normal form of words in the welded braid group was constructed in [13].

In this paper we study the structure of the virtual braid group VB_n. This is similar to the classical braid group B_n and welded braid group WB_n. The group VB_n contains the normal subgroup VP_n which is called the *virtual*

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pure braid group. The quotient group VB_n/VP_n is isomorphic to the symmetric group S_n. We find generators and defining relations of VP_n. Since VB_n is a semi-direct product of VP_n and S_n, we should study the structure of VP_n. It will be proved that VP_n is representable as the following semi-direct product:

$$VP_n = V_{n-1}^* \rtimes VP_{n-1} = V_{n-1}^* \rtimes (V_{n-2}^* \rtimes (\cdots \rtimes (V_2^* \rtimes V_1^*)) \cdots),$$

where V_i^* is some (in general infinitely generated for $i > 1$) free subgroup of VP_n. From this result it follows that there exists a normal form of words in VB_n.

In the last section we define the universal braid group UB_n which contains the braid group B_n and has as quotients the singular braid group SG_n, the virtual braid group VB_n, the welded braid group WB_n, and the braid group B_n. It is known [10] that VB_n has as its quotient the group WB_n. It will be proved that the quotient homomorphism maps VP_n into the welded pure braid group WP_n. This homomorphism agrees with the decomposition of this group into the semi-direct product given by Theorem 2 and by [2, 3].

By Artin's theorem, the group B_n is embedded into the automorphism group $\mathrm{Aut}(F_n)$ of the free group F_n. In [10] it was proved that WB_n is also embedded into $\mathrm{Aut}(F_n)$. It is not known if SG_n and VB_n are embedded into $\mathrm{Aut}(F_n)$.

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1. Different classes of braids and their properties. In this section we recall some known facts about braid groups, singular braid monoids, virtual braid groups and welded braid groups (see references from the introduction).

1.1. *The braid group and the group of conjugating automorphisms.* The braid group B_n, $n \geq 2$, on n strings can be defined as the group generated by $\sigma_1, \ldots, \sigma_{n-1}$ (see Fig. 1) with the defining relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad i = 1, \ldots, n-2, \tag{1}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \qquad |i-j| \geq 2. \tag{2}$$

There exists a homomorphism of B_n onto the symmetric group S_n on n letters. This homomorphism maps σ_i to the transposition $(i, i+1)$, $i = 1, \ldots, n-1$. Its kernel is called the *pure braid group* and denoted by P_n. The

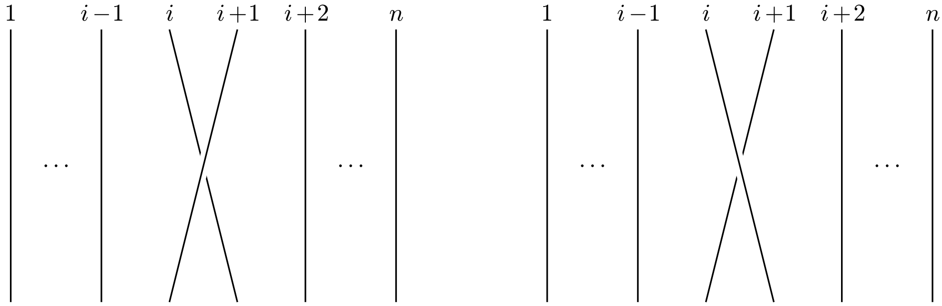

Fig. 1. Geometric braids representing σ_i and σ_i^{-1}

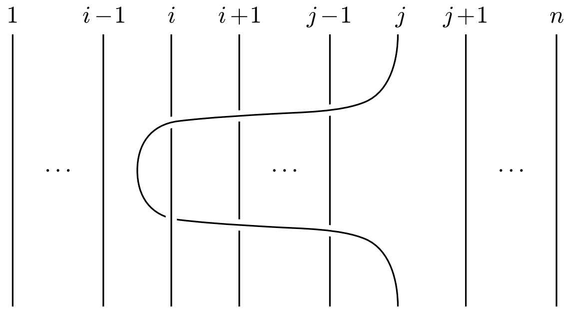

Fig. 2. The geometric braid a_{ij}

group P_n is generated by a_{ij}, $1 \leq i < j \leq n$ (see Fig. 2). These generators can be expressed by the generators of B_n as follows:

$$a_{i,i+1} = \sigma_i^2,$$

$$a_{ij} = \sigma_{j-1}\sigma_{j-2}\dots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\dots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}, \qquad i+1 < j \leq n.$$

The group P_n is the semi-direct product of the normal subgroup U_n which is a free group with free generators $a_{1n}, a_{2n}, \dots, a_{n-1,n}$, and P_{n-1}. Similarly, P_{n-1} is the semi-direct product of the free group U_{n-1} with free generators $a_{1,n-1}, a_{2,n-1}, \dots, a_{n-2,n-1}$ and P_{n-2}, and so on. Therefore, P_n is decomposable (see [17]) into the following semi-direct product:

$$P_n = U_n \rtimes (U_{n-1} \rtimes (\cdots \rtimes (U_3 \rtimes U_2))\cdots), \qquad U_i \simeq F_{i-1},\ i = 2, \dots, n.$$

The group B_n in the faithful representation in the group $\mathrm{Aut}(F_n)$ of automorphisms of the free group $F_n = \langle x_1, \dots, x_n \rangle$. In this case the generator σ_i, $i = 1, \dots, n-1$, defines the automorphism

$$\sigma_i : \begin{cases} x_i \mapsto x_i\, x_{i+1}\, x_i^{-1}, \\ x_{i+1} \mapsto x_i, \\ x_l \mapsto x_l, \quad l \neq i, i+1. \end{cases}$$

By a theorem of Artin [4, Theorem 1.9], an automorphism $\beta \in \mathrm{Aut}(F_n)$ lies in B_n if and only if β satisfies the following conditions:

$$\text{(i) } \beta(x_i) = a_i^{-1} x_{\pi(i)} a_i, \quad 1 \leq i \leq n, \qquad \text{(ii) } \beta(x_1 \cdots x_n) = x_1 \cdots x_n,$$

where π is a permutation from S_n and $a_i \in F_n$.

