FUNDAMENTA
MATHEMATICAE
216 (2012)

Dynamical characterization of C-sets and its application

by

Jian Li (Hefei)

Abstract. We set up a general correspondence between algebraic properties of $\beta\mathbb{N}$ and sets defined by dynamical properties. In particular, we obtain a dynamical characterization of C-sets, i.e., sets satisfying the strong Central Sets Theorem. As an application, we show that Rado systems are solvable in C-sets.

1. Introduction. Throughout this paper, $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$ and $\mathbb{Q}$ denote the sets of integers, non-negative integers, positive integers and rational numbers, respectively. Let us recall two celebrated theorems in combinatorial number theory.

THEOREM 1.1 (van de Waerden). *Let $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} C_i$. Then there exists $i \in \{1, \ldots, r\}$ such that C_i contains arbitrarily long arithmetic progressions.*

For a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{N}$, define the set of finite sums of $\{x_n\}_{n=1}^{\infty}$ as

$$\mathrm{FS}(\{x_n\}_{n=1}^{\infty}) = \Big\{ \sum_{n\in\alpha} x_n : \alpha \text{ is a nonempty finite subset of } \mathbb{N} \Big\}.$$

A subset F of $\mathbb{N}$ is called an *IP set* if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{N}$ such that $\mathrm{FS}(\{x_n\}_{n=1}^{\infty}) \subset F$.

THEOREM 1.2 (Hindman). *Let $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} C_i$. Then there exists $i \in \{1, \ldots, r\}$ such that C_i is an IP set.*

The original proofs of the above two theorems by combinatorial methods are somewhat complicated. In [14, 12] Furstenberg and Weiss found a new way to prove those theorems by topological dynamics methods.

A subset F of $\mathbb{N}$ is called *central* if there exists a dynamical system (X, T), a point $x \in X$, a minimal point y which is proximal to x, and an open neighborhood U of y such that $F = \{n \in \mathbb{N} : T^n x \in U\}$. The van de Waerden Theorem and the Hindman Theorem follow from the following result.

2010 *Mathematics Subject Classification*: Primary 37B20; Secondary 37B05, 05D10.
Key words and phrases: central set, C-set, Rado system.

DOI: 10.4064/fm216-3-4

Theorem 1.3 ([14, 12]).

(1) *Every central set is an IP set and contains arbitrarily long arithmetic progressions.*
(2) *Let* $r \in \mathbb{N}$ *and* $\mathbb{N} = \bigcup_{i=1}^{r} C_i$. *Then there exists* $i \in \{1, \ldots, r\}$ *such that* C_i *is a central set.*

Before going on, let us recall some notions. We call $(S, \cdot)$ a *compact Hausdorff right topological semigroup* if S is endowed with a compact Hausdorff space topology and for each $t \in S$ the right translation $s \mapsto s \cdot t$ is continuous. An *idempotent* $t \in S$ is an element satisfying $t \cdot t = t$. The *Ellis–Namakura Theorem* says that any compact Hausdorff right topological semigroup contains some idempotent. A subset I of S is called a *left ideal* of S if $SI \subset I$, a *right ideal* if $IS \subset I$, and a *two-sided ideal* (or simply an *ideal*) if it is both a left and right ideal. A *minimal left ideal* is the left ideal that does not contain any proper left ideal. Similarly, we can define a *minimal right ideal* and a *minimal ideal*. An idempotent in S is called a *minimal idempotent* if it is contained in some minimal left ideal of S.

Endowing $\mathbb{N}$ with the discrete topology, we take the points of the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$. Since $(\mathbb{N}, +)$ is a semigroup, we extend the operation $+$ to $\beta\mathbb{N}$ so that $(\beta\mathbb{N}, +)$ is a compact Hausdorff right topological semigroup. See [19] for an exhaustive treatment of the algebraic structure on $\beta\mathbb{N}$.

