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On tame embeddings of solenoids into 3-space

by

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Abstract. Solenoids are inverse limits of the circle, and the classical knot theory is the theory of tame embeddings of the circle into 3-space. We make a general study, including certain classification results, of tame embeddings of solenoids into 3-space, seen as the "inverse limits" of tame embeddings of the circle.

Some applications in topology and in dynamics are discussed. In particular, there are tamely embedded solenoids $\Sigma \subset \mathbb{R}^3$ which are strictly achiral. Since solenoids are non-planar, this contrasts sharply with the known fact that if there is a strictly achiral embedding $Y \subset \mathbb{R}^3$ of a compact polyhedron Y, then Y must be planar.

1. Introduction and motivations. The classical knot theory is the theory of tame embeddings of the circle into 3-space, which has become a central topic in mathematics. The classical theory of knots has many generalizations and variations: from the circle to graphs, from the circle to higher dimensional spheres, from tame embeddings to wild embeddings, and so on. In the present note, we try to set up a beginning of another generalization: tame embeddings of solenoids into 3-space. In such a study topology and dynamics interact well.

Solenoids were first defined in topology by Vietoris in 1927 for the 2-adic case [V] and by many others later for the general case, and introduced into dynamics by Smale in 1967 [S]. Solenoids can be presented either in a rather geometric way (intersections of nested solid tori, see Definition 2.2) or in a more algebraic way (inverse limits of self-coverings of the circle, see Definition 2.1), or in a dynamical way (mapping tori over the Cantor set, see [Mc]).

The precise definition of tame embedding of solenoids into the 3-space $\mathbb{R}^3$ will be given in §2, but the intuition is quite naive. Recall we identify S^1 with the centerline of the solid torus $S^1 \times D^2$, and say an embed-

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ding $S^1 \subset \mathbb{R}^3$ is *tame* if the embedding can be extended to an embedding $S^1 \times D^2 \subset \mathbb{R}^3$. Similarly we consider a solenoid Σ as the nested intersection of solid tori (the defining sequence of Σ), and say an embedding $\Sigma \subset \mathbb{R}^3$ is *tame* if the embedding can be extended to an embedding of those solid tori into $\mathbb{R}^3$.

Solenoids themselves are usually considered to be "wild" sets. What motivated us originally to study the tame embeddings of solenoids was the question of finding a non-planar set which admits a strictly achiral embedding into 3-space.

An embedding $A \subset \mathbb{R}^3$ is called *strictly achiral* if A stays in the fixed point set of an orientation reversing homeomorphism $r : \mathbb{R}^3 \to \mathbb{R}^3$. Obviously any planar set has a strictly achiral embedding. Indeed there is a simple relation between achirality and planarity for compact polyhedra: If there is a strictly achiral embedding $Y \subset \mathbb{R}^3$ of a compact polyhedron Y, then Y must be planar [JW]. It is natural to ask whether this is still true for continua (i.e. compact, connected metric spaces).

Solenoids are promising candidates for the question, because on the one hand they are continua realized as inverse limits of planar sets, while on the other hand they are non-planar themselves (see [Bin], and [JWZ] for a quick proof). In order to design a strictly achiral embedding $\Sigma \subset \mathbb{R}^3$ for a solenoid Σ, we need careful and deep discussions about tame embeddings of solenoids.

Another justification is that when a solenoid $\Sigma_f \subset S^3$ is realized as a hyperbolic (expanding) attractor of a diffeomorphism f on S^3 (called a Smale solenoid), the embedding $\Sigma_f \subset S^3$ is automatically tame. Our study also gives some application in this aspect.

The contents of the paper are as follows.

In §2, we give the precise definitions of tame solenoids and related notions. We present a lemma about convergence of homeomorphisms which will be repeatedly used in the paper. For comparison we also construct examples of non-tame embeddings of solenoids.

In §3, we classify the tame solenoids in the 3-sphere. The classification is based on an important notion, the "maximal" defining sequences of tame solenoids. We also give applications including (1) the knotting, linking and invariants of tame solenoids; (2) there are uncountably many unknotted 2-adic tame solenoids; (3) up to conjugacy by homeomorphisms of S^3, the number of Smale solenoids in S^3 with winding number w is finite if $|w| \leq 3$, and is infinite otherwise (see [RW] for some discussion in higher dimensions).

In §4, we give criteria of when a tame solenoid in the 3-sphere is achiral or strictly achiral in terms of its defining sequence, and construct strictly achiral tame solenoids to fulfil our original motivation. Indeed we give a simple criterion for when a solenoid has a strictly achiral embedding into the 3-sphere.

All undefined terminology is standard. For 3-manifolds, see [Ja]; for knot theory see [A]; and for braid theory, see [Bir].

2. Tame embeddings of solenoids, preliminaries

2.1. Definitions of solenoids and their tame embeddings. Let $N = D^2 \times S^1$ be the solid torus, where D^2 is the unit disk and S^1 is the unit circle. Then N admits a standard metric. A *meridian disk* of N is a D^2 slice of N. A *framing* of N is a circle on ∂N which meets each meridian disk of N in exactly one point.

DEFINITION 2.1. (1) For a sequence of maps $\{\phi_n : X_n \to X_{n-1}\}_{n\geq 1}$ between continua, the *inverse limit* is defined to be the subspace

$$\Sigma = \{(x_0, x_1, \dots, x_n, \dots) \mid x_n \in X_n,\ x_{n-1} = \phi_n(x_n)\}$$

of the product space $\prod_{n=0}^{\infty} X_n$.

(2) The inverse limit of a sequence $\{\phi_n : S^1 \to S^1\}_{n\geq 1}$ of covering maps, where ϕ_n is of degree $w_n \neq 0$, is called a *solenoid* of type $\varpi = (w_1, w_2, \dots)$.

DEFINITION 2.2. (1) Call an embedding $e : N \to \operatorname{int} N$, or just its image $e(N)$, a *thick braid of winding number* w if e preserves the D^2-fibration and descends to a covering map $S^1 \to S^1$ given by $e^{it} \mapsto e^{iwt}$, that is, $p \circ e = c_w \circ p$, where $p : N \to S^1$ denotes the projection, and $c_w : S^1 \to S^1$ is the covering of degree w. Note that the composition of finitely many thick braids is also a thick braid.

(2) Let $\{e_n : N \to N\}_{n\geq 1}$ be an infinite sequence of thick braids of winding numbers $w_n \neq 0$. Let $\psi_n = e_n \circ \dots \circ e_1$ and $N_n = \psi_n(N)$. Then we have an infinite sequence $N = N_0 \supset N_1 \supset N_2 \supset \cdots$ of thick braids. If the diameter of the meridian disk of N_n tends to zero uniformly as $n \to \infty$ then we call $\Sigma = \bigcap_{n\geq 0} \psi_n(N) = \bigcap_{n>0} N_n$ a *solenoid* of type $\varpi = (w_1, w_2, \dots)$. For a prime number p, we use $\tau_p(\varpi)$ to denote the sum of the exponents of p in w_n, $n > 1$ (which could be 0 or ∞).

There is quite a rich theory of solenoids developed in the 1960-1990's. We just list some basic facts (see [Bin], [Mc], [R] and references therein):

THEOREM 2.3.

