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Gradient otopies of gradient local maps

by

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Abstract. We introduce various classes of local maps: gradient, gradient-like, proper etc. We prove Parusiński's theorem for otopy classes of gradient local maps.

Introduction. In 1990 A. Parusiński [7] published a surprising result: if two gradient vector fields on the unit disc D^n and nonvanishing in S^{n-1} are homotopic, then they are gradient homotopic. An immediate consequence of this fact is that the inclusion of the space of gradient vector fields into the space of all vector fields on D^n nonvanishing in S^{n-1} induces a bijection between the sets of the respective homotopy classes, i.e. between the sets of path-components of these function spaces. Even though Parusiński's result does not hold for equivariant maps, his techniques can still be used to study homotopy classes of gradient equivariant maps (see [4]).

J. C. Becker and D. H. Gottlieb have introduced an extremely useful generalization of the concept of homotopy called *otopy* (see for instance [2, 3, 5]). The main advantage of using otopies is that otopy relates maps with not necessarily the same domain (so called *local maps*). Furthermore, otopy theory turns out to be fruitful in equivariant degree theory (see [1]).

The main goal of our paper is to establish a clear relation between otopy classes of gradient and usual (not necessarily gradient) local maps. More precisely, we introduce many natural classes of local maps: gradient, gradient-like, proper, proper gradient and proper gradient-like and then relate their respective otopy classes. For example, we prove some version of Parusiński's Theorem: the inclusion of the set of gradient local maps into the set of all local maps induces a bijection between the respective otopy classes of local maps. In other words, there is no better invariant in gradient otopy theory than the usual topological degree.

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The organization of the paper is as follows. Section 1 provides a brief exposition of otopy theory for local maps. Section 2 contains our main result concerning the relation between the sets of otopy classes of various classes of local maps. This result is proved in Section 7. In Section 3 we introduce standard local maps and discuss their elementary properties, which are needed in the next section. In Section 4 we prove the key lemma that says that the degree map from the set of gradient otopy classes of gradient local maps to the integers is a bijection. Sections 5 and 6 contain some additional results needed for the proof of our Main Theorem. It is worth pointing out that the nontrivial part of this proof is hidden in the Main Lemma and Proposition 6.3. The proof of Proposition 3.5 was motivated by [9].

1. Basic definitions. The notation $A \Subset B$ means that A is a compact subset of B.

DEFINITION 1.1. A continuous map $f\colon U \to \mathbb{R}^n$ is called a *local map* if

- U is an open subset of $\mathbb{R}^n$,
- $f^{-1}(0) \Subset U$.

We will often write such maps as pairs (f, U), pointing out their domains.

Recall that these maps are called *gradient* (resp. *gradient-like*) if there is a C^1-function $\varphi\colon U \to \mathbb{R}$ such that $f = \nabla\varphi$ (resp. $f(x) \cdot \nabla\varphi(x) > 0$ for all $x \in U \setminus f^{-1}(0)$), and *proper* if the preimages of compact subsets are compact.

We will consider the set of all local maps, denoted by $\mathcal{F}(n)$, and the following subsets:

$$\begin{aligned}
\mathcal{F}_\nabla(n) &:= \{f \in \mathcal{F}(n) \mid f \text{ is gradient}\},\\
\mathcal{F}_{\mathrm{gl}}(n) &:= \{f \in \mathcal{F}(n) \mid f \text{ is gradient-like}\},\\
\mathcal{P}(n) &:= \{f \in \mathcal{F}(n) \mid f \text{ is proper}\},\\
\mathcal{P}_\nabla(n) &:= \mathcal{F}_\nabla(n) \cap \mathcal{P}(n),\\
\mathcal{P}_{\mathrm{gl}}(n) &:= \mathcal{F}_{\mathrm{gl}}(n) \cap \mathcal{P}(n).
\end{aligned}$$

Immediately from the above definitions we obtain the following commutative diagram of inclusions:

$$\begin{array}{ccccc}
\mathcal{P}_\nabla(n) & \longrightarrow & \mathcal{P}_{\mathrm{gl}}(n) & \longrightarrow & \mathcal{P}(n)\\
\downarrow & & \downarrow & & \downarrow\\
\mathcal{F}_\nabla(n) & \longrightarrow & \mathcal{F}_{\mathrm{gl}}(n) & \longrightarrow & \mathcal{F}(n)
\end{array} \tag{1.1}$$

Let $I = [0,1]$.