Ellis showed that we can regard $(\beta\mathbb{N}, \mathbb{N})$ as a universal point transitive system ([10]). One may expect that there is a natural connection between algebraic properties of $\beta\mathbb{N}$ and sets defined by dynamical properties. For example, in [5] Bergelson and Hindman showed that

Theorem 1.4 ([5]). *A subset* F *of* $\mathbb{N}$ *is central if and only if there exists a minimal idempotent* $p \in \beta\mathbb{N}$ *such that* $F \in p$.

A subset F of $\mathbb{N}$ is called *quasi-central* if there exists an idempotent $p \in \beta\mathbb{N}$ with each element piecewise syndetic such that $F \in p$. Of course, every quasi-central set is central, but not conversely ([18]). The authors of [8] gave a dynamical characterization of quasi-central sets:

Theorem 1.5 ([8]). *A subset* F *of* $\mathbb{N}$ *is quasi-central if and only if there exists a dynamical system* (X, T), *a pair of points* $x, y \in X$ *where for every open neighborhood* V *of* y *the set* $\{n \in \mathbb{N} : T^n x \in V, T^n y \in V\}$ *is piecewise syndetic, and an open neighborhood* U *of* y *such that* $F = \{n \in \mathbb{N} : T^n x \in U\}$.

A subset F of $\mathbb{N}$ is called a *D-set* if there exists an idempotent $p \in \beta\mathbb{N}$ with each element having positive upper Banach density such that $F \in p$. It should be noticed that every quasi-central set is a D-set, but not conversely ([4]). There is also a dynamical characterization of D-sets:

THEOREM 1.6 ([4]). *A subset F of $\mathbb{N}$ is a D-set if and only if there exists a dynamical system (X,T), a pair of points $x,y \in X$ where for every open neighborhood V of y the set $\{n \in \mathbb{N} : T^n y \in V\}$ has positive upper Banach density and (y,y) belongs to the orbit closure of (x,y) in the product system $(X \times X, T \times T)$, and an open neighborhood U of y such that $F = \{n \in \mathbb{N} : T^n x \in U\}$.*

Central sets have substantial combinatorial content. In order to describe their properties, we first introduce some notation. By $\mathcal{P}_{\mathrm{f}}(\mathbb{N})$ we denote the set of all nonempty finite subsets of $\mathbb{N}$. For $\alpha, \beta \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})$, we write $\alpha < \beta$ if $\max \alpha < \min \beta$. Given a sequence $s_1, s_2, \dots$ in $\mathbb{Z}$ or $\mathbb{Z}^m$ and $\alpha \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})$ we let $s_\alpha = \sum_{n \in \alpha} s_n$ and call the family $(s_\alpha)_{\alpha \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})}$ an *IP-system.* A *homomorphism* $\phi : \mathcal{P}_{\mathrm{f}}(\mathbb{N}) \to \mathcal{P}_{\mathrm{f}}(\mathbb{N})$ is a map such that (1) if $\alpha \cap \beta = \emptyset$, then $\phi(\alpha) \cap \phi(\beta) = \emptyset$ and (2) $\phi(\alpha \cup \beta) = \phi(\alpha) \cup \phi(\beta)$. Evidently such a homomorphism is determined by $\phi(\{i\})$ for each $i \in \mathbb{N}$, and then $\phi(\alpha) = \bigcup_{i \in \alpha} \phi(\{i\})$. Given an IP-system $\{s_\alpha\}$, an IP-subsystem is defined by a homomorphism $\phi : \mathcal{P}_{\mathrm{f}}(\mathbb{N}) \to \mathcal{P}_{\mathrm{f}}(\mathbb{N})$ and forming $\{s_{\phi(\alpha)}\} \subset \{s_\alpha\}$. If $r \in \mathbb{Z}$, we shall denote by $\bar{r}^{(m)}$ the vector $(r, \dots, r) \in \mathbb{Z}^m$.