(1) *The above two definitions of solenoids are equivalent; each solenoid is determined by its type* ϖ.
(2) *Two solenoids* Σ *and* Σ' *of types* ϖ *and* ϖ' *respectively are homeomorphic if and only if* $\tau_p(\varpi) = \tau_p(\varpi')$ *for almost all primes* p*, except possibly for finitely many* p *with* $\tau_p(\varpi) \neq \infty \neq \tau_p(\varpi')$*. In particular* Σ *is the circle if and only if all except finitely many* w_n *are* ± 1.

(3) *Each solenoid Σ is connected, compact and has topological dimension one. Moreover, if Σ is not the circle, then Σ has uncountably many path components.*

We assume below that all winding numbers involved in the definition of solenoid are greater than 1, unless otherwise specified. In particular, a solenoid is not the circle.

DEFINITION 2.4. (1) In Definition 2.2, call $\{N_n\}_{n\geq 0}$ a *defining sequence* of the solenoid Σ, and call $\Sigma \subset N$ a *standard embedding* of Σ in the solid torus N.

(2) $\Sigma \subset N$ is called a *tame* embedding of Σ in the solid torus N if there is a homeomorphism $f : (N, \Sigma) \to (N, \Sigma')$ for some standard embedding $\Sigma' \subset N$; then call $\{f^{-1}(N'_n)\}_{n\geq 0}$ a *defining sequence* of Σ, where $\{N'_n\}_{n>0}$ is a defining sequence of Σ'.

(3) An embedding $\Sigma \subset S^3$ of a solenoid is called *tame* if it can be factored as $\Sigma \subset N \subset S^3$ in which $\Sigma \subset N$ is tame; then each defining sequence $\{N_n\}_{n\geq 0}$ of $\Sigma \subset N$ is also considered as a defining sequence of $\Sigma \subset S^3$, and we have $S^3 \supset N = N_0 \supset N_1 \supset N_2 \supset \cdots \supset \Sigma$.

REMARK 2.5. From Definition 2.4, for each defining sequence $\{N_n\}_{n\geq 0}$ of a tame solenoid in S^3, we always assume that the D^2-slices of all N_i are coherent.

DEFINITION 2.6. Call two tame solenoids $\Sigma, \Sigma' \subset S^3$ *equivalent* if there is an orientation preserving homeomorphism $f : S^3 \to S^3$ such that $f(\Sigma) = \Sigma'$.

DEFINITION 2.7. Say two defining sequences $\{N_n\}_{n\geq 0}$ and $\{N'_n\}_{n\geq 0}$ of tame solenoids in S^3 are *strongly equivalent* if there is an orientation preserving homeomorphism $f_0 : (S^3, N_0) \to (S^3, N'_0)$ and orientation preserving homeomorphisms $f_n : (N_{n-1}, N_n) \to (N'_{n-1}, N'_n)$ with $f_n|_{\partial N_{n-1}} = f_{n-1}|_{\partial N_{n-1}}$ for $n \geq 1$. Say $\{N_n\}_{n\geq 0}$ and $\{N'_n\}_{n\geq 0}$ are *equivalent* if $\{N_{k+n}\}_{n\geq 0}$ and $\{N'_{k'+n}\}_{n\geq 0}$ are strongly equivalent for some non-negative integers k and k'.

REMARK 2.8. (1) A defining sequence $\{N_n\}_{n\geq 0}$ of a tame solenoid $\Sigma \subset S^3$ carries information on the braiding of N_n in N_{n-1} and the knotting of N_n in S^3. The winding numbers w_n, the simplest invariant of the braiding of N_n in N_{n-1}, give rise to the type of the abstract solenoid Σ.

(2) Suppose $\Sigma \subset S^3$ is a tame embedding given by a defining sequence $\{N_n\}_{n\geq 0}$; then any infinite subsequence of $\{N_n\}_{n\geq 0}$ is a defining sequence of the same embedding.

(3) Definitions 2.4–2.7 also apply to other 3-manifolds in the obvious way.

In this paper we view each braid β as drawn in $D^2 \times [0,1]$ with ends staying in $D^2 \times \{0,1\}$. The closure $\overline{\beta} \subset D^2 \times S^1$ is obtained by identifying $D^2 \times 0$ and $D^2 \times 1$ via the identity. Conversely, for each closed braid $\overline{\beta} \subset D^2 \times S^1$, cutting $D^2 \times S^1$ open along a D^2-slice yields a braid $\beta \subset D^2 \times [0,1]$ up to conjugacy. Therefore, we have a 1-1 correspondence between the set of closed braids of winding number n and the set of conjugacy classes of braids on n strands.

Note that a tubular neighborhood of a closed braid in $D^2 \times S^1$ is a thick braid, and conversely, a framing of a thick braid in $D^2 \times S^1$ is a closed braid.

2.2. Convergence of homeomorphisms. The following lemma will be used repeatedly in this paper.

LEMMA 2.9. *Let X, Y be compact metric spaces and $\{f_n : X \to Y\}_{n\geq 0}$ be a sequence of homeomorphisms. If there exist subsets $\{U_n\}_{n\geq 0}$ of X and positive numbers $\{\varepsilon_n\}_{n\geq 0}$ such that*

(1) *$U_n \subset U_{n-1}$ and $f_n|_{X\setminus U_{n-1}} = f_{n-1}|_{X\setminus U_{n-1}}$,*
(2) *$\lim_{n\to\infty} \varepsilon_n = 0$ and $d(f_m(x), f_n(x)) \leq \varepsilon_n$, $d(f_m^{-1}(y), f_n^{-1}(y)) \leq \varepsilon_n$ for $x \in X$, $y \in Y$, $m \geq n$,*
(3) *neither $\bigcap_{n\geq 0} U_n$ nor $\bigcap_{n\geq 0} f_n(U_n)$ has interior points,*

then f_n uniformly converges to a homeomorphism $f : (X, \bigcap_{n\geq 0} U_n) \to (Y, \bigcap_{n\geq 0} f_n(U_n))$. Moreover, if in addition

(4) *$X = Y$ and $d(f_n(x), x) \leq \varepsilon_n$ for $x \in U_n$,*

then $\bigcap_{n\geq 0} U_n$ lies in the fixed point set of f.

Proof. Since both X and Y are compact, it follows from (2) that f_n and f_n^{-1} uniformly converge to continuous maps $f : X \to Y$ and $g : Y \to X$, respectively. By (1), we have $f|_{X\setminus U_n} = f_n|_{X\setminus U_n}$ and $g|_{Y\setminus f(U_n)} = f_n^{-1}|_{Y\setminus f(U_n)}$, hence $f|_{X\setminus \bigcap_{n\geq 0} U_n}$ is the inverse of $g|_{Y\setminus \bigcap_{n\geq 0} f_n(U_n)}$.

Since $X \setminus \bigcap_{n\geq 0} U_n$ is dense in X by (3), on which the continuous map $gf : X \to X$ acts as the identity, it follows that $gf = \mathrm{id}_X$. Similarly, we have $fg = \mathrm{id}_Y$. So the conclusion follows.

The "moreover" part is clear. ∎

2.3. Non-tame embeddings of solenoids. As there are non-tame embeddings of the circle into 3-space, there are non-tame embeddings of solenoids. In fact, any tame embedding of a solenoid can be modified in a simple way to a non-tame embedding, which we illustrate below by a concrete example.