DEFINITION 1.2. A continuous map $h\colon \Omega \to \mathbb{R}^n$ is called an *otopy* if

- Ω is an open subset of $\mathbb{R}^n \times I$,
- $h^{-1}(0) \Subset \Omega$.

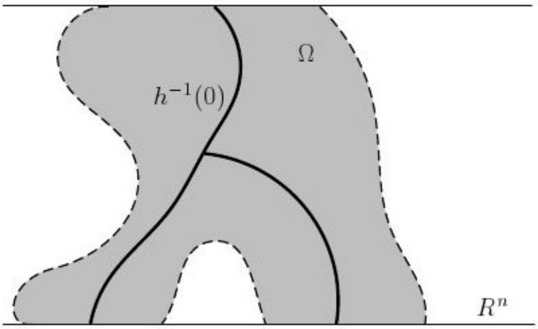

Fig. 1. Domain of an otopy and its zeros

Given an otopy (h, Ω) we can define for each $t \in I$ sets $\Omega_t = \{x \in \mathbb{R}^n \mid (x,t) \in \Omega\}$ and maps $h_t \colon \Omega_t \to \mathbb{R}^n$ with $h_t(x) = h(x,t)$. We allow h_t to be the empty map.

DEFINITION 1.3. If (h, Ω) is an otopy, we say that (h_0, Ω_0) and (h_1, Ω_1) are *otopic* (written $h_0 \sim h_1$ or $(h_0, \Omega_0) \sim (h_1, \Omega_1)$).

REMARK 1.4. Of course, otopy gives an equivalence relation on $\mathcal{F}(n)$. The set of otopy classes of local maps will be denoted by $\mathcal{F}[n]$. Observe that if (f, U) is a local map and V is open subset of U such that $f^{-1}(0) \subset V$, then (f, U) and $(f|_V, V)$ are otopic. In particular, if $f^{-1}(0) \cap U = \emptyset$ then (f, U) is otopic to the empty map.

Apart from the usual otopies, we will consider otopies that satisfy some additional conditions, namely:

- *gradient*, i.e. $h(x,t) = \nabla_x \chi(x,t)$ for some C^1-function χ,
- *gradient-like*, i.e. $h(x,t) \cdot \nabla_x \chi(x,t) > 0$ if $h(x,t) \neq 0$,
- *proper*, i.e. h is proper,
- *proper gradient*,
- *proper gradient-like*.

The sets of respective otopy classes in $\mathcal{F}_\nabla(n)$, $\mathcal{F}_{\mathrm{gl}}(n)$, $\mathcal{P}(n)$, $\mathcal{P}_\nabla(n)$, $\mathcal{P}_{\mathrm{gl}}(n)$ will be denoted by $\mathcal{F}_\nabla[n]$, $\mathcal{F}_{\mathrm{gl}}[n]$, $\mathcal{P}[n]$, $\mathcal{P}_\nabla[n]$, $\mathcal{P}_{\mathrm{gl}}[n]$.

Let $\Sigma^n = \mathbb{R}^n \cup \{*\}$ denote the one-point compactification of $\mathbb{R}^n$. It is a pointed space with base point $*$. We will write $\mathcal{M}_*\Sigma^n$ for the set of pointed continuous maps from Σ^n to Σ^n. With every map $f \in \mathcal{M}_*\Sigma^n$ one associates a proper local map $(f\restriction_{f^{-1}(\mathbb{R}^n)}, f^{-1}(\mathbb{R}^n))$. Conversely, if $(f, U) \in \mathcal{P}(n)$, then the map $f^+ \colon \Sigma^n \to \Sigma^n$ given by

$$f^+(x) = \begin{cases} f(x) & \text{if } x \in U, \\ * & \text{otherwise}, \end{cases}$$

is continuous. Using this observation we see that the map

$$\mu\colon \mathcal{P}(n) \to \mathcal{M}_*\Sigma^n, \qquad \mu((f,U)) = f^+,$$

is a bijection. Since $\mathcal{M}_*\Sigma^n$ is a metric space, $\mathcal{P}(n)$ also has the metric structure induced by the pullback of the supremum metric.