PROPOSITION 1.7 (Central Sets Theorem [12]). *Let F be a central set in $\mathbb{N}$, and for any $m \geq 1$, let $\{s_\alpha\}$ be any IP-system in $\mathbb{Z}^m$. Then there exists an IP-subsystem $\{s_{\phi(\alpha)}\}$ and an IP-system $\{r_\alpha\}$ in $\mathbb{N}$ such that the vector $\bar{r}_\alpha^{(m)} + s_{\phi(\alpha)}$ is in F^m for each $\alpha \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})$.*

Recently, the authors of [9, 20] proved a stronger version of the Central Sets Theorem, and defined C-sets to be the sets satisfying the conclusion of that stronger version. Here we will not discuss the strong Central Sets Theorem, so we adopt an alternative definition of C-sets.

A subset F of $\mathbb{N}$ is called a *J-set* if for every $m \in \mathbb{N}$ and every IP-system $\{s_\alpha\}$ in $\mathbb{Z}^m$ there exist $r \in \mathbb{N}$ and $\alpha \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})$ such that $\bar{r}^{(m)} + s_\alpha \in F^m$. Denote by $\mathcal{J}$ the collection of all J-sets. A subset F of $\mathbb{N}$ is called a *C-set* if there exists an idempotent $p \in \beta\mathbb{N}$ with each element being a J-set such that $F \in p$. Since every positive upper Banach density set is a J-set ([13]), every D-set is a C-set. But there exist C-sets with zero upper Banach density ([17]), so they are not D-sets.

In this paper, we obtain a dynamical characterization of C-sets.

THEOREM 1.8. *A subset F of $\mathbb{N}$ is a C-set if and only if there exists a dynamical system (X,T), a pair of points $x,y \in X$ where for any open neighborhood V of y the set $\{n \in \mathbb{N} : T^n y \in V\}$ is a J-set and (y,y) belongs to the orbit closure of (x,y) in the product system $(X \times X, T \times T)$, and an open neighborhood U of y such that $F = \{n \in \mathbb{N} : T^n x \in U\}$.*

In [12] Furstenberg used the Central Sets Theorem to show that any central subset of $\mathbb{N}$ contains solutions to all Rado systems. Let $A = (a_{ij})$ be a $p \times q$ matrix over $\mathbb{Q}$. The homogeneous system of linear equations

$$A(x_1, \ldots, x_q)^T = 0$$

is called *partition regular* (or a *Rado system*) if for every $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^r C_i$, there exists $i \in \{1, \ldots, r\}$ such that the system has a solution $(x_1, \ldots, x_q)$ all of whose components lie in C_i. In [26] Rado characterized when a homogeneous system of linear equations is partition regular.

THEOREM 1.9 (Rado's Theorem). *Let $A = (a_{ij})$ be a $p \times q$ matrix over $\mathbb{Q}$. Then the system $A(x_1, \ldots, x_q)^T = 0$ is partition regular if and only if the index set $\{1, \ldots, q\}$ can be divided into l disjoint subsets $I_1, \ldots, I_l$ and rational numbers c_j^r may be found for $r \in \{1, \ldots, l\}$ and $j \in I_1 \cup \cdots \cup I_r$ such that the following relations are satisfied:*

$$\begin{aligned}
&\sum_{j \in I_1} a_{ij} = 0, \\
&\sum_{j \in I_2} a_{ij} = \sum_{j \in I_1} c_j^1 a_{ij}, \\
&\cdots \\
&\sum_{j \in I_l} a_{ij} = \sum_{j \in I_1 \cup \cdots \cup I_{l-1}} c_j^{l-1} a_{ij}.
\end{aligned}$$

Let F be a subset of $\mathbb{N}$. We say that *Rado systems are solvable in F* if every Rado system $A(x_1, \ldots, x_q)^T = 0$ has a solution $(x_1, \ldots, x_q)$ all of whose components lie in F.