Let $\Sigma \subset \mathbb{R}^3$ be a tame embedding of a 2-adic solenoid with defining sequence $\{N_0, N_1, N_2, \ldots\}$, where the core of N_i is unknotted, and N_{i+1} is a 2-cable in N_i, $i = 0, 1, 2, \ldots$. Then there are two disjoint meridian disks D, D' of N_0 bounding a cylinder $U_0 \subset N_0 \subset \mathbb{R}^3$ such that $U_i = N_i \cap U_0$

consists of 2^i cylinders. For each i, pick a component of U_i denoted by U_i' such that U_j' is disjoint from U_i' for each $j > i$.

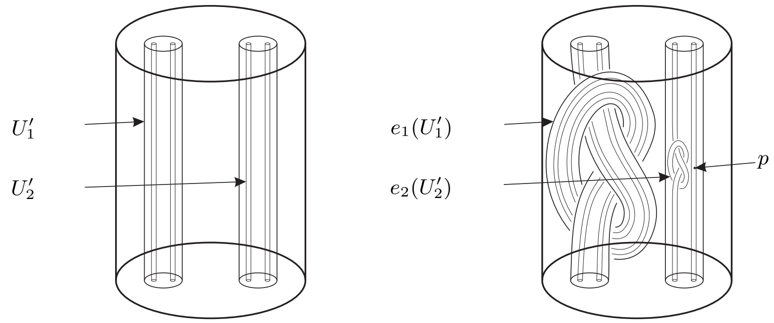

Fig. 1

On the left of Figure 1, we see the embedding sequence $U_0 \supset U_1 \supset U_2 \supset \cdots$, and U_i''s are selected. We define a re-embedding $e : \Sigma \subset \mathbb{R}^3$ by assembling the re-embeddings $\{e_n : U_n' \subset U_0\}_{n\geq 1}$ relative to $\{D, D'\}$, where e_n replaces an unknotted middle portion of U_n' by a knotted one. The knotted portion of $e_n(U_n')$ should sit in a 3-ball $B_n \subset U_{n-1}$, with the 3-balls arranged to converge to a single point p. The situation is indicated on the right of Figure 1. Note that for each small open 3-ball neighborhood B of p there is an arc component of $B \cap \Sigma$ which is knotted in B, which implies that the new embedding is not tame, since tame embeddings do not have this property.

3. Classification and applications

3.1. Maximal defining sequences. As can be easily seen, a tame solenoid $\Sigma \subset N$ has a lot of defining sequences and there is no way to choose a "minimal" one. However, a "maximal" defining sequence of a tame solenoid can be defined in the sense that any other defining sequence is equivalent to a subsequence of it (see Proposition 3.2).

Before giving the precise definition we fix some notation. For a properly embedded surface S (resp. an embedded 3-manifold P) in a 3-manifold M, we use $M \setminus S$ (resp. $M \setminus P$) to denote the manifold obtained by splitting M along S (resp. removing the interior of P).

Let M be a compact orientable 3-manifold. Call an embedded torus T in M *essential* if T is incompressible (see [Ja, p. 23]) and is not parallel to any component of ∂M.

According to the theory of Johannson and Jaco–Shalen ([Ja], [Joh], see also [Ha]), for every compact orientable irreducible 3-manifold M, there is a minimal union $\Gamma \subset M$ of disjoint essential tori of M, unique up to isotopy, such that each piece of $M \setminus \Gamma$ is either Seifert fibered (see [Ja, Chapter VI]) or contains no essential torus.

Below we will call the components of Γ the *JSJ-decomposition tori* of M.

Definition 3.1. Call a defining sequence $\{N_n\}_{n\geq 0}$ of a tame solenoid $\Sigma \subset S^3$ *maximal* if $N_n \setminus N_{n+1}$ contains no essential torus for each $n \geq 0$.

Proposition 3.2. *Every tame solenoid $\Sigma \subset S^3$ has a maximal defining sequence. Moreover, if $\{N_n\}_{n\geq 0}$ is a maximal defining sequence of Σ then every defining sequence of Σ is equivalent to a subsequence of $\{N_n\}_{n\geq 0}$. Hence all maximal defining sequences of Σ are equivalent.*

The following lemma will be repeatedly used in this paper.

Lemma 3.3. *Suppose N' is a thick braid in N and Γ is the union of the JSJ-decomposition tori of $N \setminus N'$. Then*

(1) *each component T of Γ bounds a solid torus N^* such that N^* is a thick braid in N and N' is a thick braid in N^*;*
(2) *no component of $(N \setminus N') \setminus \Gamma$ contains an essential torus;*
(3) *for each solid torus N'' with $N' \subset N'' \subset N$, $\partial N''$ is isotopic in $N \setminus N'$ to a component of $\Gamma \cup \partial N \cup \partial N'$.*

Proof. (1) Let w be the winding number of N' in N and let D be a meridian disk of N which meets N' in w meridian disks of N'. Then $(N \setminus N') \setminus D \cong P_w \times [0,1]$, where P_w is the w-punctured disk. Isotope T so that $T \cap D$ has a minimum number of components. Then a standard argument in 3-manifold topology shows that each component of $T \setminus D$ in $P_w \times [0,1]$ is a vertical annulus which separates a vertical $D^2 \times [0,1]$ from $N \setminus D$. Therefore T bounds a solid torus N^* which is a thick braid in N and clearly N' is a thick braid in N^*.

(2) Let Q be a component of $(N \setminus N') \setminus \Gamma$. By JSJ theory [Ja], Q is either simple, hence contains no essential torus by definition, or a Seifert piece. Suppose Q is a Seifert piece. Then it is also a Seifert piece of a knot complement with incompressible boundary. According to [Ja, Lemma IX.22], Q is either a torus knot space, or a $P_w \times S^1$ where P_w is the w-punctured disk with $w \geq 2$, or a cable space (see [Ja, p. 188]). Since ∂Q has at least two components, Q is not a torus knot space. By (1) and the fact that N' is connected, one can verify that there is no embedding of $P_w \times S^1$ in $N \setminus N'$ with incompressible boundary for $w \geq 2$. Therefore Q is a cable space. It is known that a cable space contains no essential torus.

(3) By (2), it suffices to show that $\partial N''$ is incompressible in $N \setminus N'$. Suppose $\partial N''$ has a compressing disk D in $N \setminus N'$. Then after surgery on D we will get a separating 2-sphere S^2 in $N \setminus N'$ such that each component of $(N \setminus N') \setminus S^2$ contains a boundary torus, which contradicts $N \setminus N'$ being irreducible. ∎

Proof of Proposition 3.2. Let $\{N_n\}_{n\geq 0}$ be a defining sequence of $\Sigma \subset S^3$. Then $S^3 \supset N = N_0 \supset N_1 \supset N_2 \supset \cdots \supset \Sigma = \bigcap_{n=1}^{\infty} N_n$ and

$$N_0 \setminus N_1 \subset N_0 \setminus N_2 \subset \cdots \subset N_0 \setminus \Sigma = \bigcup_{n=1}^{\infty} N_0 \setminus N_n.$$

Moreover, since the winding number of N_{n+1} in N_n is greater than 1 for every n, ∂N_n is not parallel to ∂N_j for $n \neq j$.