An automorphism is called a *conjugating automorphism* (or a permutation-conjugating automorphism according to the terminology of [10]) if it satisfies condition (i). The group C_n of conjugating automorphisms is generated by σ_i and the automorphisms α_i, $i = 1, \dots, n-1$, where

$$\alpha_i : \begin{cases} x_i \longmapsto x_{i+1}, \\ x_{i+1} \longmapsto x_i, \\ x_l \longmapsto x_l, \quad l \neq i, i+1. \end{cases}$$

It is not hard to see that the automorphisms α_i generate the symmetric group S_n and, hence, satisfy the following relations:

$$(3) \qquad \alpha_i \alpha_{i+1} \alpha_i = \alpha_{i+1} \alpha_i \alpha_{i+1}, \qquad i = 1, \dots, n-2,$$

$$(4) \qquad \alpha_i \alpha_j = \alpha_j \alpha_i, \qquad |i - j| \geq 2,$$

$$(5) \qquad \alpha_i^2 = 1, \qquad i = 1, \dots, n-1.$$

The group C_n is defined by relations (1)–(2) of B_n, relations (3)–(5) of S_n, and the mixed relations (see [10, 19])

$$(6) \qquad \alpha_i \sigma_j = \sigma_j \alpha_i, \qquad |i - j| \geq 2,$$

$$(7) \qquad \sigma_i \alpha_{i+1} \alpha_i = \alpha_{i+1} \alpha_i \sigma_{i+1}, \qquad i = 1, \dots, n-2,$$

$$(8) \qquad \sigma_{i+1} \sigma_i \alpha_{i+1} = \alpha_i \sigma_{i+1} \sigma_i, \qquad i = 1, \dots, n-2.$$

If we consider the group generated by the automorphisms ε_{ij}, $1 \leq i \neq j \leq n$, where

$$\varepsilon_{ij} : \begin{cases} x_i \longmapsto x_j^{-1} x_i x_j, & i \neq j, \\ x_l \longmapsto x_l, & l \neq i, \end{cases}$$

then we get the group Cb_n of *basis-conjugating automorphisms.* The elements of satisfy condition (i) for the identical permutation π, i.e., map each generator x_i to the conjugating element. J. McCool [18] proved that Cb_n is defined by the relations (from now on, different letters stand for different indices)

$$(9) \qquad \varepsilon_{ij} \varepsilon_{kl} = \varepsilon_{kl} \varepsilon_{ij},$$

$$(10) \qquad \varepsilon_{ij} \varepsilon_{kj} = \varepsilon_{kj} \varepsilon_{ij},$$

$$(11) \qquad (\varepsilon_{ij} \varepsilon_{kj}) \varepsilon_{ik} = \varepsilon_{ik} (\varepsilon_{ij} \varepsilon_{kj}).$$

The group C_n is representable as the semi-direct product $C_n = Cb_n \rtimes S_n$, where S_n is generated by the automorphisms $\alpha_1, \dots, \alpha_{n-1}$. The following equalities hold (see [19]):

$$\varepsilon_{i,i+1} = \alpha_i \sigma_i^{-1}, \qquad \varepsilon_{i+1,i} = \sigma_i^{-1} \alpha_i,$$
$$\varepsilon_{ij} = \alpha_{j-1}\alpha_{j-2}\cdots\alpha_{i+1}\varepsilon_{i,i+1}\alpha_{i+1}\cdots\alpha_{j-2}\alpha_{j-1}, \qquad i < j,$$
$$\varepsilon_{ji} = \alpha_{j-1}\alpha_{j-2}\cdots\alpha_{i+1}\alpha_i\varepsilon_{i,i+1}\alpha_i\alpha_{i+1}\cdots\alpha_{j-2}\alpha_{j-1}, \qquad i < j.$$

The structure of Cb_n was studied in [2, 3]. It was proved that Cb_n, $n \geq 2$, is decomposable into the semi-direct product

$$Cb_n = D_{n-1} \rtimes (D_{n-2} \rtimes (\cdots \rtimes (D_2 \rtimes D_1))\cdots)$$

of subgroups D_i, $i = 1, \dots, n-1$, generated by $\varepsilon_{i+1,1}$, $\varepsilon_{i+1,2}$, $\dots$, $\varepsilon_{i+1,i}$, $\varepsilon_{1,i+1}$, $\varepsilon_{2,i+1}$, $\dots$, $\varepsilon_{i,i+1}$. The elements $\varepsilon_{i+1,1}$, $\varepsilon_{i+1,2}$, $\dots$, $\varepsilon_{i+1,i}$ generate a free group of rank i. The elements $\varepsilon_{1,i+1}$, $\varepsilon_{2,i+1}$, $\dots$, $\varepsilon_{i,i+1}$ generate a free abelian group of rank i.

The pure braid group P_n is contained in Cb_n and the generators of P_n can be written in the form

$$a_{i,i+1} = \varepsilon_{i,i+1}^{-1}\varepsilon_{i+1,i}^{-1}, \qquad i = 1, \dots, n-1,$$
$$\begin{aligned} a_{ij} &= \varepsilon_{j-1,i}\varepsilon_{j-2,i}\dots\varepsilon_{i+1,i}(\varepsilon_{ij}^{-1}\varepsilon_{ji}^{-1})\varepsilon_{i+1,i}^{-1}\dots\varepsilon_{j-2,i}^{-1}\varepsilon_{j-1,i}^{-1} \\ &= \varepsilon_{j-1,j}^{-1}\varepsilon_{j-2,j}^{-1}\dots\varepsilon_{i+1,j}^{-1}(\varepsilon_{ij}^{-1}\varepsilon_{ji}^{-1})\varepsilon_{i+1,j}\dots\varepsilon_{j-2,j}\varepsilon_{j-1,j}, \quad 2 \leq i+1 < j \leq n. \end{aligned}$$

1.2. *The singular braid monoid.* The *Baez–Birman monoid* [1, 5] or the *singular braid monoid* SB_n is generated (as a monoid) by elements σ_i, σ_i^{-1}, τ_i, $i = 1, \dots, n-1$. The elements σ_i, σ_i^{-1} generate the braid group B_n. The generators τ_i satisfy the defining relations

$$\tau_i\tau_j = \tau_j\tau_i, \qquad |i-j| \geq 2, \tag{12}$$

and the other relations are mixed:

$$\tau_i\sigma_j = \sigma_j\tau_i, \qquad |i-j| \geq 2, \tag{13}$$
$$\tau_i\sigma_i = \sigma_i\tau_i, \qquad i = 1, \dots, n-1, \tag{14}$$
$$\sigma_i\sigma_{i+1}\tau_i = \tau_{i+1}\sigma_i\sigma_{i+1}, \qquad i = 1, \dots, n-2, \tag{15}$$
$$\sigma_{i+1}\sigma_i\tau_{i+1} = \tau_i\sigma_{i+1}\sigma_i, \qquad i = 1, \dots, n-2. \tag{16}$$

In [9] it was proved that the singular braid monoid SB_n is embedded into the group SG_n which is called the *singular braid group* and has the same defining relations as SB_n.

1.3. *The virtual braid group and welded braid group.* The virtual braid group VB_n was introduced in [15]. In [21] a shorter system of defining relations was found (see below). The group VB_n is generated by σ_i, ϱ_i, $i = 1, \dots, n-1$ (see Fig. 3).