Furstenberg and Weiss improved Rado's result by showing that

THEOREM 1.10 ([14, 12]). *Rado systems are solvable in central sets.*

Recently, the authors of [3] extended Furstenberg and Weiss' result to

THEOREM 1.11 ([3]). *Rado systems are solvable in D-sets.*

In this paper, we use the dynamical characterization of C-sets to show

THEOREM 1.12. *Rado systems are solvable in C-sets.*

This paper is organized as follows. In Section 2 we introduce some notions related to Furstenberg families. In Section 3 the basic properties of the Stone–Čech compactification of $\mathbb{N}$ are discussed. In Section 4 we set up a general correspondence between algebraic properties of $\beta\mathbb{N}$ and sets defined by dynamical properties. The dynamical characterizations of quasi-central sets and D-sets are special cases of our results. In Section 5, we investigate the set's forcing, that is, the dynamical properties of a point along a subset of $\mathbb{N}$. In Section 6, we consider both addition and multiplication in $\mathbb{N}$ and $\beta\mathbb{N}$. In particular we show that if F is a quasi-central set or a D-set, then for every $n \in \mathbb{N}$ both nF and $n^{-1}F$ are also quasi-central sets or D-sets. In

Section 7 using the correspondence which is set up in Section 4 and some properties of J-sets, we obtain a dynamical characterization of C-sets. In Section 8, as an application, we give a topological dynamical proof of the fact that Rado systems are solvable in C-sets.

2. Furstenberg families. Let us recall some notions related to families (for more details see [1]). Denote by $\mathcal{P} = \mathcal{P}(\mathbb{N})$ the collection of all subsets of $\mathbb{N}$. A subset $\mathcal{F}$ of $\mathcal{P}$ is called a *Furstenberg family* (or just *family*) if it is upward hereditary, i.e., $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. A family $\mathcal{F}$ is called *proper* if it is a nonempty proper subset of $\mathcal{P}$, i.e., neither empty nor all of $\mathcal{P}$. For a family $\mathcal{F}$, the *dual family* of $\mathcal{F}$, denoted by $\kappa\mathcal{F}$, is

$$\{F \in \mathcal{P} : F \cap F' \neq \emptyset, \forall F' \in \mathcal{F}\}.$$

Sometimes the dual family $\kappa\mathcal{F}$ is also denoted by $\mathcal{F}^*$.

A family $\mathcal{F}$ is called a *filter* when it is a proper family closed under intersection, i.e., if $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$. A family $\mathcal{F}$ is called a *filterdual* if its dual $\kappa\mathcal{F}$ is a filter. It is easy to see that a proper family $\mathcal{F}$ is a filterdual if and only if it has the *Ramsey property*: whenever $F_1 \cup F_2 \in \mathcal{F}$ then either $F_1 \in \mathcal{F}$ or $F_2 \in \mathcal{F}$. Since $\kappa(\kappa\mathcal{F}) = \mathcal{F}$, a family $\mathcal{F}$ is a filter if and only if $\kappa\mathcal{F}$ is a filterdual.

Of special interest are filters that are maximal with respect to inclusion. Such a filter is called an *ultrafilter*. By Zorn's Lemma every filter is contained in some ultrafilter. For any $n \in \mathbb{N}$ the family $\{A \subset \mathbb{N} : n \in A\}$ is an ultrafilter, called a *principal ultrafilter*. Any other ultrafilter is *non-principal*. The following two lemmas give basic properties of ultrafilters (see [1, 15, 19] for example).

Lemma 2.1. *Let $\mathcal{F}$ be a filter. Then the following conditions are equivalent:*

(1) *$\mathcal{F}$ is an ultrafilter;*
(2) *$\mathcal{F} = \kappa\mathcal{F}$;*
(3) *$\mathcal{F}$ is a filterdual;*
(4) *for all $F \subset \mathbb{N}$, either $F \in \mathcal{F}$ or $\mathbb{N} \setminus F \in \mathcal{F}$.*

Lemma 2.2. *Let $\mathcal{F}$ be a filterdual and $\mathcal{A} \subset \mathcal{F}$. If for any finite collection of elements $A_1, \ldots, A_n$ in $\mathcal{A}$ the intersection $\bigcap_{i=1}^n A_i$ is in $\mathcal{F}$, then there exists an ultrafilter $\mathcal{F}'$ such that $\mathcal{A} \subset \mathcal{F}' \subset \mathcal{F}$.*

For $n \in \mathbb{Z}$ and $F \subset \mathbb{N}$, denote $n + F = \{n + m \in \mathbb{N} : m \in F\}$. A family $\mathcal{F}$ is called *translation $+$ invariant* if $n + F \in \mathcal{F}$ for every $n \in \mathbb{Z}_+$ and $F \in \mathcal{F}$, *translation $-$ invariant* if $-n + F \in \mathcal{F}$ for every $n \in \mathbb{Z}_+$ and $F \in \mathcal{F}$, and *translation invariant* if it is both $+$ and $-$ invariant.