Now we refine the defining sequence $\{N_n\}_{n\geq 0}$ to a maximal one. Let Γ_n be the union of the JSJ-decomposition tori of $N_n \setminus N_{n+1}$. Note that the set $\bigcup_{0\leq n<j} \Gamma_n \cup \bigcup_{0<n<j} \partial N_n$ is the union of the JSJ-decomposition tori of $N_0 \setminus N_j$. Define $\Gamma(\Sigma) = \bigcup_{n\geq 0}(\Gamma_n \cup \partial N_n)$. By Lemma 3.3(1)&(2) we can re-index the components of $\Gamma(\Sigma)$ as $\{T_n\}_{n\geq 0}$ so that

- each T_n bounds a solid torus N_n^*,
- each N_{n+1}^* is a thick braid in N_n^*,
- no $N_n^* \setminus N_{n+1}^*$ contains an essential torus.

Clearly $\{N_n\}_{n\geq 0}$ is a subsequence of $\{N_n^*\}_{n\geq 0}$ and by definition $\{N_n^*\}_{n\geq 0}$ is a maximal defining sequence of Σ.

Let $\{N'_n\}_{n\geq 0}$ be another maximal defining sequence of $\Sigma \subset S^3$. We shall show that $\{N'_n\}_{n\geq 0}$ and $\{N_n^*\}_{n\geq 0}$ are equivalent. Since Σ is compact, we have $N_j^* \subset N'_k \subset N_0^*$ for some large integers j, k. By Lemma 3.3(3) and the construction of $\{N_n^*\}_{n\geq 0}$, $\partial N'_k$ is isotopic in $N_0^* \setminus N_j^*$ to some ∂N_m^*. Clearly, the isotopy automatically sends N'_k to N_m^*.

Similarly, one argues that N'_{k+1} can be further isotoped in N_m^* relative to Σ to some $N_{m'}^*$, where m' must be $m+1$ because $N'_k \setminus N'_{k+1}$ contains no essential torus, and so on. Hence we verify that $\{N'_{k+n}\}_{n\geq 0}$ is strongly equivalent to $\{N_{m+n}^*\}_{n\geq 0}$ and the conclusion follows. ■

3.2. Classification of tame solenoids

THEOREM 3.4. *Let $\Sigma, \Sigma' \subset S^3$ be two tame solenoids. The following statements are equivalent.*

(1) *Σ, Σ' are equivalent.*
(2) *Some defining sequences of Σ, Σ' are equivalent.*
(3) *The maximal defining sequences of Σ, Σ' are equivalent.*

Proof. (2)⇒(1). Without loss of generality, suppose the defining sequences $\{N_n\}_{n\geq 0}$, $\{N'_n\}_{n\geq 0}$ of Σ, Σ' are strongly equivalent. By definition there is an orientation preserving homeomorphism $f_0 : (S^3, N_0) \to (S^3, N'_0)$ and orientation preserving homeomorphisms $f_n : (N_{n-1}, N_n) \to (N'_{n-1}, N'_n)$ with $f_n|_{\partial N_{n-1}} = f_{n-1}|_{\partial N_{n-1}}$ for $n \geq 1$.

By Remark 2.5, we assume the D^2-slices of all N_i (respectively of all N'_i) are coherent. Then it is easy to see that we can first isotope $f_0 : (S^3, N_0) \to$

(S^3, N_0') so that $f_0| : N_0 \to N_0'$ is D^2-fibration preserving, then inductively isotope $f_n : (N_{n-1}, N_n) \to (N_{n-1}', N_n')$ for each $n \geq 1$ so that $f_n| : N_n \to N_n'$ is D^2-fibration preserving, and still $f_n|_{\partial N_{n-1}} = f_{n-1}|_{\partial N_{n-1}}$.

To apply Lemma 2.9, we set $U_n = N_n$ and extend f_n onto S^3 by setting $f_n|_{S^3 \setminus N_{n-1}} = f_{n-1}|_{S^3 \setminus N_{n-1}}$. Clearly conditions (1) and (3) of Lemma 2.9 are satisfied. Since the diameters of the meridian disks of N_n and N_n' tend to zero uniformly as $n \to \infty$ and since f_n is D^2-fibration preserving and $f_n|_{\partial N_{n-1}} = f_{n-1}|_{\partial N_{n-1}}$, condition (2) of Lemma 2.9 is also satisfied. Therefore, by Lemma 2.9, f_n uniformly converges to a homeomorphism $f : (S^3, \bigcap_{n \geq 0} N_n) \to (S^3, \bigcap_{n \geq 0} N_n')$. That is, $\Sigma = \bigcap_{n \geq 0} N_n$ and $\Sigma' = \bigcap_{n \geq 0} N_n'$ are equivalent.

(3)⇒(2) is obvious.

(1)⇒(3). Let $f : S^3 \to S^3$ be an orientation preserving homeomorphism such that $f(\Sigma) = \Sigma'$. Clearly for each maximal defining sequence $\{N_n\}_{n \geq 0}$ of Σ, $\{f(N_n)\}_{n \geq 0}$ is a maximal defining sequence of Σ' and is equivalent to $\{N_n\}_{n \geq 0}$. By Proposition 3.2, $\{f(N_n)\}_{n \geq 0}$, hence $\{N_n\}_{n \geq 0}$, is equivalent to every maximal defining sequence of Σ'. ∎

3.3. Knotting, linking and invariants. Thanks to the classification theorem, we can talk about the knotting, linking and invariants of tame solenoids.

Definition 3.5. A tame embedding of a solenoid $\Sigma \subset S^3$ with defining sequence $\{N_n\}_{n \geq 0}$ is called *knotted* if some defining solid torus $N_n \subset S^3$ is knotted; otherwise we call the embedding *unknotted*.

Note that for a defining sequence $\{N_n\}_{n \geq 0}$ of a tame solenoid, if N_n is knotted then so is $N_{n'}$ for all $n' > n$. It follows from Theorem 3.4 that the notion of knotting is well defined for equivalence classes of tame solenoids.

Definition 3.6. Let $\Sigma, \Sigma' \subset S^3$ be disjoint tame solenoids with disjoint defining sequences $\{N_n\}_{n \geq 0}$ and $\{N_j'\}_{j \geq 0}$ respectively.

- Call Σ, Σ' *algebraically linked* if some linking number $\mathrm{lk}(N_n, N_j')$ (i.e. the linking number of their centerlines) is non-zero.
- Call Σ, Σ' *linked* if some defining solid tori N_n, N_j' are linked.

Since two disjoint tame solenoids $\Sigma, \Sigma' \subset S^3$ always have disjoint defining sequences and as $\mathrm{lk}(N_n, N_j') \neq 0$ implies $\mathrm{lk}(N_{n'}, N_{j'}') \neq 0$ for all $n' \geq n$, $j' \geq j$, by Theorem 3.4 again the notion of algebraic linking is well defined.

Similarly the notion of linking is well defined, too. In particular, Σ and Σ' are linked if and only if there are no disjoint 3-balls B and B' such that $\Sigma \subset B$ and $\Sigma' \subset B'$.

To define invariants of tame solenoids, the proposition below will be of help.