The elements σ_i generate the braid group B_n with defining relations (1)–(2) and the elements ϱ_i generate the symmetric group S_n which is defined

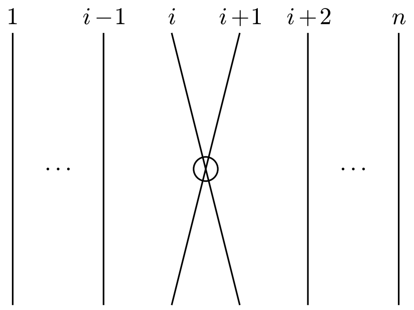

Fig. 3. The geometric virtual braid ϱ_i

by the relations

(17) $$\varrho_i \varrho_{i+1} \varrho_i = \varrho_{i+1} \varrho_i \varrho_{i+1}, \qquad i = 1, \ldots, n-2,$$

(18) $$\varrho_i \varrho_j = \varrho_j \varrho_i, \qquad |i-j| \geq 2,$$

(19) $$\varrho_i^2 = 1, \qquad i = 1, \ldots, n-1.$$

The other relations are mixed:

(20) $$\sigma_i \varrho_j = \varrho_j \sigma_i, \qquad |i-j| \geq 2,$$

(21) $$\varrho_i \varrho_{i+1} \sigma_i = \sigma_{i+1} \varrho_i \varrho_{i+1}, \qquad i = 1, \ldots, n-2.$$

Note that the last relation is equivalent to

$$\varrho_{i+1} \varrho_i \sigma_{i+1} = \sigma_i \varrho_{i+1} \varrho_i.$$

In [12] it was proved that the relations

$$\varrho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \varrho_{i+1}, \qquad \varrho_{i+1} \sigma_i \sigma_{i+1} = \varrho_i \sigma_{i+1} \sigma_i$$

are not satisfied in VB_n.

The welded braid group WB_n was introduced in [10]. This group is generated by σ_i, α_i, $i = 1, \ldots, n-1$. The elements σ_i generate the braid group B_n. The elements α_i generate the symmetric group S_n and the following mixed relations hold:

(22) $$\alpha_i \sigma_j = \sigma_j \alpha_i, \qquad |i-j| \geq 2,$$

(23) $$\sigma_i \alpha_{i+1} \alpha_i = \alpha_{i+1} \alpha_i \sigma_{i+1}, \qquad i = 1, \ldots, n-2,$$

(24) $$\sigma_{i+1} \sigma_i \alpha_{i+1} = \alpha_i \sigma_{i+1} \sigma_i, \qquad i = 1, \ldots, n-2.$$

In [10] it was proved that WB_n is isomorphic to the group C_n of conjugating automorphisms.

Comparing the defining relations of VB_n with the defining relations of WB_n, we see that WB_n can be obtained from VB_n by adding a new relation. Therefore, there exists a homomorphism

$$\varphi_{VW} : VB_n \to WB_n$$

taking σ_i to σ_i and ϱ_i to α_i for all i. Hence, WB_n is the homomorphic image of VB_n.

In [10] it was proved that the relation (symmetric to (23))

$$\sigma_{i+1}\alpha_i\alpha_{i+1} = \alpha_i\alpha_{i+1}\sigma_i,$$

is true in WB_n. But the following relation does not hold:

$$\alpha_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\alpha_i.$$

Linear representations of VB_n and WB_n by matrices from $\mathrm{GL}_n(\mathbb{Z}[t,t^{-1}])$ which extend the well known Burau representation were constructed in [21]. A linear representation of $C_n \simeq WB_n$ was constructed in [3]. This representation extends (with some conditions on parameters) the known Lawrence–Krammer representation.

2. Generators and defining relations of the virtual pure braid group. In this section we introduce a virtual pure braid group and find its generators and defining relations.

Define the map $\nu : VB_n \to S_n$ by its action on the generators:

$$\nu(\sigma_i) = \nu(\varrho_i) = \varrho_i, \qquad i = 1, \dots, n-1,$$

where S_n is the group generated by ϱ_i. Then $\ker(\nu)$ is called the *virtual pure braid group* and denoted by VP_n. It is clear that VP_n is a normal subgroup of index $n!$ of VB_n. Moreover, since $VP_n \cap S_n = e$ and $VB_n = VP_n \cdot S_n$, we have $VB_n = VP_n \rtimes S_n$.

Define

$$\lambda_{i,i+1} = \varrho_i\sigma_i^{-1}, \qquad \lambda_{i+1,i} = \varrho_i\lambda_{i,i+1}\varrho_i = \sigma_i^{-1}\varrho_i, \qquad i = 1, \dots, n-1,$$
$$\lambda_{ij} = \varrho_{j-1}\varrho_{j-2}\cdots\varrho_{i+1}\lambda_{i,i+1}\varrho_{i+1}\cdots\varrho_{j-2}\varrho_{j-1},$$
$$\lambda_{ji} = \varrho_{j-1}\varrho_{j-2}\cdots\varrho_{i+1}\lambda_{i+1,i}\varrho_{i+1}\cdots\varrho_{j-2}\varrho_{j-1}, \qquad 1 \le i < j-1 \le n-1.$$

Obviously, all these elements belong to VP_n. Their geometric interpretation is shown in Figs. 4 and 5.

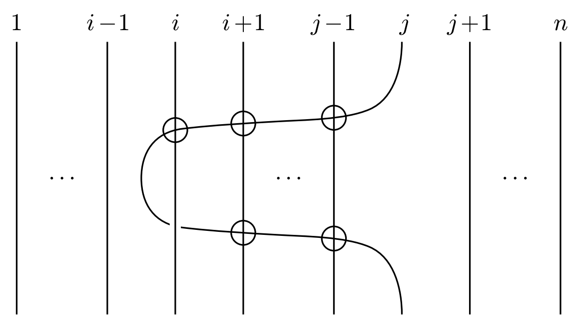

Fig. 4. The geometric virtual braid λ_{ij} $(1 \le i < j \le n)$

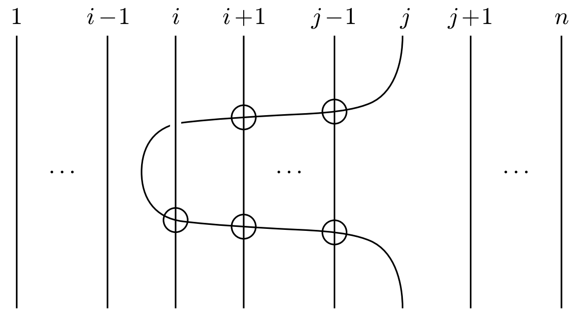

Fig. 5. The geometric virtual braid λ_{ji} $(1 \le i < j \le n)$

LEMMA 1. *Let $1 \le i < j \le n$. The following conjugation rules are satisfied in VB_n:*

