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On the maximal exact structure of an additive category

by

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Dedicated to B. V. M.

Abstract. We prove that every additive category has a unique maximal exact structure in the sense of Quillen.

Introduction. In his treatise on algebraic K-theory, Quillen [7] introduced the concept of *exact category*, which means an additive category $\mathscr{A}$ with a distinguished class of short exact sequences (= *conflations*) such that relative homological algebra can be applied to $\mathscr{A}$. The attribute "relative" comes from the fact that every exact structure on $\mathscr{A}$ can be induced from a full embedding $\mathscr{A} \hookrightarrow \mathscr{B}$ into an abelian category $\mathscr{B}$ [7, 8, 16].

In many cases, the exact structure on $\mathscr{A}$ is intrinsic. This is certainly true for the class of splitting conflations. For abelian categories, the natural exact structure is the greatest one. So by the Gabriel–Quillen embedding $\mathscr{A} \hookrightarrow \mathscr{B}$, one should expect that a canonical exact structure ought to be maximal. In fact, there is a wide class of *preabelian* categories (i.e. with kernels and cokernels) where *all* short exact sequences form an exact structure, namely, the *quasi-abelian* categories [4, 14, 10]. (For a brief history of the concept, see [11].) They are defined by the property that cokernels are stable under pullback, and kernels are stable under pushout. A (co-)kernel with this stability property is said to be *semi-stable* [9]. If a semi-stable cokernel has a semi-stable kernel, the corresponding short exact sequence is said to be *stable*.

The conflations of an exact category are stable. So it is natural to ask whether the stable exact sequences form an exact structure for any additive category. Sieg and Wegner [15] confirmed this for the preabelian case, and Crivei [2] extended the result to additive categories which are *divisive*

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(i.e. every split epimorphism has a kernel). In particular, a divisive additive category has a greatest exact structure, hence a "natural" homological algebra.

It is known that the distinguished cokernels (= *deflations*) c of a divisive additive category are *divisible*, i.e. $c = ba$ implies that b is again a deflation. In general, this implication only holds if b has a kernel. (Quillen [7] stated the latter fact as an axiom. Being redundant for exact categories, it became known as the "obscure axiom" [16].)

The aim of this note is to show that a greatest exact structure exists for any additive category $\mathscr{A}$. It remains an open question whether this greatest exact structure coincides with the class of all stable short exact sequences. More generally, we introduce *one-sided* exact structures on $\mathscr{A}$ and prove that there is always a greatest one (Corollary 1 of Theorem 2). In this context, Quillen's "obscure axiom" (Q) reappears in an essential way. Namely, we define a *left exact structure* to be a class $\mathscr{D}$ of cokernels in $\mathscr{A}$ which satisfies half of the axioms for an exact structure, plus Quillen's axiom (Q). We then prove that every left exact structure combines with every right exact one to give an exact structure on $\mathscr{A}$. Thus we reduce the existence problem to the one-sided case, getting in return the trouble with Quillen's axiom (Q).

We introduce three operators C, P, Q on morphism classes $\mathscr{D} \subset \mathscr{A}$, related to properties of a left exact structure. Specifically, $P\mathscr{D}$ consists of the morphisms in $\mathscr{D}$ which remain stable under pullback, $C\mathscr{D}$ picks up the cokernels of $\mathscr{D}$, while $Q\mathscr{D}$ has a more subtle relationship to Quillen's property (Q). For example, $PC\mathscr{A}$ consists of the semi-stable cokernels of $\mathscr{A}$. Using these operators, we prove that any subcategory $\mathscr{D}$ containing the splitting deflations gives rise to a left exact structure $PQPC\mathscr{D}$ (Theorem 2) which is maximal for $\mathscr{D} = \mathscr{A}$. If $\mathscr{A}$ is divisive, the operator Q can be dropped for suitable $\mathscr{D}$ (Theorem 3). In particular, this implies that the maximal left exact structure on a divisive additive category $\mathscr{A}$ simplifies to $PC\mathscr{A}$.