Proposition 3.7. *Up to strong equivalence, each knotted tame solenoid* $\Sigma \subset S^3$ *has a unique maximal defining sequence* $\{N_n\}_{n\geq 0}$ *such that* N_0 *is knotted and any other defining sequence* $\{N'_n\}_{n\geq 0}$ *with* N'_0 *knotted is a subsequence of* $\{N_n\}_{n\geq 0}$.

Proof. The detailed proof is similar to the proof of Proposition 3.2.

Assume N_0 is knotted. Then $S^3 \setminus N_0$ is boundary incompressible. Let Γ be the tori of the JSJ-decomposition of $S^3 \setminus N_0$. Then there is a unique torus $T \in \Gamma$ such that

(1) T bounds a solid torus N containing N_0 as a closed braid,
(2) if another $T' \in \Gamma$ has property (1), then the solid torus N' bounded by T' is a closed braid in N.

Now we get a new defining sequence $\{N, N_0, N_1, \ldots\}$ of Σ. Then refine this sequence to a maximal one. One can verify that the resulting maximal defining sequence is what we want. ■

Any knot invariant I (for example, the genus, the Gromov volume, the Alexander polynomial or the Jones polynomial) gives rise to an invariant I of tame solenoids. For a knotted tame solenoid $\Sigma \subset S^3$, let $\{N_n\}_{n\geq 0}$ be the unique maximal defining sequence from the above proposition. Then the infinite sequence $I(\Sigma) = \{I(N_0), I(N_1), \ldots\}$ depends only on the equivalence class of Σ. If a tame solenoid $\Sigma \subset S^3$ is unknotted, then for any defining sequence $\{N_n\}_{n\geq 0}$ of Σ the sequence $I(\Sigma) = \{I(N_0), I(N_1), \ldots\}$ is identically trivial, say $\{0, 0, \ldots\}$, if I is either the genus or the Gromov volume, or $\{1, 1, \ldots\}$, if I is either the Alexander polynomial or the Jones polynomial.

In general for a given numerical function g and a knot invariant I, one may organize the sequence $I(\Sigma)$ into a formal series $I(\Sigma, g) = \sum_{n=0}^{\infty} g(n)I(N_n)t^n$. We wonder if $I(\Sigma, g)$ would have interesting properties for certain g and I and for suitable classes of solenoids.

3.4. Unknotted 2-adic tame solenoids. Given an unknotted solid torus N in S^3, there are exactly two kinds of thick braid of winding number two in N that are unknotted in S^3 as shown in Figure 2, where the left one is denoted by 1, and the right one by -1. Then any maximal defining sequence of an unknotted 2-adic tame solenoids in S^3 can be presented as an infinite sequence of ± 1.

Let Z_2 be the set of infinite sequences $(a_1, a_2, \ldots)$ of ± 1's. Two such sequences are said to be *equivalent* if they can be made identical by deleting finitely many terms. By Theorem 3.4 the equivalence classes of unknotted 2-adic tame solenoids are in 1-1 correspondence with the equivalence classes in Z_2. In particular, there are uncountably many equivalence classes of unknotted 2-adic tame solenoids.

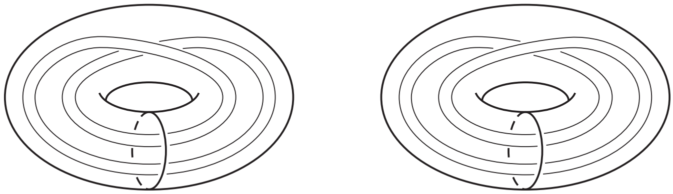

Fig. 2

3.5. Smale solenoids. The solenoids were introduced into dynamics by Smale [S] as hyperbolic attractors.

Definition 3.8. Let M be a 3-manifold and $f : M \to M$ be a homeomorphism. If there is a solid torus $N \subset M$ such that $f|_N$ (resp. $f^{-1}|_N$) defines a thick braid as in Definition 2.2(1), we call the hyperbolic (expanding) attractor $\Sigma_f = \bigcap_{n=1}^{\infty} f^n(N)$ (resp. the hyperbolic repeller $\Sigma = \bigcap_{n=1}^{\infty} f^{-n}(N)$) a *Smale solenoid.* Define the *winding number* w of $f(N)$ in N to be the winding number of the Smale solenoid Σ_f.

Clearly each Smale solenoid $\Sigma_f \subset S^3$ is tame. It is known that a Smale solenoid $\Sigma_f \subset S^3$ must be unknotted [JNW]. Moreover, it is proved in [JNW] that if the non-wandering set $\Omega(f)$ of a dynamics f consists of finitely many disjoint Smale solenoids, then $\Omega(f)$ consists of two solenoids (indeed they are algebraically linked).

Definition 3.9. Let $w_1, \ldots, w_k$ be integers greater than 1. Call $\Sigma \subset S^3$ a Smale solenoid of *type* $(w_1, \ldots, w_k)$ if (1) there is a dynamics f taking Σ as an attractor, (2) there is a defining sequence $\{N_n\}_{n\geq 0}$ of Σ such that f sends N_n to N_{k+n} for all $n \geq 0$ and (3) w_n is the winding number of N_n in N_{n-1} for $1 \leq n \leq k$.

Proposition 3.10. *Any given type $(w_1, \ldots, w_k)$ is realized by a Smale solenoid $\Sigma \subset S^3$. Moreover, the number of Smale solenoids $\Sigma \subset S^3$ of type $(w_1, \ldots, w_k)$ is finite if $w_n \leq 3$ for all n, and is countably infinite otherwise.*

Proof. Take an unknotted solid torus $N_0 \subset S^3$. Inductively construct $e_i : (S^3, N_{i-1}) \to (S^3, N_i)$ so that $e_i(N_{i-1})$ is the thick w_i-braid in N_{i-1} representing the "$1/w_i$-twist". Let $f = e_k \circ \cdots \circ e_1 : (S^3, N_0) \to (S^3, N_k)$. Then $\Sigma_f = \bigcap_{n=1}^{\infty} f^n(N)$ is a Smale solenoid of type $(w_1, \ldots, w_k)$.

The "moreover" part follows from the fact that each Smale solenoid in S^3 must be unknotted, Theorem 3.4, and the lemma below. ∎

Lemma 3.11. *Let W_n be the set of n-strand braids whose closures are unknotted in S^3. Then*

(1) *W_n has two conjugacy classes as pictured in Figure 2 for $n = 2$;*

(2) *W_n has three conjugacy classes as pictured in Figure* 3 *for* $n = 3$*;*
(3) *W_n has infinitely many conjugacy classes for* $n > 3$.

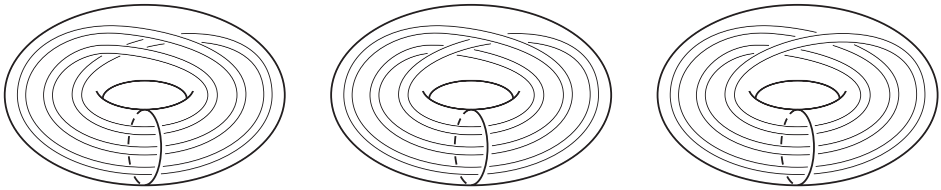

Fig. 3

Proof. (1) is well-known. (2) is proved in [MP, Theorem 1]. The closed braids in (1) and (2) are presented in Figures 2 and 3 respectively.