(i) *for $k < i-1$, $i < k < j-1$ and $k > j$,*

$$\varrho_k \lambda_{ij} \varrho_k = \lambda_{ij}, \ \ \varrho_k \lambda_{ji} \varrho_k = \lambda_{ji};$$

(ii) $\varrho_{i-1} \lambda_{ij} \varrho_{i-1} = \lambda_{i-1,j}, \ \ \varrho_{i-1} \lambda_{ji} \varrho_{i-1} = \lambda_{j,i-1};$

(iii) *for $i < j-1$,*

$$\varrho_i \lambda_{i,i+1} \varrho_i = \lambda_{i+1,i}, \quad \varrho_i \lambda_{ij} \varrho_i = \lambda_{i+1,j},$$
$$\varrho_i \lambda_{i+1,i} \varrho_i = \lambda_{i,i+1}, \quad \varrho_i \lambda_{ji} \varrho_i = \lambda_{j,i+1};$$

(iv) *for $i+1 < j$,*

$$\varrho_{j-1} \lambda_{ij} \varrho_{j-1} = \lambda_{i,j-1}, \qquad \varrho_{j-1} \lambda_{ji} \varrho_{j-1} = \lambda_{j-1,i};$$

(v) $\varrho_j \lambda_{ij} \varrho_j = \lambda_{i,j+1}, \quad \varrho_j \lambda_{ji} \varrho_j = \lambda_{j+1,i}.$

Proof. We consider only the rules involving λ_{ij} for $i < j$ (the remaining ones can be considered analogously). Recall that

$$\lambda_{ij} = \varrho_{j-1} \varrho_{j-2} \cdots \varrho_{i+1} \lambda_{i,i+1} \varrho_{i+1} \cdots \varrho_{j-2} \varrho_{j-1}.$$

If $k < i-1$ or $k > j$ then ϱ_k is permutable with $\varrho_i, \varrho_{i+1}, \dots, \varrho_{j-1}$ in view of relation (18) and with σ_i in view of relation (20). Hence, ϱ_k is permutable with λ_{ij}.

Let $i < k < j-1$. Then

$$\varrho_k \lambda_{ij} \varrho_k = \varrho_k (\varrho_{j-1} \cdots \varrho_{k+2} \varrho_{k+1} \varrho_k \cdots \varrho_{i+1} \lambda_{i,i+1} \varrho_{i+1} \cdots \varrho_k \varrho_{k+1} \varrho_{k+2} \cdots \varrho_{j-1}) \varrho_k.$$

Permuting ϱ_k and $\lambda_{i,i+1}$ whenever possible, we get

$$\varrho_{j-1} \cdots \varrho_{k+2} (\varrho_k \varrho_{k+1} \varrho_k) \cdots \varrho_{i+1} \lambda_{i,i+1} \varrho_{i+1} \cdots (\varrho_k \varrho_{k+1} \varrho_k) \varrho_{k+2} \cdots \varrho_{j-1}.$$

Using the relation $\varrho_k \varrho_{k+1} \varrho_k = \varrho_{k+1} \varrho_k \varrho_{k+1}$, we rewrite the last formula as follows:

$$\varrho_{j-1} \cdots \varrho_{k+1} \varrho_k (\varrho_{k+1} \varrho_{k-1} \cdots \varrho_{i+1} \lambda_{i,i+1} \varrho_{i+1} \cdots \varrho_{k-1} \varrho_{k+1})$$
$$\times \varrho_k \varrho_{k+1} \cdots \varrho_{j-1} = \varrho_{j-1} \cdots \varrho_k (\varrho_{k+1} \lambda_{i,k} \varrho_{k+1}) \varrho_k \cdots \varrho_{j-1}.$$

In view of the case considered earlier, we have

$$\varrho_{k+1}\lambda_{ik}\varrho_{k+1} = \lambda_{ik}$$

and, hence,

$$\varrho_{j-1}\cdots\varrho_k(\varrho_{k+1}\lambda_{ik}\varrho_{k+1})\varrho_k\cdots\varrho_{j-1} = \lambda_{ij}.$$

Thus, the first rule of (i) is proven.

(ii) Consider

$$\varrho_{i-1}\lambda_{ij}\varrho_{i-1} = \varrho_{i-1}(\varrho_{j-1}\varrho_{j-2}\cdots\varrho_{i+1}\lambda_{i,i+1}\varrho_{i+1}\cdots\varrho_{j-2}\varrho_{j-1})\varrho_{i-1}.$$

Using (18), let us permute ϱ_{i-1} and $\lambda_{i,i+1}$ whenever possible. We get

$$\varrho_{i-1}\lambda_{ij}\varrho_{i-1} = \varrho_{j-1}\ldots\varrho_{i+2}\varrho_{i+1}(\varrho_{i-1}\lambda_{i,i+1}\varrho_{i-1})\varrho_{i+1}\varrho_{i+2}\ldots\varrho_{j-2}. \tag{25}$$

The expression in brackets can be rewritten as

$$\varrho_{i-1}\lambda_{i,i+1}\varrho_{i-1} = \varrho_{i-1}\varrho_i\sigma_i^{-1}\varrho_{i-1} = \varrho_{i-1}\varrho_i\sigma_i^{-1}\varrho_{i-1}\varrho_i\varrho_i.$$

Using the relation $\sigma_i^{-1}\varrho_{i-1}\varrho_i = \varrho_{i-1}\varrho_i\sigma_{i-1}^{-1}$ (following from (21)) and (18), (19), we obtain

$$\begin{aligned}\varrho_{i-1}\varrho_i(\sigma_i^{-1}\varrho_{i-1}\varrho_i)\varrho_i &= \varrho_{i-1}(\varrho_i\varrho_{i-1}\varrho_i)\sigma_{i-1}^{-1}\varrho_i\\ &= (\varrho_{i-1}\varrho_{i-1})\varrho_i\varrho_{i-1}\sigma_{i-1}^{-1}\varrho_i = \varrho_i\lambda_{i-1,i}\varrho_i.\end{aligned}$$

Then from (25) we obtain

$$\varrho_{i-1}\lambda_{ij}\varrho_{i-1} = \lambda_{i-1,j}.$$

Thus, the desired relations are proven.