1. Left exact structures. Let $\mathscr{A}$ be an additive category. A sequence

$$A \xrightarrow{a} B \xrightarrow{b} C \tag{1}$$

of morphisms $a, b \in \mathscr{A}$ is said to be *short exact* if $a = \ker b$ and $b = \operatorname{cok} a$. We call $\mathscr{A}$ *divisive* [12] if every split monomorphism has a cokernel, or equivalently, if any pair of morphisms $i\colon A \to B$ and $p\colon B \to A$ with $pi = 1_A$ can be completed to a biproduct [6] $B \cong A \oplus C$. Thus $\mathscr{A}$ is divisive if and only if $\mathscr{A}^{\mathrm{op}}$ is. Note that any triangulated category is divisive, but not necessarily Karoubian (i.e. with splitting idempotents).

In [13, Definition 4], we introduced left exact structures on an additive category, necessarily divisive, so that two-sided exactness is equivalent to exactness in the sense of Gabriel and Roiter [3]. For our present purpose, we stick to the original concept given by Quillen [7]. For the one-sided case, we require Quillen's "obscure axiom" [16, Appendix A], which is known to be redundant for two-sided exact categories [17, 5].

DEFINITION 1. We define a *left exact* structure on an additive category $\mathscr{A}$ to be a class $\mathscr{D} \subset \mathscr{A}$ of cokernels, called *deflations*, which satisfies

(C) $\mathscr{D}$ is a subcategory with $\mathrm{Ob}\,\mathscr{D} = \mathrm{Ob}\,\mathscr{A}$.
(P) The pullback of any $c \in \mathscr{D}$ along an arbitrary morphism exists and belongs to $\mathscr{D}$.
(Q) If $A \xrightarrow{a} B \xrightarrow{b} C$ belongs to $\mathscr{D}$ and b has a kernel, then $b \in \mathscr{D}$.

We indicate deflations by two-head arrows ($\twoheadrightarrow$). Note that the pullback of a deflation $b\colon B \twoheadrightarrow C$ along $0 \to C$ yields its kernel $a\colon A \to B$, which shows that every deflation b gives rise to a short exact sequence (1). We refer to such sequences as *conflations*. Kernels of deflations will be called *inflations* and written as $A \rightarrowtail B$.

Note that by (C), every identical morphism 1_C is a deflation. Hence Quillen's axiom (Q) implies that every split short exact sequence is a conflation. In particular, deflations are closed under isomorphism.

An additive category $\mathscr{A}$ with a left exact structure $\mathscr{D}$ will be called a *left exact category*. Dually, a *right exact* structure on $\mathscr{A}$ is given by a class $\mathscr{I}$ of *inflations* which satisfies the axioms of Definition 1 in $\mathscr{A}^{\mathrm{op}}$. Thus, an *exact* category in the sense of Quillen [7] is the same as a left and right exact category. The following two lemmata will be used frequently. Their proofs are left to the reader.

LEMMA 1. *Let $\mathscr{A}$ be an additive category. For a pullback*

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "b"'] & B \arrow[d, "c"] \\ C \arrow[r, "d"] & D \end{tikzcd}$$

the morphism a has a kernel if and only if d has a kernel. If $k = \ker a$, then $bk = \ker d$.

LEMMA 2. *Let $\mathscr{A}$ be an additive category, and let*

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "d"'] & E \arrow[r, "h"] \arrow[d, "e"'] \arrow[dr, phantom, "\mathrm{PB}"] & F \arrow[d, "f"] \\ A \arrow[r, "a"] & B \arrow[r, "b"] & C \end{tikzcd} \tag{2}$$

be a commutative diagram in $\mathscr{A}$ such that the right-hand square is a pullback. The whole rectangle is a pullback if and only if the left-hand square is a pullback.

Our first result shows that any pair of a left and a right exact category determines an exact category.