(3) should be also known. We give a simple construction: Let σ_i's be the standard generators of the braid groups. For each $n > 3$, let $\beta(n,k) = (\sigma_2)^{-k}\sigma_1\sigma_2(\sigma_2)^k\sigma_3\sigma_4\cdots\sigma_{n-1}$. Then one sees directly that the closure $\overline{\beta}(n,k)$ is a trivial knot in 3-space. On the other hand the trace of the reduced Burau representation of $\beta(n,k)$ takes value $1-k(k+1)$ at $t=-1$. Hence $\beta(n,k)$ and $\beta(n,k')$ are not conjugate for $k \neq k'$. ∎

A simplified version of Proposition 3.10 is

Corollary 3.12. *Up to conjugacy by homeomorphisms of S^3, the number of Smale solenoids in S^3 with winding number w is finite if $|w| \leq 3$, and is infinite otherwise.*

4. Chirality of tame solenoids

Definition 4.1. Call a subset $A \subset S^3$ *achiral* if there is an orientation reversing homeomorphism $r : S^3 \to S^3$ such that $r(A) = A$. Call A *strictly achiral* if there is an orientation reversing homeomorphism $r : S^3 \to S^3$ such that $r(x) = x$ for every $x \in A$.

In Definition 4.1, "achiral" means setwise achiral, and "strictly achiral" means pointwise achiral. They are two opposite extremes among various shades of achirality in the real world.

4.1. Criteria. By definition, a tame solenoid in S^3 is achiral if and only if it is equivalent to its mirror image. Therefore, by Theorem 3.4 we have the following criterion of the chirality of tame solenoids.

Theorem 4.2. *A tame solenoid given by the maximal defining sequence $\{N_n\}_{n\geq 0}$ is achiral if and only if $\{N_n\}_{n\geq 0}$ is equivalent to its mirror image.*

Example 4.3. Recall the example in §3.4. The mirror image of a maximal defining sequence of an unknotted 2-adic tame solenoid presented by

$(a_1, a_2, \dots)$ is presented by $(-a_1, -a_2, \dots)$. By Theorem 4.2, the unknotted 2-adic tame solenoid presented by $(+1, -1, +1, -1, \dots)$ is achiral but the solenoids presented by $(+1, +1, +1, \dots)$ or $(-1, -1, -1, \dots)$ are not achiral.

Below we focus on the strict achirality of tame solenoids. For a map $f : X \to X$, we use $\mathrm{Fix}(f)$ to denote the fixed point set of f.

DEFINITION 4.4. Suppose A is a subset of the solid torus N, and l is a given framing of N. Call A *strictly achiral with respect to* l if there exists an orientation reversing homeomorphism $f : N \to N$ such that $A \cup l \subset \mathrm{Fix}(f)$.

To prove the main theorem of this subsection, we need the following two lemmas.

LEMMA 4.5. *Suppose l' is a braid in N. If it is strictly achiral with respect to a framing l of N, then there exists an orientation reversing and D^2-fibration preserving homeomorphism $r : N \to N$ such that $l \cup l' \subset \mathrm{Fix}(f)$.*

Proof. Since l' is strictly achiral with respect to l, there exists an orientation reversing homeomorphism $r : N \to N$ such that $l \cup l' \subset \mathrm{Fix}(r)$.

Let w be the winding number of l' in N. Fix a base point $* \in S^1$ and denote by D_* the disk $D^2 \times *$ which intersects l' in w points. Without loss of generality, we may assume that ∂D_* is invariant under r. Furthermore, we can take a small tubular neighborhood B of l' such that the D^2-foliation of N restricted to B induces a disk foliation on B. Isotoping r near l', we can assume that B is invariant under r, and r restricted to B preserves the disk foliation. Denote the w-punctured disk $D_* \setminus B$ by P.

Let $p : \tilde{N} \to N$ be the infinite cyclic covering, $\tau : \tilde{N} \to \tilde{N}$ the deck translation, and $\tilde{B} = p^{-1}(B) \subset \tilde{N}$ the preimage of B. Then $\tilde{N} \setminus \tilde{B}$ is homeomorphic to $P \times \mathbb{R}$.

Fix a component $\tilde{D}_*$ of the preimage of D_*. Then $\tilde{P}_0 = \tilde{D}_* \setminus \tilde{B}$ is a component of the preimage of P. Let $\tilde{r} : \tilde{N} \to \tilde{N}$ be the lift of r such that $\tilde{r}(\partial\tilde{D}_*) = \partial\tilde{D}_*$. By sliding r along l' we may further assume that, upstairs, the lift $\tilde{r}$ leaves each component of $\partial\tilde{P}_0$ invariant.

We assume that $r : N \to N$, subject to the above conditions, has been isotoped so that $r(P)$ intersects $\mathrm{int}\, P$ transversely, and $r(P) \cap \mathrm{int}\, P$ has the minimal number of circle components. We claim that $r(P) \cap \mathrm{int}\, P = \emptyset$.

Let $\tilde{P}_k = \tau^k(\tilde{P}_0)$. Then $p^{-1}(P) = \bigcup_{k=-\infty}^{\infty} \tilde{P}_k$. Clearly $\tilde{r}(\tilde{P}_0)$ intersects $\mathrm{int}\, \tilde{P}_k$ transversely for all k. Denote the portion of $\tilde{N} \setminus \tilde{B}$ between $\tilde{P}_k$ and $\tilde{P}_{k+1}$ by $\tilde{N}_k$.

Suppose $r(P) \cap \mathrm{int}\, P \neq \emptyset$. Let $k \geq 0$ be the largest integer such that $\tilde{F} = \tilde{r}(\tilde{P}_0) \cap \tilde{N}_k$ is not empty (otherwise consider the smallest negative integer). Clearly $\partial\tilde{F} \subset \tilde{P}_k$. Moreover, $\tilde{F}$ must be incompressible in $\tilde{N}_k$. Otherwise, there would be an essential circle c on $\tilde{F}$ which bounds a disk $D \subset \tilde{N}_k$ with $D \cap \tilde{F} = c$. Since $\tilde{r}(\tilde{P}_0)$ is a proper incompressible surface in $\tilde{N} \setminus \tilde{B}$, c must bound a sub-disk D' of $\tilde{r}(\tilde{P}_0)$. This D' must intersect $\tilde{P}_k$ because it is not in $\tilde{F}$. Since $D \cup D'$ is an embedded 2-sphere and $\tilde{N} \setminus \tilde{B}$ is irreducible, $D \cup D'$ bounds a 3-ball D^3 whose interior misses $\tilde{r}(\tilde{P}_0)$. Clearly r can be isotoped so that D replaces D' and therefore $r(P) \cap P$ gets reduced, contradicting the minimality assumption on r.