(iii) The first formula follows from the definitions of $\lambda_{i,i+1}$ and $\lambda_{i+1,i}$. Consider

$$\varrho_i\lambda_{ij}\varrho_i = \varrho_i(\varrho_{j-1}\varrho_{j-2}\cdots\varrho_{i+1}\lambda_{i,i+1}\varrho_{i+1}\cdots\varrho_{j-2}\varrho_{j-1})\varrho_i.$$

Permuting ϱ_i and $\lambda_{i,i+1}$ whenever possible, we obtain

$$\varrho_i\lambda_{ij}\varrho_i = \varrho_{j-1}\cdots\varrho_{i+2}(\varrho_i\varrho_{i+1}\lambda_{i,i+1}\varrho_{i+1}\varrho_i)\varrho_{i+2}\cdots\varrho_{j-1}.$$

Rewrite the expression in brackets as follows:

$$\begin{aligned}\varrho_i\varrho_{i+1}\lambda_{i,i+1}\varrho_{i+1}\varrho_i &= \varrho_i\varrho_{i+1}\varrho_i(\sigma_i^{-1}\varrho_{i+1}\varrho_i) = \varrho_i\varrho_{i+1}(\varrho_i\varrho_{i+1}\varrho_i)\sigma_{i+1}^{-1}\\ &= \varrho_i\varrho_{i+1}\varrho_{i+1}\varrho_i\varrho_{i+1}\sigma_{i+1}^{-1} = \varrho_{i+1}\sigma_{i+1}^{-1}.\end{aligned}$$

Hence,

$$\varrho_i\lambda_{ij}\varrho_i = \varrho_{j-1}\ldots\varrho_{i+2}(\varrho_{i+1}\sigma_{i+1}^{-1})\varrho_{i+2}\ldots\varrho_{j-1} = \lambda_{i+1,j}.$$

Thus, the desired relations are proven.

(iv) follows from the relation $\varrho_{j-1}^2 = e$ and the definition of λ_{ij}.

(v) is an immediate consequence of the definition of λ_{ij}. ∎

COROLLARY 1. *The group S_n acts by conjugation on the set $\{\lambda_{kl} \mid 1 \leq k \neq l \leq n\}$. This action is transitive.*

In view of Lemma 1, the subgroup $\langle \lambda_{kl} \mid 1 \leq k \neq l \leq n \rangle$ of VP_n is normal in VB_n. Let us prove that this group coincides with VP_n and let us find its generators and defining relations. For this purpose we use the Reidemeister–Schreier method (see, for example, [16, Ch. 2.2]).

Let $m_{kl} = \varrho_{k-1}\varrho_{k-2}\ldots\varrho_l$ for $l < k$ and $m_{kl} = 1$ in the other cases. Then the set

$$\Lambda_n = \left\{ \prod_{k=2}^{n} m_{k,j_k} \,\middle|\, 1 \leq j_k \leq k \right\}$$

is a Schreier set of coset representatives of VP_n in VB_n.

THEOREM 1. *The group VP_n admits a presentation with the generators λ_{kl}, $1 \leq k \neq l \leq n$, and the defining relations*

$$\lambda_{ij}\lambda_{kl} = \lambda_{kl}\lambda_{ij}, \tag{26}$$

$$\lambda_{ki}(\lambda_{kj}\lambda_{ij}) = (\lambda_{ij}\lambda_{kj})\lambda_{ki}, \tag{27}$$

where distinct letters stand for distinct indices.

Proof. Define the map $^{-} : VB_n \to \Lambda_n$ which takes an element $w \in VB_n$ to its representative $\overline{w}$ from Λ_n. In this case the element $w\overline{w}^{-1}$ belongs to VP_n. By Theorem 2.7 of [16] the group VP_n is generated by

$$s_{\lambda,a} = \lambda a \cdot (\overline{\lambda a})^{-1},$$

where λ runs over the set Λ_n and a runs over the set of generators of VB_n.

It is easy to establish that $s_{\lambda,\varrho_i} = e$ for all representatives λ and generators ϱ_i. Consider the generators

$$s_{\lambda,\sigma_i} = \lambda\sigma_i \cdot (\overline{\lambda\varrho_i})^{-1}.$$

For $\lambda = e$ we get $s_{e,\sigma_i} = \sigma_i\varrho_i = \lambda_{i,i+1}^{-1}$. Note that $\lambda\varrho_i$ is equal to $\overline{\lambda\varrho_i}$ in S_n. Therefore,

$$s_{\lambda,\sigma_i} = \lambda(\sigma_i\varrho_i)\lambda^{-1}.$$

From Lemma 1 it follows that each generator s_{λ,σ_i} is equal to some λ_{kl}, $1 \leq k \neq l \leq n$. By Corollary 1, the converse is also true, i.e., each λ_{kl} is equal to some s_{λ,σ_i}. The first part of the theorem is proven.

To find defining relations of VP_n we define a rewriting process τ. It allows us to rewrite a word which is written in the generators of VB_n and present an element in VP_n as a word in the generators of VP_n. Let us associate to the reduced word

$$u = a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_\nu^{\varepsilon_\nu}, \qquad \varepsilon_l = \pm 1,\ a_l \in \{\sigma_1, \ldots, \sigma_{n-1}, \varrho_1, \ldots, \varrho_{n-1}\},$$

the word

$$\tau(u) = s_{k_1,a_1}^{\varepsilon_1} s_{k_2,a_2}^{\varepsilon_2} \cdots s_{k_\nu,a_\nu}^{\varepsilon_\nu}$$

in the generators of VP_n, where k_j is a representative of the $(j-1)$th initial segment of the word u if $\varepsilon_j = 1$, and a representative of the jth initial segment of u if $\varepsilon_j = -1$.

By [16, Theorem 2.9], the group VP_n is defined by the relations

$$r_{\mu,\lambda} = \tau(\lambda r_\mu \lambda^{-1}), \qquad \lambda \in \Lambda_n,$$

where r_μ is the defining relation of VB_n.

Denote by

$$r_1 = \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1}$$

the first relation of VB_n. Then

$$\begin{aligned} r_{1,e} = \tau(r_1) &= s_{e,\sigma_i} s_{\overline{\sigma_i},\sigma_{i+1}} s_{\overline{\sigma_i\sigma_{i+1}},\sigma_i} s^{-1}_{\overline{\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}},\sigma_{i+1}} s^{-1}_{\overline{\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}},\sigma_i} s^{-1}_{\overline{r_1},\sigma_{i+1}} \\ &= \lambda_{i,i+1}^{-1} (\varrho_i \lambda_{i+1,i+2}^{-1} \varrho_i)(\varrho_i \varrho_{i+1} \lambda_{i,i+1}^{-1} \varrho_{i+1} \varrho_i) \\ &\quad \cdot (\varrho_{i+1} \varrho_i \lambda_{i+1,i+2} \varrho_i \varrho_{i+1})(\varrho_{i+1} \lambda_{i,i+1} \varrho_{i+1}) \lambda_{i+1,i+2}. \end{aligned}$$

Using the conjugating rules from Lemma 1, we get

$$r_{1,e} = \lambda_{i,i+1}^{-1} \lambda_{i,i+2}^{-1} \lambda_{i+1,i+2}^{-1} \lambda_{i,i+1} \lambda_{i,i+2} \lambda_{i+1,i+2}.$$

Therefore, the relation

$$\lambda_{i,i+1} (\lambda_{i,i+2} \lambda_{i+1,i+2}) = (\lambda_{i+1,i+2} \lambda_{i,i+2}) \lambda_{i,i+1}$$

holds in VP_n. The remaining relations $r_{1,\lambda}$, $\lambda \in \Lambda_n$, can be obtained from this relation using conjugation by λ^{-1}. By the formulas of Lemma 1, we obtain (27).