THEOREM 1. *Let $\mathscr{A}$ be an additive category with a left exact structure $\mathscr{D} \subset \mathscr{A}$ and a right exact structure $\mathscr{I} \subset \mathscr{A}$. The class of short exact sequences* (1) *with $a \in \mathscr{I}$ and $b \in \mathscr{D}$ makes $\mathscr{A}$ into an exact category.*

Proof. Let $\mathscr{D}_s$ be the class of deflations $b \in \mathscr{D}$ with $\ker b \in \mathscr{I}$. By symmetry, it is enough to verify (C) and (P) for $\mathscr{D}_s$ instead of $\mathscr{D}$. Thus let $A \overset{a}{\rightarrowtail} B \overset{b}{\twoheadrightarrow} C$ and $E \overset{e}{\rightarrowtail} C \overset{c}{\twoheadrightarrow} F$ be short exact sequences with $b, c \in \mathscr{D}_s$. To show that $cb \in \mathscr{D}_s$, consider the pullback of b and e, which gives a commutative diagram

$$\begin{tikzcd} A \arrow[r, tail, "q"] \arrow[d, equal] & D \arrow[r, two heads, "p"] \arrow[d, tail, "d"] \arrow[dr, phantom, "\mathrm{PB}"] & E \arrow[d, tail, "e"] \\ A \arrow[r, tail, "a"] & B \arrow[r, two heads, "b"] \arrow[d, tail, "cb"] & C \arrow[d, two heads, tail, "c"] \\ & F \arrow[r, equal] & F \end{tikzcd}$$

with $p \in \mathscr{D}$ and $q = \ker p$. Furthermore, the pullback property implies that $d = \ker cb$. By (C), we have $cb \in \mathscr{D}$. Hence $cb = \operatorname{cok} d$. Now it is straightforward to verify that the diagonal squares in the commutative diagram

$$\begin{tikzcd} A \arrow[r, tail, "q"] \arrow[d, tail, "a"] \arrow[dr, phantom, "\mathrm{PO}"] & D \arrow[r, tail, "d"] \arrow[d, tail, "\binom{d}{p}"] & B \arrow[d, tail, "\binom{1}{b}"] \\ B \arrow[r, tail, "\binom{1}{0}"] & B \oplus E \arrow[r, tail, "\left(\begin{smallmatrix}1 & 0\\ 0 & e\end{smallmatrix}\right)"] & B \oplus C \\ & E \arrow[u, tail, "\binom{0}{1}"] \arrow[r, tail, "e"] \arrow[ur, phantom, "\mathrm{PO}"] & C \arrow[u, tail, "\binom{0}{1}"] \end{tikzcd}$$

are pushouts. Therefore, the morphisms $\binom{d}{p}$ and $\left(\begin{smallmatrix}1 & 0\\ 0 & e\end{smallmatrix}\right)$ belong to $\mathscr{I}$. Thus $\binom{1}{b}d = \left(\begin{smallmatrix}1 & 0\\ 0 & e\end{smallmatrix}\right)\binom{d}{p} \in \mathscr{I}$. Since $cb = \operatorname{cok} d$, we get $d \in \mathscr{I}$ by the dual of (Q). Hence $cb \in \mathscr{D}_s$.

Next, let

$$\begin{tikzcd} K \arrow[r, tail, "i"] \arrow[d, equal] & A \arrow[r, two heads, "a"] \arrow[d, tail, "b"] \arrow[dr, phantom, "\mathrm{PB}"] & B \arrow[d, tail, "c"] \\ K \arrow[r, tail, "j"] & C \arrow[r, two heads, "d"] & D \end{tikzcd}$$

be a pullback with $d \in \mathscr{D}_s$. Then (P) gives $a \in \mathscr{D}$. Since $a = \operatorname{cok} i$ and $bi = j \in \mathscr{I}$, we get $i \in \mathscr{I}$ by the dual of (Q). Hence $a \in \mathscr{D}_s$. ■

We conclude this section with some examples of left exact categories which need not be right exact.

