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Cyclic branched coverings and homology 3-spheres with large group actions

by

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Abstract. We show that, if the covering involution of a 3-manifold M occurring as the 2-fold branched covering of a knot in the 3-sphere is contained in a finite nonabelian simple group G of diffeomorphisms of M, then M is a homology 3-sphere and G isomorphic to the alternating or dodecahedral group $\mathbb{A}_5 \cong \mathrm{PSL}(2,5)$. An example of such a 3-manifold is the spherical Poincaré sphere. We construct hyperbolic analogues of the Poincaré sphere. We also give examples of hyperbolic $\mathbb{Z}_2$-homology 3-spheres with $\mathrm{PSL}(2,q)$-actions, for various small prime powers q. We note that the groups $\mathrm{PSL}(2,q)$, for odd prime powers q, are the only candidates for being finite nonabelian simple groups which possibly admit actions on $\mathbb{Z}_2$-homology 3-spheres (but the exact classification remains open).

1. Introduction. By [Z1], the only finite nonabelian simple group acting on a homology 3-sphere is the alternating or dodecahedral group $\mathbb{A}_5 \cong \mathrm{PSL}(2,5)$. Up to conjugation, there are two orthogonal actions of $\mathbb{A}_5$ on the 3-sphere, in each case the quotient orbifold is the 3-sphere, and the singular sets of the two actions are the graphs shown in Figures 1a and b (where edges are labelled with their branching orders, and edges without label have branching order two). Apart from this, the most prominent example of a homology 3-sphere with an $\mathbb{A}_5$-action is the Poincaré sphere; again the quotient orbifold is the 3-sphere, and the singular set the tetrahedral graph shown in Figure 1c. The Poincaré sphere is a spherical manifold, and the group $\mathbb{A}_5$ acts by isometries. The cyclic subgroups of orders two, three and five of $\mathbb{A}_5$, unique up to conjugation, are the covering groups of the Poincaré sphere as the 2-fold branched covering of the $(3,5)$-torus knot, the 3-fold cyclic branched covering of the $(2,5)$-torus knot and the 5-fold cyclic branched covering of the $(2,3)$-torus knot; on the other hand, the isometry group of the Poincaré sphere is isomorphic to the orthogonal group $SO(3)$, and the

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three covering groups are conjugate in the isometry group of the Poincaré sphere to subgroups of a single cyclic group of order 30, and consequently these three knots are related by the second standard abelian construction described in [RZ1]. We note that, in general, it is quite a rare phenomenon that the same 3-manifold occurs as a cyclic branched covering of different knots in S^3 (see [RZ1], [RZ2]).

Our first result is the following

THEOREM 1. *Let M be the 2-fold branched covering of a knot in the 3-sphere such that the covering involution is an element of a finite nonabelian simple group G of diffeomorphisms of M. Then M is a homology 3-sphere, and G is isomorphic to the dodecahedral group $\mathbb{A}_5$.*

We believe that Theorem 1 also remains true for p-fold cyclic branched coverings, for primes $p > 2$.

The Poincaré sphere is the only spherical homology 3-sphere (and, as part of the 3-manifold geometrization program, conjectured to be the only homology 3-sphere with finite fundamental group). There are no euclidean homology 3-spheres (in fact, not a single example of a closed euclidean homology sphere seems to be known in any dimension). There are many hyperbolic homology 3-spheres, and the question arises if there exists one with properties similar to the Poincaré sphere. We have the following

THEOREM 2. *There exists a hyperbolic homology 3-sphere M (of approximate volume 32.01607) which is the 2-fold branched covering of a knot K_2 in S^3 and the 3-fold cyclic branched covering of a knot K_3 in S^3, such that the covering groups generate a group $\mathbb{A}_5$ of isometries of M. The quotient orbifold $M/\mathbb{A}_5$ is the 3-sphere, and its singular set is the Kuratowski graph shown in Figure 1d (the complete bi-partite graph on six vertices).*

In the language of [RZ3] this means that the hyperbolic 3-manifold M has many hidden symmetries with respect to both the 2-fold and the 3-fold cyclic branched coverings (i.e., symmetries which are not lifts of symmetries of the knots K_2 resp. K_3; this is again a quite exceptional situation). The manifold M is also the 5-fold cyclic branched covering of a knot K_5 in a homology 3-sphere different from S^3; at present we do not have an example of a hyperbolic homology 3-sphere with an $\mathbb{A}_5$-action such that all three knots K_2, K_3 and K_5 are knots in S^3 (as in the case of the Poincaré sphere).