2. Main Theorem. The diagram (1.1) induces the following commutative diagram of maps between sets of otopy classes (all the maps are induced by inclusions):

$$\begin{array}{ccccc} \mathcal{P}_\nabla[n] & \xrightarrow{\mathfrak{a}} & \mathcal{P}_{\mathrm{gl}}[n] & \xrightarrow{\mathfrak{b}} & \mathcal{P}[n] \\ \mathfrak{c}\big\downarrow & & \mathfrak{d}\big\downarrow & & \mathfrak{e}\big\downarrow \\ \mathcal{F}_\nabla[n] & \xrightarrow{\mathfrak{f}} & \mathcal{F}_{\mathrm{gl}}[n] & \xrightarrow{\mathfrak{g}} & \mathcal{F}[n] \end{array} \tag{2.1}$$

Let us formulate the main result of this paper.

Main Theorem 2.1. *The maps $\mathfrak{b}$, $\mathfrak{d}$, $\mathfrak{e}$, $\mathfrak{f}$ and $\mathfrak{g}$ in the diagram* (2.1) *are bijections and the maps $\mathfrak{a}$, $\mathfrak{c}$ are surjections.*

Remark 2.2. We expect that $\mathfrak{c}$ and hence $\mathfrak{a}$ are also bijections, although at the moment we do not have a complete proof of that fact due to some technical difficulties. It is worth pointing out that our result includes some version of Parusiński's Theorem: the inclusion $\mathcal{F}_\nabla(n) \subset \mathcal{F}(n)$ induces a bijection $\mathcal{F}_\nabla[n] \to \mathcal{F}[n]$. However, our proof makes no appeal to the original proof of Parusiński.

Remark 2.3. It is clear from the classical degree theory that all the maps in the following commutative diagram are bijections:

$$\begin{array}{ccccc} \mathcal{M}_*[\Sigma^n] & \longleftarrow & \mathcal{P}[n] & \longrightarrow & \mathcal{F}[n] \\ & \searrow^{\deg} & \big\downarrow \deg & \swarrow_{\deg} & \\ & & \mathbb{Z} & & \end{array}$$

Consequently, the map $\mathfrak{e}$ in (2.1) is bijective.

The true difficulty in proving the Main Theorem lies in the following

Main Lemma 1. $\deg\colon \mathcal{F}_\nabla[n] \to \mathbb{Z}$ *is bijective.*

We will see that, in fact, only injectivity causes a problem. Another nontrivial result is injectivity of $\mathfrak{d}$.

3. Standard local maps. Let us denote by $B(p,r)$ the open r-ball around p and by R the reflection defined by

$$\mathrm{R}(x_1, x_2, \ldots, x_n) = (-x_1, x_2, \ldots, x_n).$$

Any map $f\colon B(p,r) \to \mathbb{R}^n$ of the form $f(x) = \mathrm{Id}(x-p)$ (resp. $f(x) = \mathrm{R}(x-p)$) is called a *t-identity* (resp. *t-reflection*) at p. The letter t stands here for "moved by translation". Clearly, any two t-identities (resp. t-reflections) are gradient otopic, since both translations and restrictions of gradient maps induce gradient otopies.

The following terminology is borrowed from [8], but is used in a slightly different context.

Definition 3.1. We say that a local map (f, U) is *standard of type* (m, l) if

- U is a finite disjoint union of open balls,
- on each of these balls, f is either a t-identity or a t-reflection at its center,
- f consists of exactly m t-identities and l t-reflections.

A standard map of type $(m, 0)$ (resp. $(0, l)$) will be called *standard of type* m (resp. $-l$) for brevity.

Observe that standard maps are gradient. Moreover, we see at once that any two standard maps of type m are gradient otopic (the same reasoning as above).

Reduction of standard maps of type (m, l) to standard maps of type $m-l$ is established by the following Annihilation Lemma.