Examples. Let $\mathscr{A}$ be an additive category with kernels and cokernels. The class of all short exact sequences makes $\mathscr{A}$ into a left exact category if and only if $\mathscr{A}$ is *left quasi-abelian* (= left almost abelian [10]), i.e. if condition (P) of Definition 1 holds for arbitrary cokernels c. This follows by [10, Proposition 2 and Corollary 1 of Proposition 1]. For example, let $\mathbf{Mod}(R)$ denote the category of left modules over a ring R. Then every full subcategory $\mathscr{A}$ of $\mathbf{Mod}(R)$ which is closed with respect to submodules and products is left quasi-abelian. By [10, Theorem 2 and Lemma 7], every left quasi-abelian category $\mathscr{A}$ admits a full embedding into an abelian category $\mathrm{Q}_l(\mathscr{A})$ such that the left exact structure of $\mathscr{A}$ is exact if and only if $\mathscr{A}$ is closed with respect to extensions.

Further examples are given in [1, Examples 4.6 and 4.7].

2. The operators C, P, and Q. In this section, we introduce three operators on morphism classes which will be used for a subsequent construction of left exact structures from any subcategory.

Definition 2. Let $\mathscr{A}$ be an additive category, and let $\mathscr{D}$ be a class of morphisms in $\mathscr{A}$. We define $P\mathscr{D}$ to be the class of morphisms $f\colon C \to F$ such that for every $h\colon E \to F$ in $\mathscr{A}$, the pullback

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, "e"] & C \arrow[d, "f"] \\ E \arrow[r, "h"] & F \end{tikzcd} \tag{3}$$

exists and satisfies $e \in \mathscr{D}$. We write $Q\mathscr{D}$ for the class of morphisms $f \in \mathscr{D}$ such that for every pullback (3), the implication $b \in \mathscr{D} \Rightarrow h \in \mathscr{D}$ holds. The class of cokernels in $\mathscr{D}$ will be denoted by $C\mathscr{D}$.

The operators P and Q are related, respectively, to the properties (P) and (Q) in Definition 1. For (Q), this will be shown in Proposition 3. The special case $h\colon 0 \to F$ in (3) shows that every morphism in $P\mathscr{D}$ has a kernel.

In general, $C\mathscr{D}$ need not satisfy (C) even if $\mathscr{D}$ does. The application of P, however, recovers this property.

Proposition 1. *Let $\mathscr{A}$ be an additive category, and let $\mathscr{D}$ be a class of morphisms in $\mathscr{A}$. Then $PP\mathscr{D} = P\mathscr{D}$. If* (C) *holds for $\mathscr{D}$, then it also holds for $P\mathscr{D}$ and $PC\mathscr{D}$.*

Proof. Since $P\mathscr{D} \subset \mathscr{D}$, the equality $PP\mathscr{D} = P\mathscr{D}$ says that $P\mathscr{D}$ satisfies condition (P) of Definition 1. Consider a pullback (3) with $f \in P\mathscr{D}$. For any $g\colon D \to E$, the pullback of f along hg exists, which leads to a commutative diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "d"] & B \arrow[r, "b"] \arrow[d, "e"] & C \arrow[d, "f"] \\ D \arrow[r, "g"] & E \arrow[r, "h"] & F \end{tikzcd} \tag{4}$$

where the whole rectangle and the right-hand square are pullbacks. By Lemma 2, so is the left-hand square. Since $d \in \mathscr{D}$, it follows that $e \in P\mathscr{D}$.

For the rest of the proof, we assume $\mathscr{D}$ satisfies (C). First, let $g\colon D \to E$ and $h\colon E \to F$ in $P\mathscr{D}$ be given. For an arbitrary morphism $f\colon C \to F$, the pullback of hg along f can be obtained in two steps and leads to a pullback (4) consisting of two pullback squares. Hence $a, b \in \mathscr{D}$, and thus $ba \in \mathscr{D}$. This proves that $P\mathscr{D}$ satisfies (C).