We note that the covering groups of K_2 and K_3 in Theorem 2 can be chosen to lie in a dihedral subgroup $\mathbb{D}_3$ of order six of $\mathbb{A}_5$, and that consequently we are in the standard dihedral situation described in [RZ2] for two knots with the same 2-fold resp. 3-fold cyclic branched covering.

In Section 5, we describe other hyperbolic homology 3-spheres with an isometric $\mathbb{A}_5$-action which occur as cyclic branched coverings of knots in the 3-sphere; one of these is a 3-fold and also a 5-fold cyclic branched covering of knots in the 3-sphere. This seems to be the first example of two knots in the 3-sphere which have the same hyperbolic 3-manifold as a cyclic branched covering but which are not related in a solvable way (i.e., the covering groups cannot be chosen to generate a solvable group of isometries of the manifold; see also [RZ1]).

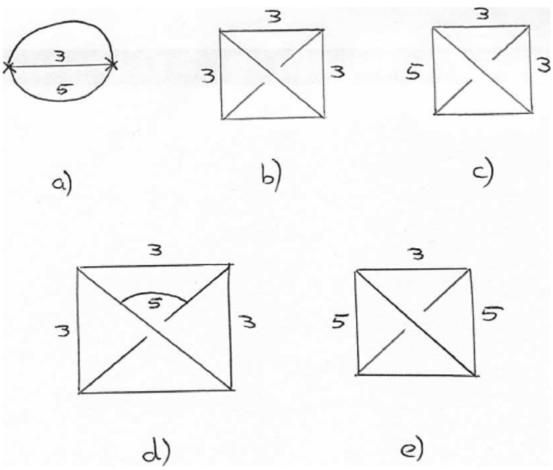

Fig. 1

The best known hyperbolic 3-manifold with an isometric $\mathbb{A}_5$-action is the Seifert–Weber dodecahedral manifold. The singular set of the quotient orbifold by the $\mathbb{A}_5$-action is the tetrahedral graph shown in Figure 1e; the fact that the edges of order five are disconnected implies that the Seifert–Weber manifold is not a $\mathbb{Z}_5$-homology 3-sphere (i.e., with the homology of the 3-sphere with coefficients in the integers mod five; in fact, by [Br, Theorem 7.9] the fixed point set of a periodic transformation of order five of a $\mathbb{Z}_5$-homology 3-sphere is connected). The Seifert–Weber manifold is a 5-fold cyclic branched covering of the 2-component Whitehead link (which again implies that it is not a $\mathbb{Z}_5$-homology 3-sphere; in fact, its first homology is $(\mathbb{Z}_5)^3$); it is also the 2-fold and 3-fold cyclic branched covering of knots K_2 resp. K_3 in lens spaces with fundamental group $\mathbb{Z}_5$.

Considering the more general class of $\mathbb{Z}_2$-homology 3-spheres, it is shown in [MZ] that the only finite simple groups which possibly admit actions on $\mathbb{Z}_2$-homology 3-spheres are the linear fractional groups $\mathrm{PSL}(2,q)$, for odd prime powers q; however, it remains open for which odd prime powers q such actions really occur. In Section 4, we give examples of $\mathrm{PSL}(2,q)$-actions on $\mathbb{Z}_2$-homology 3-spheres, for various small values of q. These examples are obtained as regular branched coverings of suitable 3-orbifolds, and we calculate the homology of these coverings by computer using the group-theory package GAP (we thank M. Conder for his help with these examples). We think that all such groups $\mathrm{PSL}(2,q)$ admit actions on $\mathbb{Z}_2$-homology 3-spheres but have no method to prove this at present.

2. Proof of Theorem 1. It is well known that, for any prime power p^n, the p^n-fold cyclic branched covering of a knot in S^3 is a $\mathbb{Z}_p$-homology 3-sphere (see e.g. [G]), so M is a $\mathbb{Z}_2$-homology 3-sphere. It follows from [MZ, Theorem 1] that a finite simple group G which admits an action on a $\mathbb{Z}_2$-homology 3-sphere is isomorphic to a linear fractional groups $\mathrm{PSL}(2,q)$, for an odd prime power q. Alternatively, one may apply [RZ4], where it is shown that a finite simple group G acting on a closed 3-manifold and containing an involution h with nonempty connected fixed point set is isomorphic to a group $\mathrm{PSL}(2,q)$, for an odd prime power q.