Lemma 3.2 (Annihilation Lemma). *A disjoint union of a t-identity and a t-reflection is gradient otopic to the empty local map.*

Proof. As observed before, both translations and restrictions of gradient maps induce gradient otopies. So it remains to find a gradient otopy for some convenient maps. For $t \in I$ let:

- $e_t = (t, 0, \dots, 0) \in \mathbb{R}^n$,
- $\Omega_t = B(-e_t, 1) \cup B(e_t, 1)$,
- $\chi(x,t) = \frac{1}{2}x_1|x_1| - tx_1 + \frac{1}{2}x_2^2 + \dots + \frac{1}{2}x_n^2$.

Finally set:

- $\Omega = \bigcup_{t\in I} \Omega_t \times t$,
- $h(x,t) = \nabla_x \chi(x,t) = (|x_1| - t, x_2, \dots, x_n)$.

Then (h, Ω) is almost the desired gradient otopy. What is left is to "lift" the local map (h_0, Ω_0) a little (i.e. "move up" its first component) to get a local map without zeros, which is evidently gradient otopic to the empty local map (see Figure 2). ■

Repeated application of the Annihilation Lemma yields the following conclusion.

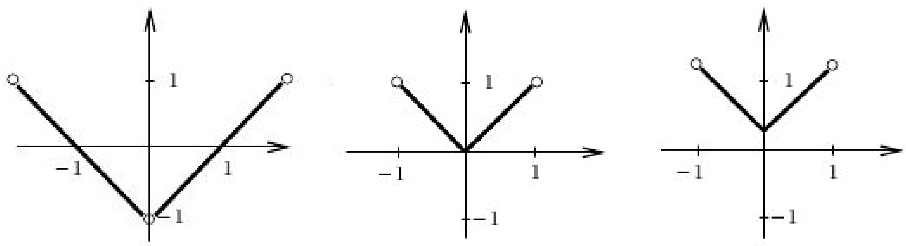

Fig. 2. Annihilation of t-identity and minus t-identity

Corollary 3.3. *Two standard maps of type (m,l) and (m',l') are gradient otopic iff $m-l=m'-l'$. In other words, two standard maps of degree k are gradient otopic.*

We will also need the following two results.

Proposition 3.4 (turning a helmet inside-out). *Any t-identity in the plane is gradient otopic to any minus t-identity.*

Proof. Consider the following three gradient local maps:

$$\begin{aligned} f(x_1,x_2) &= (x_1+2,x_2) && \text{on } U_1 = B((-2,0),1),\\ g(x_1,x_2) &= (-x_1,x_2) && \text{on } U_2 = B((0,0),1),\\ h(x_1,x_2) &= (-x_1+2,-x_2) && \text{on } U_3 = B((2,0),1). \end{aligned}$$

If $\sim$ denotes for a moment the relation of being gradient otopic, then analysis similar to that in the proof of the Annihilation Lemma shows that

$$(f,U_1)=(f,U_1)\sqcup\emptyset\sim(f,U_1)\sqcup(g,U_2)\sqcup(h,U_3)\sim\emptyset\sqcup(h,U_3)=(h,U_3). \ \blacksquare$$

Proposition 3.5. *If $A\colon\mathbb{R}^n\to\mathbb{R}^n$ is a linear map represented by a nonsingular symmetric matrix, then $(A,\mathbb{R}^n)$ is gradient otopic either to $(\mathrm{Id},\mathbb{R}^n)$ or to $(\mathrm{R},\mathbb{R}^n)$.*

Proof. The statement is trivial for $n=1$, so suppose $n\ge 2$. Denote by $Symm(n)$ the set of $n\times n$ symmetric matrices. By abuse of notation, we will use the same letter A for both a linear map and its matrix representation. Let $A\in GL_n(\mathbb{R})\cap Symm(n)$. We describe a procedure which allows us to reduce A to the required form. It is worth pointing out that we cannot carry out this procedure in the space of nonsingular symmetric matrices, because the signature is constant on connected components of this space. Therefore we have to leave the "universe" of linearity for a moment constructing our gradient otopy.

The proof falls naturally into four steps.

Step 1 (diagonalization). There exists $P\in SO(n)$ such that $PAP^{-1}=\mathrm{diag}(\alpha_1,\ldots,\alpha_n)$. Since $SO(n)$ is path-connected, P is homotopic to Id in

$SO(n)$ by some homotopy P_t, i.e. $P_t \in SO(n)$, $P_0 = \mathrm{Id}$ and $P_1 = P$. Therefore, the homotopy

$$H_t = P_t A P_t^{-1} \in GL_n(\mathbb{R}) \cap Symm(n)$$

connects A to $\mathrm{diag}(\alpha_1, \ldots, \alpha_n)$.