Next, let $g\colon D \to E$ and $h\colon E \to F$ be in $PC\mathscr{D}$. Then the kernels $k = \ker g$ and $q = \ker h$ exist, and the pullback of g and q gives a commutative diagram

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[d, equal] & G \arrow[r, "p"] \arrow[d, "j"] \arrow[dr, phantom, "\mathrm{PB}"] & H \arrow[d, "q"] \\ K \arrow[r, "k"] & D \arrow[r, "g"] \arrow[d, "hg"] & E \arrow[d, "h"] \\ & F \arrow[r, equal] & F \end{tikzcd}$$

with $i = \ker p$. Since $g, h \in C\mathscr{D}$, we have $g = \operatorname{cok} k$ and $h = \operatorname{cok} q$. Furthermore, $g \in PC\mathscr{D}$ implies that $p \in C\mathscr{D}$. Hence $hg = \operatorname{cok} j$, and thus $hg \in C\mathscr{D}$. Moreover, for any pullback (4), the morphisms a, b belong to $PC\mathscr{D}$. Hence $ba \in C\mathscr{D}$, which yields $hg \in PC\mathscr{D}$. ■

Now we turn to Quillen's property (Q) and its relationship to the operator Q. First, we prove

Proposition 2. *Let $\mathscr{A}$ be an additive category, and let $\mathscr{D}$ be a class of morphisms which satisfies* (C) *and* (P). *Assume that every zero epimorphism $A \to 0$ belongs to $\mathscr{D}$. Then $Q\mathscr{D}$ satisfies* (C) *and* (Q).

Proof. Let $g\colon D \to E$ and $h\colon E \to F$ in $Q\mathscr{D}$ be given. Then $hg \in \mathscr{D}$. Since $\mathscr{D}$ satisfies (P), any pullback of a morphism $f\colon C \to F$ along hg exists and splits into a pair of pullback squares (4). If $d \in \mathscr{D}$, then $g \in Q\mathscr{D}$ implies that $e \in \mathscr{D}$, and $h \in Q\mathscr{D}$ yields $f \in \mathscr{D}$. Hence $hg \in Q\mathscr{D}$. This proves (C).

To verify (Q), assume that $c \in Q\mathscr{D}$ has a factorization $c = hg$ such that h has a kernel $k\colon K \to E$. Then there is a pullback

$$\begin{array}{ccc} K \oplus D & \xrightarrow{(0\ 1)} & D \\ \Big\downarrow{\scriptstyle (k\ g)} & & \Big\downarrow{\scriptstyle c} \\ E & \xrightarrow{h} & F \end{array}$$

The canonical morphism $(0\ 1)\colon K \oplus D \to D$ belongs to $\mathscr{D}$ since it can be obtained as a pullback of $K \to 0$ along $D \to 0$. Hence $h \in \mathscr{D}$.

Now consider a pullback (3). The pullback of f along hg gives rise to a commutative diagram (4), where the left-hand square is a pullback by Lemma 2. Assume that $e \in \mathscr{D}$. Then (P) implies that $d \in \mathscr{D}$. Hence $hg \in Q\mathscr{D}$ yields $f \in \mathscr{D}$. Thus $h \in Q\mathscr{D}$. ∎

As a consequence, we can express (Q) by means of the operator Q.

Proposition 3. *Let $\mathscr{A}$ be an additive category, and let $\mathscr{D}$ be a class of morphisms which satisfies* (C) *and* (P). *Assume that every zero epimorphism $A \to 0$ belongs to $\mathscr{D}$. Then $\mathscr{D}$ satisfies* (Q) *if and only if $Q\mathscr{D} = \mathscr{D}$.*

Proof. Assume $\mathscr{D}$ satisfies (Q). Consider a pullback (3) with $b, f \in \mathscr{D}$. Since (C) holds for $\mathscr{D}$, we get $he = fb \in \mathscr{D}$. As $\mathscr{D}$ satisfies (P), the morphism b has a kernel k, and thus $ek = \ker h$ by Lemma 1. Hence $h \in \mathscr{D}$. This proves that $Q\mathscr{D} = \mathscr{D}$. The converse follows by Proposition 2. ∎

Proposition 4. *Let $\mathscr{A}$ be an additive category, and let $\mathscr{D}$ be a class of morphisms which satisfies* (C) *and* (P). *Assume that the zero epimorphisms belong to $\mathscr{D}$. Then $PQ\mathscr{D}$ satisfies* (C), (P), *and* (Q).