So we can assume that $G = \mathrm{PSL}(2,q)$, where q is an odd prime power. In $G = \mathrm{PSL}(2,q)$, all involutions are conjugate, and in particular conjugate to the covering involution h. Then the quotient of M by any involution in G is the 3-sphere with a knot as branch set.

A Sylow 2-subgroup of $G = \mathrm{PSL}(2,q)$ is a dihedral group which contains a subgroup $U \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The quotient of M by each of the three involutions h_1, h_2 and h_3 in U is the 3-sphere. The group U projects to an involution $\bar{h}$ of $M/h_1 \cong S^3$, with nonempty fixed point set. By the solution of the Smith conjecture for involutions ([W]), the involution $\bar{h}$ of S^3 is standard (conjugate to an orthogonal one), and hence $S^3/\bar{h} = M/U$ is again the 3-sphere. So the space of the quotient orbifold M/U is S^3, and its singular set is a θ-curve T in S^3 (i.e. a graph with two vertices and three connecting edges), with branching order two associated to the three edges (because h_2 and h_3 act as strong inversions on the fixed point set of h_1). The three edges of such a θ-curve define three knots and hence three 2-fold branched coverings of S^3 along these knots. In our situation, these three 2-fold branched coverings are the 3-manifolds M/h_1, M/h_2 and M/h_3, all homeomorphic to the 3-sphere. Also, M is the regular $\mathbb{Z}_2 \times \mathbb{Z}_2$-covering of S^3 branched along the θ-curve T.

By [N1] or [N2], the first homology of the regular $\mathbb{Z}_2 \times \mathbb{Z}_2$-covering of a θ-curve in the 3-sphere is the product of the first homologies of the three 3-manifolds which are the 2-fold branched coverings associated to the three constituent knots of the θ-curve. In our case, we get three times the 3-sphere, which implies that M is an integer homology 3-sphere. By [Z1], the only finite simple group acting on an integer homology 3-sphere is the alternating or dodecahedral group $\mathbb{A}_5 \cong \mathrm{PSL}(2,5)$.

This finishes the proof of Theorem 1.

3. Proof of Theorem 2. We denote by $\mathcal{K}$ the 3-orbifold whose space is the 3-sphere and whose singular set is the Kuratowski graph shown in Figure 1d resp. 2a. Using the orbifold geometrization theorem ([BP], [CHK]) we will show that $\mathcal{K}$ is a hyperbolic orbifold (alternatively, one may try to construct explicitly its universal covering group in hyperbolic 3-space).