Let us consider the next relation of VB_n:

$$r_2 = \sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1}, \qquad |i-j| \geq 2.$$

We have

$$r_{2,e} = \tau(r_2) = s_{e,\sigma_i} s_{\overline{\sigma_i},\sigma_j} s^{-1}_{\overline{\sigma_i\sigma_j\sigma_i^{-1}},\sigma_i} s^{-1}_{\overline{r_2},\sigma_j} = \lambda_{i,i+1}^{-1} \lambda_{j,j+1}^{-1} \lambda_{i,i+1} \lambda_{j,j+1}.$$

Hence, the relation

$$\lambda_{i,i+1} \lambda_{j,j+1} = \lambda_{j,j+1} \lambda_{i,i+1}, \qquad |i-j| \geq 2,$$

holds in VP_n. Conjugating this relation by all representatives from Λ_n, we obtain (26).

Let us prove that only trivial relations follow from all other relations of VB_n. This is evident for relations (17)–(19) defining the group S_n because $s_{\lambda,\varrho_i} = e$ for all $\lambda \in \Lambda_n$ and ϱ_i.

Consider the mixed relation (21) (relation (20) can be considered similarly):

$$r_3 = \sigma_{i+1} \varrho_i \varrho_{i+1} \sigma_i^{-1} \varrho_{i+1} \varrho_i.$$

Using the rewriting process, we get

$$r_{3,e} = \tau(r_3) = s_{e,\sigma_{i+1}} s^{-1}_{\sigma_{i+1}\varrho_i\varrho_{i+1}\sigma_i^{-1},\sigma_i} = \lambda^{-1}_{i+1,i+2}(\varrho_i\varrho_{i+1}\lambda_{i,i+1}\varrho_{i+1}\varrho_i)$$
$$= \lambda^{-1}_{i+1,i+2}\lambda_{i+1,i+2} = e.$$

Thus, VP_n is defined by relations (26)–(27). ∎

3. The structure of the virtual braid group. From the definition of VP_n and Lemma 1 it follows that $VB_n = VP_n \rtimes S_n$, i.e., VB_n is the splittable extension of the group VP_n by S_n. Consequently, we have to study the structure of the virtual pure braid group VP_n. Let us define the subgroups

$$V_i = \langle \lambda_{1,i+1}, \lambda_{2,i+1}, \dots, \lambda_{i,i+1}; \lambda_{i+1,1}, \lambda_{i+1,2}, \dots, \lambda_{i+1,i} \rangle, \qquad i = 1, \dots, n-1,$$

of VP_n. Each V_i is a subgroup of VP_{i+1}. Let V_i^* be the normal closure of V_i in VP_{i+1}. The following theorem is the main result of this section.

THEOREM 2. *The group VP_n, $n \geq 2$, is representable as the semi-direct product*

$$VP_n = V^*_{n-1} \rtimes VP_{n-1} = V^*_{n-1} \rtimes (V^*_{n-2} \rtimes (\dots \rtimes (V_2^* \rtimes V_1^*))\dots),$$

where V_1^* is a free group of rank 2 and V_i^*, $i = 2, \dots, n-1$, are free infinitely generated subgroups.

Let us prove the theorem by induction on n. For $n = 2$, we have

$$VP_2 = V_1 = V_1^*$$

and, by Theorem 1, the group V_1 is free generated by λ_{12} and λ_{21}.

To make the general case clearer consider the case $n = 3$.

3.1. *The structure of VP_3.* By Theorem 1, the group VP_3 is generated by the subgroups V_1, V_2 and defined by the relations

$$\lambda_{12}(\lambda_{13}\lambda_{23}) = (\lambda_{23}\lambda_{13})\lambda_{12}, \qquad \lambda_{21}(\lambda_{23}\lambda_{13}) = (\lambda_{13}\lambda_{23})\lambda_{21},$$
$$\lambda_{13}(\lambda_{12}\lambda_{32}) = (\lambda_{32}\lambda_{12})\lambda_{13}, \qquad \lambda_{31}(\lambda_{32}\lambda_{12}) = (\lambda_{12}\lambda_{32})\lambda_{31},$$
$$\lambda_{23}(\lambda_{21}\lambda_{31}) = (\lambda_{31}\lambda_{21})\lambda_{23}, \qquad \lambda_{32}(\lambda_{31}\lambda_{21}) = (\lambda_{21}\lambda_{31})\lambda_{32}.$$

From these relations we obtain the next lemma.

LEMMA 2. *In VP_3 the following equalities hold*:

(i)
$$\lambda_{13}^{\lambda_{12}} = \lambda_{32}^{\lambda_{12}}\lambda_{13}\lambda_{32}^{-1}, \quad \lambda_{31}^{\lambda_{12}} = \lambda_{32}\lambda_{31}\lambda_{32}^{-\lambda_{12}}, \quad \lambda_{23}^{\lambda_{12}} = \lambda_{13}\lambda_{23}\lambda_{32}\lambda_{13}^{-1}\lambda_{32}^{-\lambda_{12}},$$
$$\lambda_{13}^{\lambda_{12}^{-1}} = \lambda_{32}^{-1}\lambda_{13}\lambda_{32}^{\lambda_{12}^{-1}}, \quad \lambda_{31}^{\lambda_{12}^{-1}} = \lambda_{32}^{-\lambda_{12}^{-1}}\lambda_{31}\lambda_{32}, \quad \lambda_{23}^{\lambda_{12}^{-1}} = \lambda_{32}^{-\lambda_{12}^{-1}}\lambda_{13}^{-1}\lambda_{32}\lambda_{23}\lambda_{13},$$

(ii)
$$\lambda_{23}^{\lambda_{21}} = \lambda_{31}^{\lambda_{21}}\lambda_{23}\lambda_{31}^{-1}, \quad \lambda_{32}^{\lambda_{21}} = \lambda_{31}\lambda_{32}\lambda_{31}^{-\lambda_{21}}, \quad \lambda_{13}^{\lambda_{21}} = \lambda_{23}\lambda_{13}\lambda_{31}\lambda_{23}^{-1}\lambda_{31}^{-\lambda_{21}},$$
$$\lambda_{23}^{\lambda_{21}^{-1}} = \lambda_{31}^{-1}\lambda_{23}\lambda_{31}^{\lambda_{21}^{-1}}, \quad \lambda_{32}^{\lambda_{21}^{-1}} = \lambda_{31}^{-\lambda_{21}^{-1}}\lambda_{32}\lambda_{31}, \quad \lambda_{13}^{\lambda_{21}^{-1}} = \lambda_{31}^{-\lambda_{21}^{-1}}\lambda_{23}^{-1}\lambda_{31}\lambda_{13}\lambda_{23},$$

where a^b stands for $b^{-1}ab$.