Proof. By Proposition 1, $PQ\mathscr{D}$ satisfies (P). Furthermore, Propositions 1 and 2 imply that (C) holds for $PQ\mathscr{D}$. To show that $PQ\mathscr{D}$ satisfies (Q), let $c\colon A \to C$ be a morphism in $PQ\mathscr{D}$, and let $c = ba$ be a factorization such that b has a kernel. Since $c \in Q\mathscr{D}$, it follows that $b \in \mathscr{D}$. By assumption, $\mathscr{D}$ satisfies (P). Therefore, the pullback of any morphism $f\colon F \to C$ along b or c exists, which yields a commutative diagram (2) where the whole rectangle and the right-hand square are pullbacks. Now $c \in PQ\mathscr{D}$

implies that $hg \in Q\mathscr{D}$. Since b has a kernel, Lemma 1 implies that h has a kernel. Therefore, Proposition 2 implies that $h \in Q\mathscr{D}$. This shows that $b \in PQ\mathscr{D}$. ∎

3. The maximal exact structure. By the results of the preceding section, we are now in a position to show how any left exact structure arises from a combination of the operators C, P, and Q.

THEOREM 2. *Let $\mathscr{A}$ be an additive category, and let $\mathscr{D}$ be a subcategory which contains every split epimorphism with kernel. Then $PQPC\mathscr{D}$ is a left exact structure on $\mathscr{A}$.*

Proof. By Proposition 1, $PC\mathscr{D}$ satisfies (C) and (P). Since $\mathscr{D}$ contains the split epimorphisms with kernel, the same holds for $PC\mathscr{D}$. Therefore, Proposition 4 implies that $PQPC\mathscr{D}$ satisfies (C), (P), and (Q). Since $C\mathscr{D}$ consists of cokernels, the same is true for $PQPC\mathscr{D}$. Thus $PQPC\mathscr{D}$ is a left exact structure on $\mathscr{A}$. ∎

As a first application, we get

COROLLARY 1. *Let $\mathscr{A}$ be an additive category. Then $PQPC\mathscr{A}$ is the unique maximal left exact structure on $\mathscr{A}$.*

Proof. Theorem 2 implies that $PQPC\mathscr{A}$ is a left exact structure. By Proposition 3, any left exact structure $\mathscr{D}$ satisfies $Q\mathscr{D} = \mathscr{D}$. Hence $\mathscr{D} = PQPC\mathscr{D} \subset PQPC\mathscr{A}$, which proves the claim. ∎

Combining Corollary 1 with Theorem 1 yields

COROLLARY 2. *Every additive category $\mathscr{A}$ admits a unique maximal exact structure.*

Proof. By Corollary 1, every left exact structure is contained in the maximal left exact structure $PQPC\mathscr{A}$. Thus, by symmetry, Theorem 1 shows that the maximal exact structure consists of the short exact sequences (1) where b belongs to $PQPC\mathscr{A}$, and a belongs to the maximal right exact structure. ∎

Sieg and Wegner [15] and Crivei [2] have shown that Corollary 2 holds under the assumption that $\mathscr{A}$ is divisive. In this case, the existence of a kernel in Quillen's property (Q) can be dropped (see [13, Definition 4]).