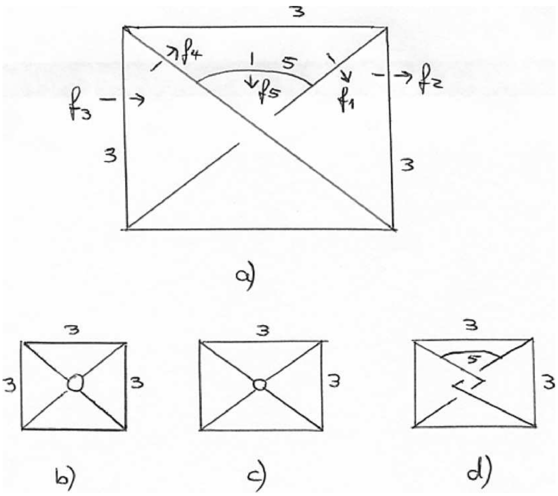

Fig. 2

We consider the spherical 3-orbifold $\mathcal{O}$ whose space is the 3-sphere and whose singular set is the graph shown in Figure 1b. Let $p : S^3 \to \mathcal{O}$ be its univeral covering, with covering group $\mathbb{A}_5$ acting orthogonally on the 3-sphere. Note that the singular graph of the orbifold $\mathcal{K}$ is obtained by adjoining the unique edge $\bar{L}$ of branching order five to the singular graph of $\mathcal{O}$ (see Figures 1d and 1b). The complement of a regular neighbourhood of the edge $\bar{L}$ in $\mathcal{K}$ is the pyramidal orbifold $\widetilde{\mathcal{P}}$ with one cusp whose singular set is shown in Figure 2b. Now $\widetilde{\mathcal{P}}$ is a complete hyperbolic 3-orbifold of finite volume; in fact, $\widetilde{\mathcal{P}}$ is a 2-fold (branched) covering of the Picard orbifold $\mathcal{P}$ shown in Figure 2c, i.e. the quotient of hyperbolic 3-space by the Picard group $\mathrm{PSL}(2, \mathbb{Z}[i])$ (see e.g. [F]).

Let L be the link in S^3 which is the preimage of the edge $\bar{L}$ under the covering $p : S^3 \to \mathcal{O}$; the complement of L is hyperbolic since it is a finite cover of the hyperbolic orbifolds $\widetilde{\mathcal{P}}$ and $\mathcal{P}$, so L is a hyperbolic link invariant under the $\mathbb{A}_5$-action on S^3 (at present we do not have an explicit projection of this "dodecahedral link" L; the volume of its complement is 36.63852, which is 120 times the volume 0.305321 of the Picard orbifold). Let $\mathcal{L}_5$ be the closed 3-orbifold with singular set L obtained by associating branching order five to all components of the link L; by the orbifold geometrization theorem, and in particular [BP, Corollary 3], $\mathcal{L}_5$ is a hyperbolic orbifold. By a second application of the orbifold geometrization theorem we can suppose that the group $\mathbb{A}_5$ of symmetries of the link L acts by hyperbolic isometries of the orbifold $\mathcal{L}_5$. The quotient of $\mathcal{L}_5$ by the $\mathbb{A}_5$-action is the orbifold $\mathcal{K}$, so also $\mathcal{K}$ is a hyperbolic orbifold (at the end of this section, we will compute the volume of $\mathcal{K}$).

We will calculate a presentation of the orbifold fundamental group $\pi_1\mathcal{K}$ of $\mathcal{K}$. The fundamental group of the complement in S^3 of the singularity graph of $\mathcal{K}$ can be computed from its projection in Figure 2a by the Wirtinger algorithm, in a similar way to the fundamental group of the complement of a knot in S^3, calculated from a projection of the knot. The fundamental group of the complement is generated by the meridians of all edges of $\mathcal{K}$; in Figure 2a, we have indicated some of these meridian generators. For each vertex and each crossing of the singular set we obtain a relation (as in the case of knots or links one of these relations is redundant). Then a presentation of the orbifold fundamental group $K := \pi_1\mathcal{K}$ is obtained by raising all meridian generators to the powers given by the branching orders. Using the relations to eliminate some of the meridian generators, we obtain the following presentation of K:

$$\langle f_1, f_2, f_3, f_4, f_5 \mid f_1^2, f_2^3, f_3^3, f_4^2, f_5^5, (f_1f_2)^3, f_3f_4f_1f_2, (f_4f_5)^2, (f_1f_5)^2, (f_4f_5f_2)^2\rangle.$$

We define a surjection $\phi : K \to \mathbb{A}_5$ by

$$\phi(f_1) = (1,4)(2,3), \quad \phi(f_2) = (2,4,3), \quad \phi(f_3) = (2,3,4),$$
$$\phi(f_4) = (1,4)(2,3), \quad \phi(f_5) = (1,2,3,4,5).$$

The kernel U of ϕ is torsionfree (the torsion elements of K are conjugate to the meridians of the singular edges of $\mathcal{K}$). Now $M := \mathbb{H}^3/U$ is a closed hyperbolic 3-manifold, with fundamental group isomorphic to U, and with an induced action of $\mathbb{A}_5 \cong K/U$ such that $M/\mathbb{A}_5 = \mathbb{H}^3/K = \mathcal{K}$. Using computational techniques, and in particular the group theory package GAP, we find that the abelianization of $U \cong \pi_1 M$ is trivial, which implies that M is an integer homology 3-sphere.