Proof. The first and second relations of (i) immediately follow from the third and fourth relations of VP_3 (see the relations before the lemma). Similarly, the first and second relations of (ii) immediately follow from the fifth and sixth relations of VP_3.

Further, from the first and second relations of VP_3 we obtain

$$\lambda_{23}^{\lambda_{1,2}} = \lambda_{13}\lambda_{23}\lambda_{13}^{-\lambda_{12}}, \qquad \lambda_{13}^{\lambda_{21}} = \lambda_{23}\lambda_{13}\lambda_{23}^{-\lambda_{21}}.$$

Using the already proved formulas for $\lambda_{13}^{\lambda_{12}}$ and $\lambda_{23}^{\lambda_{21}}$, we get the third formulas of (i) and (ii) respectively.

The formulas for conjugation by λ_{12}^{-1} and λ_{21}^{-1} can be obtained analogously. ∎

Note that there exists an epimorphism $\varphi_3 : VP_3 \to VP_2$ which takes the generators of $V_2 = \langle \lambda_{13}, \lambda_{23}, \lambda_{31}, \lambda_{32} \rangle$ to the unit and fixes the generators of $V_1 = \langle \lambda_{12}, \lambda_{21} \rangle$. The kernel of this epimorphism is the normal closure of V_2 in VP_3, i.e., $\ker(\varphi_3) = V_2^*$.

Let u be the empty word or a reduced word beginning with a non-zero power of λ_{12} and representing an element from V_1. Let $\lambda_{32}(u) = \lambda_{32}^u = u^{-1}\lambda_{32}u$. We call this element the *reduced power of the generator* λ_{32} with power u. Analogously, if v is the empty word or a reduced word beginning with a non-zero power of λ_{21} and representing an element of V_1, then we put $\lambda_{31}(v) = \lambda_{13}^v$ and call it the *reduced power of the generator* λ_{31} with power v.

Lemma 3. *The group V_2^* is a free group with generators λ_{13}, λ_{23} and all reduced powers of λ_{31} and λ_{32}.*

Proof. To prove the lemma we can use the Reidemeister–Schreier method, but it is easier to use the definitions of normal closure and semi-direct product. Clearly, the group V_2^* is generated by the elements

$$\lambda_{13}^w,\ \lambda_{23}^w,\ \lambda_{31}^w,\ \lambda_{32}^w, \qquad w \in V_1.$$

In view of Lemma 2, it is sufficient to take from these elements only λ_{13}, λ_{23} and all reduced powers of the generators λ_{31} and λ_{32}.

The freedom of V_2^* follows from the representation of VP_3 as a semi-direct product. Indeed, since $V_1 \cap V_2^* = e$, $V_1V_2^* = VP_3$, it follows that $VP_3 = V_2^* \rtimes V_1$. In this case the defining relations of VP_3 are equivalent to the conjugating rules of Lemma 2. Therefore, all relations define the action of the group V_1 on the group V_2^*. Since there are no other relations, this means that V_1 and V_2^* are free groups. ∎

As a consequence of this lemma, we obtain the normal form of words in VP_3. Any element w from VP_3 can be written in the form $w = w_1w_2$, where w_1 is a reduced word over the alphabet $\{\lambda_{12}^{\pm1}, \lambda_{21}^{\pm1}\}$ and w_2 is a reduced word over the alphabet $\{\lambda_{13}^{\pm1}, \lambda_{23}^{\pm1}, \lambda_{31}(u)^{\pm1}, \lambda_{32}(v)^{\pm1}\}$, where $\lambda_{31}(u)$, $\lambda_{32}(v)$ are reduced powers of the generators λ_{31} and λ_{32} respectively.

3.2. *The proof of Theorem 2.* Let λ_{ij}^* denote either λ_{ij} or λ_{ji} from VP_n.

LEMMA 4. *For every $n \geq 2$ there exists a homomorphism $\varphi_n : VP_n \to VP_{n-1}$ which takes the generators λ_{ij}^*, $i = 1, \dots, n-1$, to the unit and fixes the other generators.*

Proof. It is sufficient to prove that all defining relations go to the defining relations under φ. For the defining relations of VP_{n-1} this is evident. If the commutativity relation (see (26)) contains some generator of V_{n-1} then φ_n turns it into the trivial relation. Consider the left hand side of (27). We see that it contains every index two times. Hence, if this part includes some generator of V_{n-1} (i.e., one of the indices is n) then some other generator involves the index n. Therefore, there are two generators of V_{n-1} on the left hand side of the relation. Since the right hand side contains all generators from the left hand side, φ_n turns this relation into the trivial relation. ■

LEMMA 5. *The following formulas are satisfied in the group VP_n:*

(i) $\lambda_{kl}^{\lambda_{ij}^{\varepsilon}} = \lambda_{kl}$, $\max\{i,j\} < \max\{k,l\}$, $\varepsilon = \pm 1$;

(ii) $\lambda_{ik}^{\lambda_{ij}} = \lambda_{kj}^{\lambda_{ij}} \lambda_{ik} \lambda_{kj}^{-1}$, $\lambda_{ik}^{\lambda_{ij}^{-1}} = \lambda_{kj}^{-1} \lambda_{ik} \lambda_{kj}^{\lambda_{ij}^{-1}}$, $i < j < k$ *or* $j < i < k$;

(iii) $\lambda_{ki}^{\lambda_{ij}} = \lambda_{kj} \lambda_{ki} \lambda_{kj}^{-\lambda_{ij}}$, $\lambda_{ki}^{\lambda_{ij}^{-1}} = \lambda_{kj}^{-\lambda_{ij}^{-1}} \lambda_{ki} \lambda_{kj}$, $i < j < k$ *or* $j < i < k$;

(iv) $\lambda_{jk}^{\lambda_{ij}} = \lambda_{ik} \lambda_{jk} \lambda_{kj} \lambda_{ik}^{-1} \lambda_{kj}^{-\lambda_{ij}}$, $\lambda_{jk}^{\lambda_{ij}^{-1}} = \lambda_{jk}^{-\lambda_{ik}^{-1}} \lambda_{ij}^{-1} \lambda_{jk} \lambda_{kj} \lambda_{ij}$, $i < j < k$ *or* $j < i < k$,

where, as usual, different letters stand for different indices.