Step 2 (normalization). The homotopy

$$H_t = \mathrm{diag}(t\alpha_1 + (1-t)\,\mathrm{sgn}\,\alpha_1, \ldots, t\alpha_n + (1-t)\,\mathrm{sgn}\,\alpha_n) \in GL_n(\mathbb{R}) \cap Symm(n)$$

connects $\mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ to a diagonal matrix with all diagonal elements equal to ± 1, i.e. to $B = \mathrm{diag}(\beta_1, \ldots, \beta_n)$ with $\beta_i^2 = 1$. If $B = \mathrm{Id}$, then we are done. So suppose it is not the case. If $B = -\mathrm{Id}$, go to Step 4. Otherwise, go to the next step.

Step 3 (swapping pairs of adjacent 1 and -1 on the diagonal). The homotopy

$$H_t = \begin{pmatrix} \cos(\pi t) & \sin(\pi t) \\ \sin(\pi t) & -\cos(\pi t) \end{pmatrix}$$

shows that $\mathrm{diag}(1, -1)$ and $\mathrm{diag}(-1, 1)$ are homotopic in $GL_2(\mathbb{R}) \cap Symm(2)$. Repeated application of this observation enables us to gather all -1's at the top of the diagonal. Precisely, $\mathrm{diag}(\beta_1, \ldots, \beta_n)$ with $\beta_i^2 = 1$ is homotopic in $GL_n(\mathbb{R}) \cap Symm(n)$ to $C = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$ with $\gamma_i \leq \gamma_j$ for $i < j$ (i.e. all -1's followed by all 1's). If $C = \mathrm{R}$, then we are done, but otherwise we have to elaborate a little.

Step 4 (replacement of pairs of adjacent -1's on the diagonal with pairs of adjacent 1's). It should be emphasized that all the above steps used "very nice" homotopies in $GL_n(\mathbb{R}) \cap Symm(n)$, which, of course, are also gradient otopies. But the last step, as we have mentioned above, cannot be realized in $GL_n(\mathbb{R}) \cap Symm(n)$ because of the invariance of signature. However, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ are gradient otopic by Proposition 3.4. Consequently, both the matrix C obtained in the previous step and the matrix $-\mathrm{Id}$ from Step 2 are gradient otopic either to Id or to R, which completes the proof. ∎

4. Proof of the Main Lemma. The proof of the Main Lemma will consist of two parts.

Surjectivity. By definition, any standard map of type n is gradient and has degree n.

Injectivity. By Corollary 3.3 it is sufficient to show that any gradient local map of degree k is gradient otopic to some standard map of type (m, l) with $m - l = k$. For clarity, this will be proved in four steps. Let $(f, U) \in \mathcal{F}_\nabla(n)$ (i.e. $f = \nabla\varphi$) and $\deg(f, U) = k$. Of course, there is no loss of generality in assuming that U is bounded.

Step 1. We deform the potential φ to a Morse function ψ. By density and openness of the set of Morse functions (see for instance [6, Ch. 6]) we can choose a Morse function ψ such that the straight-line homotopy of potentials $\chi(x,t) = (1-t)\varphi + t\psi$ induces a gradient homotopy on U (hence also a gradient otopy) between $\nabla\varphi$ and $\nabla\psi$.

Step 2. The gradient otopy class of the local map $(\nabla\psi, U)$ is uniquely determined by its restriction to a disjoint union of open balls around non-degenerate critical points of ψ, since any such restriction map induces a gradient otopy.

Step 3. Next we replace $\nabla\psi$ on each of these balls by its linear approximation around the center of the ball. Precisely, since the derivative of the gradient $\nabla\psi$ is the Hessian matrix of the potential ψ, we have

$$\nabla\psi(x) = \operatorname{Hess}\psi(p)[x-p] + o(|x-p|) \qquad \text{for } x \in B(p,r).$$

If we make the radius r sufficiently small, then the straight-line homotopy of the potential

$$\chi(x,t) = (1-t)\psi(x) + \frac{1}{2}t[x-p]^T \operatorname{Hess}(p)[x-p]$$

induces the gradient homotopy on the reduced ball around p, which is also a gradient otopy.