We consider the involution $h = \phi(f_1)$ in $\mathbb{A}_5$ (note that all involutions in $\mathbb{A}_5$ are conjugate). The fixed point set of h is nonempty (because $\mathcal{K}$ has singular edges of order two), and by [Br, Theorem 7.9] the fixed point set of h is connected (because M is a homology 3-sphere). Then M is a 2-fold branched covering of the 3-manifold $M_2 := M/h$, branched along a knot K_2 in M_2. We denote by U_2 the subgroup of K generated by U and the element f_1 of order two; then U is a subgroup of index two of U_2, and $M_2 = M/h = \mathbb{H}^3/U_2$. By [A], the fundamental group of M_2 is obtained by dividing out the torsion of U_2, that is, by taking the quotient of U_2 by the normal subgroup generated by f_1.

GAP produces a presentation of the kernel U of ϕ with 10 generators (given as words in the generators f_i of K) and 16 relations. We add f_1 to these generators and obtain a generating system of the subgroup U_2 of K. GAP produces a presentation of U_2 with 11 generators and 15 relations. Then a presentation of the fundamental group of M_2 is obtained by adding the relator f_1 (that is, the relation $f_1 = 1$) to this presentation. Using Tietze transformations, GAP simplifies this presentation to the trivial presentation, so M_2 has trivial fundamental group. The normalizer of h in $\mathbb{A}_5$ contains involutions different from h; these project to involutions of M_2 with nonempty fixed point set. By the orbifold geometrization theorem, M_2 is the 3-sphere, so M is the 2-fold branched covering of a knot K_2 in S^3.

Starting the construction with the element $h = \phi(f_2)$ of order three in $\mathbb{A}_5$, one shows in a completely analogous way that M is also the 3-fold cyclic branched covering of a knot K_3 in S^3. We note that M is also the 5-fold cyclic branched covering of a knot K_5 in a homology 3-sphere M_5, which, however, has nontrivial fundamental group. At present, we do not have projections of the knots K_2 and K_3.

Finally, we compute the volume of the hyperbolic homology 3-sphere M. The orbifold $\mathcal{K}$ is obtained by $(5,5)$-surgery on the cusp of the orbifold $\widetilde{\mathcal{P}}$, and $\mathcal{K}$ is a 2-fold branched covering of the orbifold $\mathcal{P}(5,10)$ shown in Figure 2d, obtained by $(5,10)$-surgery on the cusp of the Picard orbifold $\mathcal{P}$ (see [MVZ]; see also [DM] for the notion of surgery on orbifolds). The complement of the Borromean rings $\mathcal{B}$ in S^3 is a regular 24-fold covering, with covering group $\mathbb{S}_4$, of the Picard orbifold (see e.g. [MVZ]). By [MVZ, Theorem 2.1 and Remark (A)], the hyperbolic orbifold $\mathcal{K} = \mathcal{P}(5,10)$ has the hyperbolic cone manifold $\mathcal{B}(15/2, 5/4)$, obtained by $(15/2, 5/4)$-surgery on each component of the Borromean rings, as a regular $\mathbb{S}_4$-covering. The volume of such a cone manifold can be computed by Weeks' program SnapPea, and one finds the approximate volume $v = 6.403214$ for $\mathcal{B}(15/2, 5/4)$. Then $\mathcal{K}$ has volume $2v/24$, and the regular $\mathbb{A}_5$-covering M of $\mathcal{K}$ has volume $120v/24 = 32.01607$.

This finishes the proof of Theorem 2. There remains the following

PROBLEM. Find regular projections of the dodecahedral link L, and of the knots K_2 and K_3.

4. $\mathrm{PSL}(2,q)$**-actions on** $\mathbb{Z}_2$**-homology 3-spheres.** By [MZ, Thm. 1], the only finite simple groups which possibly admit actions on $\mathbb{Z}_2$-homology 3-spheres are the linear fractional groups $\mathrm{PSL}(2,q)$, for odd prime powers q. In the following, we show how to construct such actions, for various small values of q.