PROPOSITION 5. *Let $\mathscr{A}$ be a divisive additive category, and let $\mathscr{D}$ be a class of morphisms which satisfies* (P). *If $ba \in \mathscr{D}$, then b has a kernel.*

Proof. For $c := ba$, there exists a pullback

$$\begin{tikzcd} D \arrow[r, "e"] \arrow[d, "d"] & B \arrow[d, "b"] \\ A \arrow[r, "c"] & C \end{tikzcd}$$

Since $c \cdot 1_A = b \cdot a$, there exists a unique $s\colon A \to D$ with $ds = 1_A$ and $es = a$. As $\mathscr{A}$ is divisive, this implies that d has a kernel. Hence b has a kernel by Lemma 1. ∎

THEOREM 3. *Let $\mathscr{A}$ be a divisive additive category, and let $\mathscr{D}$ be a class of morphisms which satisfies* (C) *and* (Q)*. Then $PC\mathscr{D}$ is a left exact structure on $\mathscr{A}$.*

Proof. By Proposition 1, $PC\mathscr{D}$ satisfies (P) and (C). To verify (Q), let $c\colon A \to C$ be a morphism in $PC\mathscr{D}$. For any factorization $c = ba$, we show first that $b \in C\mathscr{D}$. By Proposition 5, b has a kernel $k\colon K \to B$. Furthermore, c has a kernel $q\colon H \to A$. So we get a commutative diagram

$$\begin{tikzcd} & H \arrow[r, equal] \arrow[d, "\binom{p}{q}"] & H \arrow[d, "q"] \\ K \arrow[r, "\binom{1}{0}"] \arrow[d, equal] & K \oplus A \arrow[r, "(0\ 1)"] \arrow[d, "(k\ a)"] & A \arrow[d, "c"] \\ K \arrow[r, "k"] & B \arrow[r, "b"] & C \end{tikzcd}$$

with $(k\ a)\binom{p}{q} = 0$. In fact, $b \cdot aq = cq = 0$ implies that $-aq = kp$ for some $p\colon H \to K$. Since $c \in PC\mathscr{D}$, it follows that $(k\ a)$ is a cokernel and $c = \operatorname{cok} q$. Now let $u\colon B \to X$ be a morphism with $uk = 0$. Then $ua \cdot q = u(k\ a)\binom{p}{q} = 0$. Hence $ua = vc$ for some $v\colon C \to X$. Therefore, $(u - vb) \cdot (k\ a) = 0$, and thus $u - vb = 0$ since $(k\ a)$ is a cokernel. Furthermore, b is epic, which yields $b = \operatorname{cok} k$. Since $\mathscr{D}$ satisfies (C) and (Q), every split epimorphism belongs to $\mathscr{D}$. Hence $b \cdot (k\ a) = c \cdot (0\ 1) \in \mathscr{D}$, and thus $b \in \mathscr{D}$. Consequently, $b \in C\mathscr{D}$.

It remains to show that $b \in PC\mathscr{D}$. For any $f\colon F \to C$ in $\mathscr{A}$, the morphism c factors through $(b\ f)\colon B \oplus F \to C$. Hence $(b\ f)$ has a kernel by Proposition 5. Therefore, the pullback of f along c or b exists, and by Lemma 2, we get a commutative diagram (2) where both squares are pullbacks. By Proposition 1, we have $hg \in PC\mathscr{D}$, and the above shows that $h \in C\mathscr{D}$. Hence $b \in PC\mathscr{D}$. ∎

As a consequence, Corollary 1 simplifies as follows if $\mathscr{A}$ is divisive.

COROLLARY 3. *Let $\mathscr{A}$ be a divisive additive category. Then $PC\mathscr{A}$ is the maximal left exact structure on $\mathscr{A}$.*

Proof. For any left exact structure $\mathscr{D}$, we have $\mathscr{D} = PC\mathscr{D} \subset PC\mathscr{A}$, and $PC\mathscr{A}$ is a left exact structure by Theorem 3. ∎

The morphisms in $PC\mathscr{A}$ are called *semi-stable cokernels* [9]. Dually, the *semi-stable kernels* in $\mathscr{A}$ are defined to be the semi-stable cokernels in $\mathscr{A}^{\mathrm{op}}$. In particular, we get the main result of Crivei [2].

COROLLARY 4. *Let $\mathscr{A}$ be a divisive additive category. The maximal exact structure on $\mathscr{A}$ consists of the short exact sequences* (1) *with a semi-stable kernel a and a semi-stable cokernel b.*

Proof. This follows by Corollary 3 and Theorem 1. ∎

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