Step 4. By Proposition 3.5 any derivative of the gradient map (i.e. the Hessian matrix of the potential) is gradient otopic either to the identity or to the reflection. Thus any map obtained in the previous step is gradient otopic to some standard map of type (m,l) with $m-l=k$, which finishes the proof of the lemma. ∎

Corollary 4.1. *The map* $\mathfrak{g}\circ\mathfrak{f}\colon \mathcal{F}_\nabla[n] \to \mathcal{F}[n]$ *is a bijection.*

Proof. This follows immediately from Remark 2.3, Main Lemma and the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{F}_\nabla[n] & \xrightarrow{\ \mathfrak{gf}\ } & \mathcal{F}[n] \\ & {\scriptstyle\deg}\searrow \quad \swarrow{\scriptstyle\deg} & \\ & \mathbb{Z} & \end{array}$$

∎

It is now easy to show that the maps in the bottom row of the diagram (2.1) are bijections.

Proposition 4.2. *The map* $\mathfrak{f}\colon \mathcal{F}_\nabla[n] \to \mathcal{F}_{\mathrm{gl}}[n]$ *is a surjection.*

Proof. Observe that it suffices to make the following observation. If (f,U) is gradient-like and φ is a potential appearing in the definition of being gradient-like, then (f, U) is gradient-like otopic to $(\nabla\varphi, U)$. The required gradient-like otopy is given simply by the straight-line homotopy

$$h_t(x) = (1-t)f(x) + t\nabla\varphi(x).$$

Consequently, $\mathfrak{f}\colon \mathcal{F}_\nabla[n] \to \mathcal{F}_{\mathrm{gl}}[n]$ is surjective, since $[(f, U)] = [(\nabla\varphi, U)]$ in $\mathcal{F}_{\mathrm{gl}}[n]$. ∎

Let us mention an easy consequence of Corollary 4.1 and Proposition 4.2.

COROLLARY 4.3. *The maps* $\mathfrak{f}$ *and* $\mathfrak{g}$ *in the diagram* (2.1) *are bijections.*

5. Proper gradient local maps

PROPOSITION 5.1. *The map* $\mathfrak{c}\colon \mathcal{P}_\nabla[n] \to \mathcal{F}_\nabla[n]$ *is a surjection.*

Proof. By the Main Lemma, we know that $\deg\colon \mathcal{F}_\nabla[n] \to \mathbb{Z}$ is bijective. It remains to prove that $\deg\colon \mathcal{P}_\nabla[n] \to \mathbb{Z}$ is surjective. But that is indeed the case, since we can construct a proper gradient local map of degree ± 1 on the open unit ball around 0. Namely, an easy computation shows that

$$\varphi(x_1, x_2, \ldots, x_n) = \frac{\pm x_1^2 + x_2^2 + \cdots + x_n^2}{1 - x_1^2 - x_2^2 - \cdots - x_n^2}$$

gives the desired potential. ∎

By the commutativity of the diagram (2.1), we easily get the following result.

COROLLARY 5.2. *The map* $\mathfrak{d}\colon \mathcal{P}_{\mathrm{gl}}[n] \to \mathcal{F}_{\mathrm{gl}}[n]$ *is a surjection.*

6. Proper gradient-like local maps. The proof of Proposition 6.3 below is based on the following two lemmas. In what follows, we make use of the otopy notation introduced in Section 1.

LEMMA 6.1. *If* (h, Ω) *is a gradient-like otopy, then*

(1) *there are open sets* $\widetilde{\Omega}$, $\widehat{\Omega}$ *in* $\mathbb{R}^n \times I$ *such that*

$$h^{-1}(0) \Subset \widetilde{\Omega} \subset \operatorname{cl}\widetilde{\Omega} \Subset \widehat{\Omega} \subset \operatorname{cl}\widehat{\Omega} \Subset \Omega,$$

(2) *there is a proper gradient-like otopy* $\widehat{h}\colon \widehat{\Omega} \to \mathbb{R}^n$ *such that*

(a) $h^{-1}(0) = \widehat{h}^{-1}(0)$,
(b) $h\restriction_{\operatorname{cl}\widetilde{\Omega}} = \widehat{h}\restriction_{\operatorname{cl}\widetilde{\Omega}}$.