So $\tilde{F}$ is incompressible in $\tilde{N}_k$, and $\partial\tilde{F} \subset \tilde{P}_k$. Note that the pair $(\tilde{N}_k, \tilde{P}_k)$ is homeomorphic to $(P \times [0,1], P \times 0)$. By a classical fact in 3-manifold topology ([Wa, Proposition 3.1]), each component $\tilde{F}_i$ of $\tilde{F}$ is parallel to $\tilde{P}_k$. Hence $F_i = p(\tilde{F}_i)$ is parallel to P in $N \setminus B$, i.e., there is a product region between F_i and P. So we can push F_i via isotopy to P, then go slightly further to reduce $r(P) \cap P$, again contradicting the minimality assumption on r. This finishes the proof of the claim that $r(P) \cap \operatorname{int} P = \emptyset$.

Now $r(P)$ is an incompressible proper surface in $N \setminus (B \cup P) = P \times [0,1]$ with $\partial r(P)$ staying in, say, $P \times 0$. For the same reason as in the last paragraph, $r(P)$ is parallel to $P \times 0$, and we can isotope r so that $r(P) = P$. Finally we can isotope rel $P \cup B \cup \partial N$ so that r is D^2-fibration preserving. ∎

LEMMA 4.6. *Suppose $\Sigma \subset S^3$ is a tame embedding given by a defining sequence $\{N_n\}_{n\geq 0}$. If $\Sigma \subset \operatorname{Fix}(r)$ for some homeomorphism $r : S^3 \to S^3$, then there exists $k > 0$ such that $r(N_n) \subset \operatorname{int} N_0$, and moreover $r(N_n)$ and N_n have the same winding number in N_0, for $n \geq k$.*

Proof. Since S^3 is compact, r is uniformly continuous. We proceed as follows:

(i) Let $\epsilon = d(N_1, \partial N_0)/2$.
(ii) Choose $0 < \delta < \epsilon$ such that if $d(x,x') < \delta$ then $d(r(x), r(x')) < \epsilon$.
(iii) Choose $k > 0$ such that $\max_{x\in N_k} d(x,\Sigma) < \delta$.

Now fix an integer $n \geq k$. For any $x \in N_n$, by (i) we have $d(x, \partial N_0) \geq 2\epsilon$ and by (iii) we can choose $x' \in \Sigma$ such that $d(x,x') < \delta$, hence by (ii),

$$d(x, r(x)) \leq d(x,x') + d(x', r(x')) + d(r(x'), r(x)) < \delta + 0 + \epsilon < 2\epsilon \leq d(x, \partial N_0).$$

It follows that the unique geodesic $\alpha(x)$ connecting x and $r(x)$ lies in $\operatorname{int} N_0$. Therefore, $\{\alpha(x) \mid x \in N_n\}$ gives rise to a homotopy from N_n to $r(N_n)$ in N_0. In particular, $r(N_n) \subset \operatorname{int} N_0$, and $r(N_n)$ and N_n have the same winding number in N_0. ∎

THEOREM 4.7. *Let $\Sigma \subset S^3$ be a tame solenoid with defining sequence $\{N_n\}_{n\geq 0}$ and let l_n denote a zero framing of N_n in S^3, that is, l_n is null-homologous in $S^3 \setminus N_n$. Then Σ is strictly achiral if and only if there exists $k \geq 0$ such that l_k is strictly achiral in S^3, and l_{n+1} is strictly achiral in N_n with respect to l_n for all $n \geq k$.*

Proof. Sufficiency. Without loss of generality, we assume $k = 0$. Since l_0 is strictly achiral, there is an orientation reversing homeomorphism f_0 :

$(S^3, N_0) \to (S^3, N_0)$ such that $f_0|_{N_0}$ is D^2-fibration preserving and fixes l_0 pointwise.

Since l_1 is strictly achiral in N_0 with respect to l_0, by Lemma 4.5 there is an orientation reversing and D^2-fibration preserving homeomorphism $f_1 : (N_0, N_1) \to (N_0, N_1)$ such that $l_1 \cup l_0$ stays in $\mathrm{Fix}(f_1)$. Since both $f_0|_{N_0}$ and f_1 are orientation reversing and D^2-fibration preserving homeomorphisms of N_0 and both fix l_0 pointwise, we may assume $f_1|_{\partial N_0} = f_0|_{\partial N_0}$. So f_1 may be extended onto S^3 by setting $f_1|_{S^3 \setminus N_0} = f_0|_{S^3 \setminus N_0}$.

Then for $n > 1$, for the same reason we can recursively define a homeomorphism $f_n : (S^3, N_n) \to (S^3, N_n)$ such that $f_n|_{S^3 \setminus N_{n-1}} = f_{n-1}|_{S^3 \setminus N_{n-1}}$ and $f_n|_{N_{n-1}}$ is D^2-fibration preserving and fixes $l_n \cup l_{n-1}$ pointwise.

To apply Lemma 2.9, set $U_n = N_n$. Clearly conditions (1) and (3) of Lemma 2.9 are satisfied. Since $f_n|_{N_n}$ is D^2-fibration preserving and the diameters of the meridian disks of N_n tend to zero uniformly as $n \to \infty$, conditions (2) and (4) of Lemma 2.9 are also satisfied. By Lemma 2.9, f_n uniformly converges to an orientation reversing homeomorphism $f : S^3 \to S^3$ with $\bigcap_{n \geq 0} N_n \subset \mathrm{Fix}(f)$.

Necessity. Suppose the strict achirality of Σ is witnessed by an orientation reversing homeomorphism r and let $k > 0$ be given by Lemma 4.6. Then for any $n \geq k$ both N_n and $r(N_n)$ are contained in the interior of N_0.

Fix $n \geq k$ and choose a large integer j so that $N_j \subset N_n \cap r(N_n)$. By Lemma 3.3(3), both ∂N_n and $r(\partial N_n)$ are isotopic in $N_0 \setminus N_j$ to some components of $\Gamma \cup \partial N_0 \cup \partial N_j$ where Γ is the union of the JSJ-decomposition tori of $N_0 \setminus N_j$. Therefore, we can isotope r with support in $N_0 \setminus \Sigma$ so that either $r(N_n) = N_n$, or by Lemma 3.3(1), $r(N_n) \subset N_n$ or $N_n \subset r(N_n)$ is a thick braid of winding number greater than 1. By Lemma 4.6, $r(N_n)$ and N_n have the same winding number in N_0, so the latter case cannot happen, and moreover we may assume the zero framing l_n of N_n lies in $\mathrm{Fix}(r)$.

By the same argument, r can be further isotoped with support in $N_n \setminus \Sigma$ so that $r(N_{n+1}) = N_{n+1}$ and $l_{n+1} \subset \mathrm{Fix}(r)$. Therefore, l_n is strictly achiral in S^3 and l_{n+1} is strictly achiral in N_n with respect to l_n. ■

4.2. Examples. Thanks to Theorem 4.7, the strict achirality of tame solenoids breaks up into the problem of strict achirality of knots in S^3 and the problem of strict achirality of closed braids in the solid torus. Below we fix a point $* \in \partial D^2$ and let l_* denote the framing $* \times S^1$ of $D^2 \times S^1$.

DEFINITION 4.8. Call a braid β *achiral* if β is conjugate to its mirror image β^*.

Note that achirality is well defined on conjugacy classes of braids. Also note that the closure $\overline{\beta} \subset D^2 \times S^1$ of a braid β is connected if and only if β is cyclic, i.e. β permutes its ends cyclically.