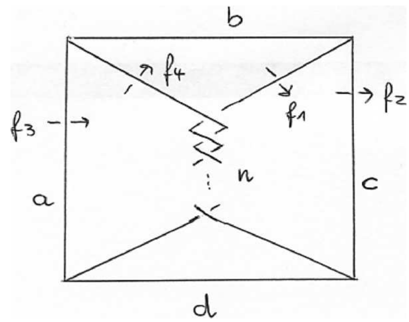

Fig. 3

We consider the orbifold $\mathcal{O}(a,b,c,d;n)$ whose space is the 3-sphere and whose singular set is shown in Figure 3 (where a, b, c and d denote branching orders and n the number of half twists between the diagonal edges). Using the Wirtinger algorithm as in Section 2, one obtains the following presentation of the orbifold fundamental group of $\mathcal{O}(a,b,c,d;n)$ (in the meridian generators indicated in Figure 3):

$$\langle f_1, f_2, f_3, f_4 \mid f_1^2, f_2^c, f_3^b, f_4^2, (f_1f_2)^b, f_3f_4f_1f_2, ((f_4f_1)^n f_1f_2)^d\rangle.$$

Let F_1 denote the orbifold fundamental group of $\mathcal{O}(3,3,3,2;2)$. It follows from the orbifold geometrization theorem that $\mathcal{O}(3,3,3,2;2)$ is hyperbolic (by the lists in [D] it does not belong to any of the other 3-dimensional geometries). By the Low Index Subgroup Program of GAP we compute the subgroups up to index 30 of F_1, their normal cores (the largest normal subgroups contained in them), the abelianizations of the cores and their quotient groups. Among other things, we find the following finite quotients G and abelianizations of the corresponding cores (a sequence of integers stands for the product of cyclic groups of these orders). The cores are torsionfree and define hyperbolic 3-manifolds with isometric G-actions; note that, in the cases listed below, these 3-manifolds are $\mathbb{Z}_2$-homology 3-spheres (equivalently, the abelianizations of their fundamental groups are finite of odd order).

PSL(2,7) (order 168):	$[7,7,7] =: [7^3]$;
PSL(2,11) (order 660):	$[11^3]$;
PSL(2,13) (order 1092):	$[13^3]$;
PSL(2,17) (order 2448):	$[3^{16}, 17^3]$;
PSL(2,23) (order 6072):	$[5^{23}, 23^3]$.

Now let F_2 denote the fundamental group of the tetrahedral orbifold $\mathcal{O}(3,5,3,2;1)$ (its singular set is the 1-skeleton of a hyperbolic tetrahedron, so this orbifold is also hyperbolic). By applying GAP as above we obtain the following quotients G and abelianizations of the corresponding cores:

PSL(2, 9) (order 360):	$[3,3,3,3,3,3] = [3^6]$;
PSL(2, 19) (order 3420):	$[3^{19}, 19^3]$;
PSL(2, 29) (order 12180):	$[3^{28}, 5^5, 29^3]$.

Using other orbifolds of type $\mathcal{O}(a,b,c,d;n)$ one easily finds more examples of $\mathrm{PSL}(2,q)$-actions on $\mathbb{Z}_2$-homology 3-spheres. We close with the

Conjecture. (i) For all odd prime powers q, the linear fractional group $\mathrm{PSL}(2,q)$ acts on some $\mathbb{Z}_2$-homology 3-sphere.
(ii) All finite groups of odd order act on some $\mathbb{Z}_2$-homology 3-sphere.

5. Other examples of hyperbolic homology 3-spheres with isometric $\mathbb{A}_5$-actions. Considering the orbifolds $\mathcal{O}(a,b,c,d;n)$ in Section 4 and Figure 3, and surjections of their fundamental groups onto $\mathbb{A}_5$, we find the following examples of hyperbolic homology 3-spheres with isometric $\mathbb{A}_5$-actions.

5.1. Let F denote the orbifold fundamental group $\pi_1\mathcal{O}(5,3,3,2;3)$ of the hyperbolic 3-orbifold $\mathcal{O}(5,3,3,2;3)$ (see Section 4); then F admits a surjection onto $\mathbb{A}_5$, and the kernel defines a hyperbolic 3-manifold M with isometric $\mathbb{A}_5$-action. Using GAP as in Sections 3 and 4, we have checked that M is a homology 3-sphere, and that M is the 3-fold and 5-fold cyclic branched covering of knots in the 3-sphere. M is also the 2-fold branched covering of a knot in a homology 3-sphere which seems to have nontrivial fundamental group (we do not have a proof of this at the moment). In any case, the covering groups of the 3-fold and 5-fold cyclic branched coverings generate the isometric $\mathbb{A}_5$-action, and no conjugates of them generate a solvable subgroup of the isometry group of M (this follows easily from the list in [MZ] of the possible nonsolvable finite groups acting on homology 3-spheres). So this seems to be the first example of a hyperbolic 3-manifold which is the 3-fold and 5-fold cyclic branched covering of knots in the 3-sphere which are not related in a solvable way (see also [RZ1]).