Proof. (i) immediately follows from the first relation of Theorem 1.

Consider relation (27) from Theorem 1:

$$\lambda_{ki}(\lambda_{kj}\lambda_{ij}) = (\lambda_{ij}\lambda_{kj})\lambda_{ki}.$$

Note that the indices of the generators are connected by one of the inequalities:

$$\text{(a) } k < j < i, \quad \text{(b) } j < k < i, \quad \text{(c) } i < j < k,$$
$$\text{(d) } j < i < k, \quad \text{(e) } k < i < j, \quad \text{(f) } i < k < j.$$

In cases (a) and (b), from (27) we obtain

$$\lambda_{ki}^{\lambda_{kj}} = \lambda_{ij}^{\lambda_{kj}} \lambda_{ki} \lambda_{ij}^{-1},$$

which is the first formula of (ii).

In cases (c) and (d) we obtain

$$\lambda_{ki}^{\lambda_{ij}} = \lambda_{kj} \lambda_{ki} \lambda_{kj}^{-\lambda_{ij}},$$

which is the first formula of (iii).

In cases (e) and (f),

$$\lambda_{ij}^{\lambda_{ki}} = \lambda_{kj} \lambda_{ij} \lambda_{kj}^{-\lambda_{ki}}.$$

Using (ii), we obtain

$$\lambda_{ij}^{\lambda_{ki}} = \lambda_{kj}\lambda_{ij}\lambda_{ji}\lambda_{kj}^{-1}\lambda_{ji}^{-\lambda_{ki}},$$

which is the first formula of (iv).

The formulas for conjugations by λ_{ij}^{-1} can be established similarly. ∎

Assume that the theorem is proven for the group VP_{n-1}. Hence, any element $w \in VP_{n-1}$ can be written in the form

$$w = w_1 w_2 \dots w_{n-2}, \qquad w_i \in V_i^*,$$

where each w_i is a reduced word over the alphabet consisting of the generators $\lambda_{ki}^{\pm 1}$, $1 \leq k \leq i-1$, and reduced powers of the generators λ_{ki}, $1 \leq k \leq i-1$, and their inverses. Let us define reduced powers of generators in the group V_{n-1}^*. We say that the element $\lambda_{nk}(w) = \lambda_{nk}^{w}$ is the *reduced power of the generator* λ_{nk} if w is the empty word or a word written in the normal form and beginning with a reduced power of some generator λ_{lk} or its inverse.

The statement about decomposition as the semi-direct product $VP_n = V_n^* \rtimes VP_{n-1}$ is quite evident. It remains to find generators of V_n^* and prove its freedom.

LEMMA 6. *The group V_{n-1}^* is free. It is generated by $\lambda_{1n}, \lambda_{2n}, \dots, \lambda_{n-1,n}$ and all reduced powers of the generators $\lambda_{n1}, \lambda_{n2}, \dots, \lambda_{n,n-1}$.*

Proof. The proof is similar to that of Lemma 3. From Lemma 5 it follows that the indicated elements generate V_{n-1}^*. Further, since the set of defining relations of VP_n is equivalent to the set of conjugating formulas defining the action of VP_{n-1} on V_{n-1}^*, only trivial relations are satisfied in V_{n-1}^*. ∎

Theorem 2 follows from these results.

As a consequence, we obtain the normal form of words in VB_n.

COROLLARY 2. *Every element of VB_n can be uniquely written in the form*

$$w = w_1 \dots w_{n-1}\lambda, \qquad \lambda \in \Lambda_n,\ w_i \in V_i^*,$$

where w_i is a reduced word in the generators, reduced powers of the generators and their inverses.

The homomorphism defined above of the virtual braid group onto the welded braid group agrees with the decomposition from Theorem 2 and with the decomposition of $C_n \simeq WB_n$ described in the first section.

COROLLARY 3. *The homomorphism $\varphi_{VW} : VB_n \to WB_n$ agrees with the decompositions of these groups, i.e., it maps VP_n onto $Cb_n \simeq WP_n$ and the factors V_i^* onto the factors D_i, $i = 1, \dots, n-1$.*

4. The universal braid group. Let us define the *universal braid group* UB_n as the group with generators $\sigma_1, \dots, \sigma_{n-1}$, $c_1, \dots, c_{n-1}$, defining relations (1)–(2), the relations

$$c_i c_j = c_j c_i, \qquad |i - j| \geq 2,$$

and the mixed relations

$$c_i \sigma_j = \sigma_j c_i, \qquad |i - j| \geq 2.$$

Recall (see [6]) that *Artin's group* of the type I is the group A_I with generators a_i, $i \in I$, and defining relations

$$a_i a_j a_i \ldots = a_j a_i a_j \ldots, \qquad i, j \in I,$$

where the words on the left and right hand sides consist of m_{ij} alternating letters a_i and a_j.

PROPOSITION 1. (i) *UB_n has the braid group B_n as a subgroup.*

(ii) *There exist surjective homomorphisms*

$$\varphi_{US} : UB_n \to SG_n, \qquad \varphi_{UV} : UB_n \to VB_n, \qquad \varphi_{UB} : UB_n \to B_n.$$

(iii) *UB_n is Artin's group.*

Proof. (i) Evidently, there exists a homomorphism $B_n \to UB_n$. On the other hand, setting $\psi(\sigma_i) = \sigma_i$, $\psi(c_i) = e$, $i = 1, \dots, n-1$, we obtain a retraction ψ of UB_n onto B_n. Therefore, the subgroup $\langle \sigma_1, \dots, \sigma_{n-1} \rangle$ of UB_n is isomorphic to the braid group B_n.

(ii) Define the map φ_{US} as follows:

$$\varphi_{US}(\sigma_i) = \sigma_i, \qquad \varphi_{US}(c_i) = \tau_i, \qquad i = 1, \dots, n-1.$$

Comparing the defining relations of UB_n and SG_n, we see that this map is a homomorphism. Analogously, we can show that the map

$$\sigma_i \longmapsto \sigma_i, \qquad c_i \longmapsto \varrho_i,$$

extends to a homomorphism φ_{UV}, and the map

$$\sigma_i \longmapsto \sigma_i, \qquad c_i \longmapsto e,$$

extends to a homomorphism φ_{UB}.

(iii) immediately follows from the defining relations of UB_n and the definition of Artin's group. ∎

It should be noted that none of the groups SG_n, VB_n, WB_n (in the natural presentations) is Artin's group.

The following questions naturally arise in the context of the results obtained above.

PROBLEMS. (i) Solve the word and conjugacy problems in UB_n, $n > 2$.

(ii) Is it possible to give some geometric interpretation for elements of UB_n similar to the geometric interpretation for elements of the braid groups B_n, SG_n, VB_n, UB_n?

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