LEMMA 4.9. *For every braid β, the closure $\overline{\beta} \subset D^2 \times S^1$ is strictly achiral with respect to l_* if and only if β is achiral.*

Proof. Sufficiency follows from the easy fact that, for conjugate braids β and $\alpha^{-1}\beta\alpha$, the closed braids $\overline{\beta}$ and $\overline{\alpha^{-1}\beta\alpha}$ are always isotopic in $D^2 \times S^1$ relative to the boundary torus $\partial(D^2 \times S^1)$. The lower-right arrow of Figure 4 is a good illustration.

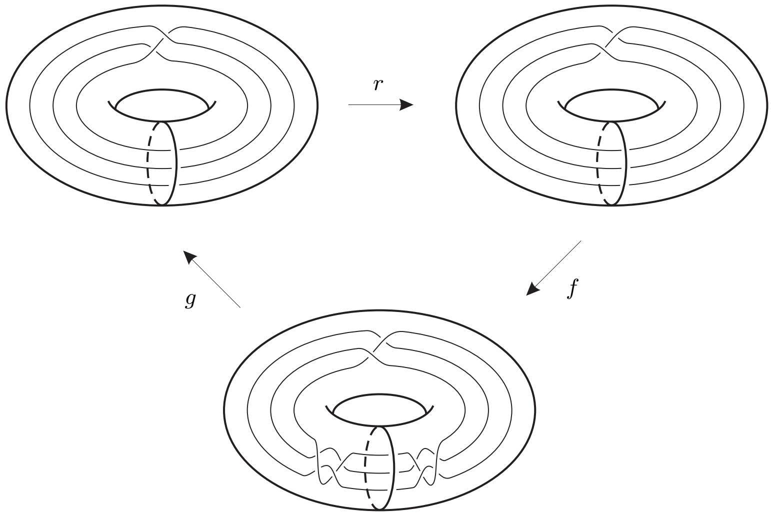

Fig. 4

Necessity. If $\overline{\beta}$ is strictly achiral with respect to l_* then $\overline{\beta}$ is isotopic to its mirror image, therefore β is conjugate to its mirror image β^*. ■

EXAMPLE 4.10 (Examples of cyclic, achiral braids). Let σ_i's be the standard generators of the braid groups.

(1) $\beta = \sigma_1\sigma_2^{-1}$ is cyclic and achiral. Setting $\alpha = \sigma_2\sigma_1^2\sigma_2^{-1}$, one can verify the equality $\beta = \alpha^{-1}\beta^*\alpha$ by the substitution of the braid relation $\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1$ as follows:

$$\begin{aligned}\alpha^{-1}\beta^*\alpha &= (\sigma_2\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_2)(\sigma_2\sigma_1\sigma_1\sigma_2^{-1}) \\ &= \sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_2\sigma_2\sigma_1\sigma_1\sigma_2^{-1} \\ &= \sigma_2\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_2\sigma_2\sigma_1\sigma_1\sigma_2^{-1} = \sigma_1\sigma_2^{-1} = \beta.\end{aligned}$$

One can also directly verify the equality $\beta = \alpha^{-1}\beta^*\alpha$ by a braid move indicated by g on the lower-left arrow in Figure 4, where $r : D^2 \times S^1 \to D^2 \times S^1$ is the reflection about the "page" (assume l_* lies on it). The composition $g \circ f \circ r : D^2 \times S^1 \to D^2 \times S^1$ is an orientation reversing homeomorphism which preserves each D^2-slice and fixes both the closure $\overline{\beta}$ and the framing l_* pointwise.

(2) For any braid β, the product $\beta\beta^*$ is achiral, since

$$\beta^{*-1}(\beta\beta^*)^*\beta^* = \beta^{*-1}\beta^*\beta\beta^* = \beta\beta^*.$$

Moreover, for any cyclic braid β with an odd number of strands, $\beta\beta^*$ is also cyclic. Hence for each cyclic braid β with an odd number of strands, $\beta\beta^*$ is cyclic and achiral.

(3) If β is an achiral braid, then so is β^k for any integer k. Moreover, if β is cyclic, then so is β^k for every integer k relatively prime to the number of strands.

PROPOSITION 4.11.

(1) *If a connected closed braid* $\overline{\beta} \subset D^2 \times S^1$ *is strictly achiral with respect to* l_*, *then the writhe of* $\overline{\beta}$ *is zero* (for the definition of writhe, see [A, p. 152]).
(2) *If a connected closed braid* $\overline{\beta} \subset D^2 \times S^1$ *is strictly achiral with respect to* l_*, *then* $\overline{\beta}$ *is neither of even winding number, nor a cable.*

Proof. (1) By Lemma 4.9, $\overline{\beta}$ strictly achiral implies $\beta = \alpha^{-1}\beta^*\alpha$ for some braid α. Clearly $\mathrm{wr}(\overline{\beta}) = \mathrm{wr}(\overline{\alpha^{-1}\beta^*\alpha}) = \mathrm{wr}(\overline{\beta^*}) = -\mathrm{wr}(\overline{\beta})$. It follows that $\mathrm{wr}(\overline{\beta}) = 0$.

(2) Suppose $\overline{\beta}$ is connected and has an even winding number. Then the braid β has an even number of strands and permutes their end points cyclically. Since a cyclic permutation of an even number of points always consists of an odd number of swaps, the number of crossings of β is odd. Hence $\mathrm{wr}(\overline{\beta})$ must be odd, which contradicts (1).

Suppose $\overline{\beta}$ is a cable. Then all the crossings of $\overline{\beta}$ have the same sign, so $\mathrm{wr}(\overline{\beta})$ is non-zero, which contradicts (1). ∎

Now we state the main result of this subsection.

THEOREM 4.12. *A solenoid of type* $\varpi = (w_1, w_2, \dots)$ *has a strictly achiral tame embedding into* S^3 *if and only if all except finitely many* w_n *are odd.*

Proof. Necessity is immediate from Theorem 4.7 and Proposition 4.11(2).

Sufficiency. Assume all w_n are odd. Let N_0 be a tubular neighborhood of a strictly achiral knot in S^3 and let N_n be a tubular neighborhood of $\overline{\beta_n\beta_n^*}$ in N_{n-1} where β_n is an arbitrary cyclic braid on w_n strands. By Theorem 4.7, Lemma 4.9 and Example 4.10(2) the defining sequence $\{N_n\}_{n\ge 0}$ gives rise to a strictly achiral tame embedding of the solenoid of type $\varpi = (w_1, w_2, \dots)$. ∎

EXAMPLE 4.13. (1) The 2-adic solenoid has no strictly achiral tame embedding into S^3.

(2) By Lemma 3.11(2), Example 4.10(1) and Proposition 4.11(2), up to equivalence the 3-adic solenoid has a unique unknotted strictly achiral tame embedding into S^3, which is obtained by nesting the thick braid pictured in the middle of Figure 3.

(3) The embedding in Theorem 4.12 can be chosen to be either knotted or unknotted, by letting N_0 in the proof be a either tubular neighborhood of the figure-8 knot, or a tubular neighborhood of the unknot and $\beta_n = \sigma_1\sigma_2\cdots\sigma_{w_n-1}$ (we leave it to the reader to verify that the closure $\overline{\beta_n\beta_n^*}$ is unknotted in S^3).

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