5.2. Now let $F = \pi_1\mathcal{O}(2,5,3,3;9)$. Again F admits a surjection onto $\mathbb{A}_5$ whose kernel is a hyperbolic homology 3-sphere M with isometric $\mathbb{A}_5$-action; now M is the 2-fold and 3-fold cyclic branched covering of knots in the 3-sphere (and the 5-fold cyclic branched covering of a knot in a homology 3-sphere), and these two knots are related by a standard dihedral construction. We note that the manifold M is a maximally symmetric $\mathbb{A}_5$-manifold in the sense of [Z2]: it admits a Heegaard splitting of genus $g = 6$ invariant under the $\mathbb{A}_5$-action, where $\mathbb{A}_5$ realizes the maximal order $12(g-1)$ of a finite group of orientation-preserving diffeomorphisms of a handlebody of genus 6.

5.3. Other examples of hyperbolic homology 3-spheres with $\mathbb{A}_5$-actions can be obtained by the orbifolds $\mathcal{O}(2,3,3,3;5)$, $\mathcal{O}(2,3,3,3;21)$ and $\mathcal{O}(3,3,3,2;4)$. Again the first two manifolds are maximally symmetric $\mathbb{A}_5$-manifolds.

5.4. Let $\mathcal{K}(3,3,3,2;n)$ denote the series of 3-orbifolds in Figure 4 parametrized by the number of half-twists n which simultaneously generalizes the orbifolds $\mathcal{O}(3,3,3,2;n)$ and the Kuratowski orbifold $\mathcal{K}$ of Section 3. Our impression (checked for many small values of n) is that for all $n \geq 0$ there is a surjection of the orbifold fundamental group onto $\mathbb{A}_5$ whose kernel has trivial abelianization and hence defines a homology 3-sphere with $\mathbb{A}_5$-action (for $n = 0$ one gets a homology 3-sphere which is the $\mathbb{A}_5$-equivariant connected sum of two copies of the Poincaré sphere, by identifying the boundaries of regular neighbourhoods of the unique global fixed points of the two $\mathbb{A}_5$-actions).

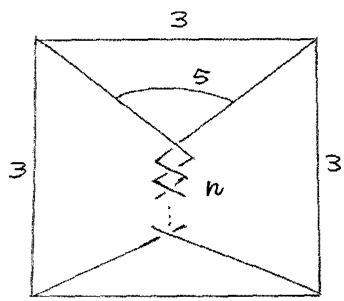

Fig. 4

5.5. Let p be an odd prime. Except for the ubiquitous $\mathbb{A}_5$ we have not found a single example of a nonabelian simple group G acting on a 3-manifold M which is a p-fold cyclic branched covering of a knot or link in the 3-sphere whose covering group is contained in G. The closest we found in this direction is the following. The fundamental group of the hyperbolic tetrahedral orbifold $\mathcal{O}(3,5,3,2;1)$ admits a surjection onto the alternating or linear fractional group $\mathbb{A}_6 \cong \mathrm{PSL}(2,9)$ of order 360 such that the kernel has abelianization $(\mathbb{Z}_3)^6$ and hence defines a hyperbolic $\mathbb{Z}_2$-homology 3-sphere M with an isometric $\mathbb{A}_6$-action. The Sylow 3-subgroup of $\mathbb{A}_6$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$, and we have checked as in Section 3 (i.e. using the orbifold geometrization theorem) that the quotient $M/\mathbb{Z}_3 \times \mathbb{Z}_3$ is the 3-sphere, so M is a regular $\mathbb{Z}_3 \times \mathbb{Z}_3$-covering of a link in the 3-sphere (note that the involution corresponding to the lower edge of the tetrahedral orbifold $\mathcal{O}(3,5,3,2;1)$ normalizes the two elements of order 3 corresponding to the two vertical edges). It would be interesting to have an example of this kind with a cyclic covering group but we believe that such an example does not exist if G is different from $\mathbb{A}_5$.

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