Proof. Put

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in [0, \alpha], \\ \alpha/(2\alpha - x) & \text{if } x \in [\alpha, 2\alpha). \end{cases}$$

Now let W be an open set such that $h^{-1}(0) \Subset W \subset \operatorname{cl} W \Subset \Omega$. Define

- $2\alpha := \min\{|h(x,t)| \mid (x,t) \in \partial W\}$,
- $W^\beta := \{(x,t) \in W \mid |h(x,t)| < \beta\}$, where $\beta = \alpha$ or $\beta = 2\alpha$,

- $\widetilde{\Omega} := W^{\alpha}$, $\widehat{\Omega} := W^{2\alpha}$,
- $\widehat{h}(x,t) := \varphi(|h(x,t)|) \cdot h(x,t)$.

It is easily seen that $\widetilde{\Omega}$, $\widehat{\Omega}$ and $(\widehat{h}, \widehat{\Omega})$ satisfy the assertion of the lemma. ∎

Lemma 6.2. *Let* $(f,U), (\widehat{f}, \widehat{U}) \in \mathcal{P}_{\mathrm{gl}}(n)$. *Assume that*

(1) $f^{-1}(0) = \widehat{f}^{-1}(0)$,
(2) *there is an open set* V *such that*
   (a) $f^{-1}(0) \Subset V \subset \operatorname{cl} V \Subset U \cap \widehat{U}$,
   (b) $f\restriction_{\operatorname{cl} V} = \widehat{f}\restriction_{\operatorname{cl} V}$.

Then (f,U) *is proper gradient-like otopic to* $(\widehat{f}, \widehat{U})$.

Proof. We will make use of a family of auxiliary functions $\varphi_t\colon [0,\infty) \to [1, 1/t]$ indexed by the real parameter $t \in (0,1]$ and given by

$$\varphi_t(x) = \begin{cases} 1 & \text{if } x \in [0,\alpha], \\ \dfrac{t(2\alpha - x) + \alpha}{2\alpha - x + t\alpha} & \text{if } x \in [\alpha, 2\alpha], \\ 1/t & \text{if } x \in [2\alpha, \infty), \end{cases}$$

where $\alpha > 0$ (see Figure 3). Observe that

- φ_t is increasing on $[\alpha, 2\alpha]$ for each $t \in (0,1]$,
- the map $\varphi(x,t) := \varphi_t(x)$ is continuous both in x and t.

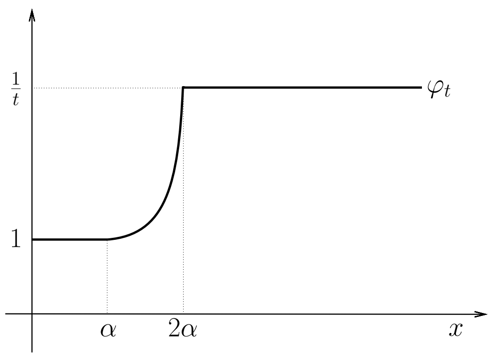

Fig. 3. The function φ_t

Moreover, since

$$\lim_{t \to 0^+} \varphi(x,t) = \begin{cases} 1 & \text{if } x \in [0,\alpha], \\ \alpha/(2\alpha - x) & \text{if } x \in [\alpha, 2\alpha), \\ \infty & \text{if } x \in [2\alpha, \infty), \end{cases}$$

we can consider φ as defined on $[0,\infty) \times (0,1] \sqcup [0, 2\alpha) \times 0$. As in the proof of Lemma 6.1, set

- $2\alpha := \min\{|f(x)| \mid x \in \partial V\}$,
- $V^\beta := \{x \in V \mid |f(x)| < \beta\}$, where $\beta = \alpha$ or $\beta = 2\alpha$.

Finally, define

- $h(x,t) := \varphi_t(|f(x)|) \cdot f(x)$,
- $\Omega_t := \begin{cases} U & \text{for } t \in (0,1], \\ V^{2\alpha} & \text{for } t = 0, \end{cases}$
- $\Omega := \bigcup_{t \in I} \Omega_t \times t$.

We check at once that (h, Ω) is a proper gradient-like otopy connecting (f, U) and $(h_0, V^{2\alpha})$. In the same way we can define a proper gradient-like otopy $(\widehat{h}, \widehat{\Omega})$ connecting $(\widehat{f}, \widehat{U})$ and $(\widehat{h}_0, \widehat{V}^{2\alpha})$. But since $f\restriction_{\mathrm{cl}\,V} = \widehat{f}\restriction_{\mathrm{cl}\,V}$, we have $(h_0, V^{2\alpha}) = (\widehat{h}_0, \widehat{V}^{2\alpha})$, which proves the lemma. ■

Proposition 6.3. *The map $\mathfrak{d}\colon \mathcal{P}_{\mathrm{gl}}[n] \to \mathcal{F}_{\mathrm{gl}}[n]$ is a bijection.*

Proof. By Proposition 5.2, $\mathfrak{d}$ is surjective. So we only need to prove it is injective. Let $(f_0, U_0), (f_1, U_1) \in \mathcal{P}_{\mathrm{gl}}(n)$ be gradient-like otopic. We will show that they are also proper gradient-like otopic. By assumption, there is a gradient-like otopy (h, Ω) such that $(h_i, \Omega_i) = (f_i, U_i)$ for $i = 0, 1$. Let $\sim$ denote for a moment the relation of being proper gradient-like otopic. Then combining Lemmas 6.1 and 6.2 we get

$$(f_0, U_0) = (h_0, \Omega_0) \overset{6.2}{\sim} (\widehat{h}_0, \widehat{\Omega}_0) \overset{6.1}{\sim} (\widehat{h}_1, \widehat{\Omega}_1) \overset{6.2}{\sim} (h_1, \Omega_1) = (f_1, U_1). \ ■$$

By the commutativity of the diagram (2.1), we immediately obtain the following result.

Corollary 6.4. *The map $\mathfrak{b}\colon \mathcal{P}_{\mathrm{gl}}[n] \to \mathcal{P}[n]$ is a bijection.*

Once again, the commutativity of the diagram (2.1), Corollary 4.3, Proposition 5.1 and Corollary 6.3 yields the following conclusion.

Corollary 6.5. *The map $\mathfrak{a}\colon \mathcal{P}_{\nabla}[n] \to \mathcal{P}_{\mathrm{gl}}[n]$ is a surjection.*

Remark 6.6. It is worth pointing out that Lemmas 6.1 and 6.2 make sense also for parametrized local maps. Namely, let

$$\mathcal{F}(n,k) := \{f\colon U \to \mathbb{R}^n \mid U \subset \mathbb{R}^{n+k} \text{ is open, } f^{-1}(0) \Subset U\}$$

be the set of parametrized local maps (with parameter in $\mathbb{R}^k$). Similarly, we can define

- the sets $\mathcal{P}(n,k)$, $\mathcal{F}_{\nabla}(n,k)$, etc.,
- otopies of parametrized local maps,
- the sets of otopy classes: $\mathcal{F}[n,k]$, $\mathcal{P}[n,k]$, etc.

Of course, $\mathcal{P}[n,k]$ is isomorphic to $\pi_{n+k} S^n$. The analogues of Lemmas 6.1 and 6.2 imply that the inclusion $\mathcal{P}(n,k) \subset \mathcal{F}(n,k)$ induces an injection $\mathcal{P}[n,k] \to \mathcal{F}[n,k]$. The same is true with subscript gl. The surjectivity of the above maps is an immediate consequence of two obvious observations:

- any local map (gl local map) can be easily perturbed to a close proper one,
- the straight-line homotopy between a given local map (gl local map) and its close proper perturbation establishes the required otopy.

7. Proof of the Main Theorem. We just gather our partial results:

- $\mathfrak{b}$ is a bijection by Corollary 6.4,
- $\mathfrak{d}$ is a bijection by Proposition 6.3,
- $\mathfrak{e}$ is a bijection by Remark 2.3,
- $\mathfrak{f}$ and $\mathfrak{g}$ are bijections by Corollary 4.3,
- $\mathfrak{a}$ is a surjection by Corollary 6.5,
- $\mathfrak{c}$ is a surjection by Proposition 5.1. ∎

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