Any nonempty collection $\mathcal{A}$ of subsets of $\mathbb{N}$ naturally generates a family

$$\mathcal{F}(\mathcal{A}) = \{F \subset \mathbb{N} : F \supset A \text{ for some } A \in \mathcal{A}\}.$$

A collection $\mathcal{A}$ of subsets of $\mathbb{N}$ is said to have the *finite intersection property* if the intersection of any finite collection of elements in $\mathcal{A}$ is not empty. In this case, the family generated by $\mathcal{A}$ is a filter.

Let $\mathcal{F}$ be a family. The *block family* of $\mathcal{F}$, denote by $b\mathcal{F}$, is the family consisting of sets $F \subset \mathbb{N}$ for which there exists some $F' \in \mathcal{F}$ such that for every finite subset W of F' one has $m + W \subset F$ for some $m \in \mathbb{Z}_+$. It is easy to see that $F \in b\mathcal{F}$ if and only if there exists a sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{Z}_+$ and $F' \in \mathcal{F}$ such that $\bigcup_{n=1}^{\infty}(a_n + F' \cap [1, n]) \subset F$. Clearly, $b(b\mathcal{F}) = b\mathcal{F}$ and $b\mathcal{F}$ is translation + invariant.

LEMMA 2.3 ([7, 22]). *If $\mathcal{F}$ is a filterdual, then so is $b\mathcal{F}$.*

Now let us recall some important sets and families. Let $\mathcal{F}_{\mathrm{inf}}$ be the family of all infinite subsets of $\mathbb{Z}_+$. It is easy to see that its dual family $\kappa\mathcal{F}_{\mathrm{inf}}$ is the family of all cofinite subsets, denoted by $\mathcal{F}_{\mathrm{cf}}$.

A subset F of $\mathbb{Z}_+$ is called *thick* if it contains arbitrarily long runs of positive integers, i.e., there exists a sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{Z}_+$ such that $\bigcup_{n=1}^{\infty}(a_n + [1, n]) \subset F$; *syndetic* if there exists $N \in \mathbb{N}$ such that $[n, n + N] \cap F \neq \emptyset$ for every $n \in \mathbb{N}$; *piecewise syndetic* if it is the intersection of a thick set and a syndetic set. The families of all thick sets, all syndetic sets and all piecewise syndetic sets are denoted by $\mathcal{F}_{\mathrm{t}}$, $\mathcal{F}_{\mathrm{s}}$ and $\mathcal{F}_{\mathrm{ps}}$, respectively. It is easy to see that $\kappa\mathcal{F}_{\mathrm{s}} = \mathcal{F}_{\mathrm{t}}$.

Let F be a subset of $\mathbb{N}$. The *upper density of F* is

$$\bar{d}(F) = \limsup_{n\to\infty} \frac{|F \cap [1, n]|}{n},$$

where $|\cdot|$ denotes cardinality, and the *upper Banach density* of F is

$$BD^*(F) = \limsup_{|I|\to\infty} \frac{|F \cap I|}{|I|}$$

where I runs over all nonempty finite intervals of $\mathbb{N}$. Using density we can define lots of interesting families. For example, $\mathcal{F}_{\mathrm{pud}}$ and $\mathcal{F}_{\mathrm{pubd}}$ are the families of sets with positive upper density and positive upper Banach density respectively.

Denote by $\mathcal{F}_{\mathrm{ip}}$ and $\mathcal{F}_{\mathrm{cen}}$ the family of all IP sets and all central sets respectively. We have the following basic properties of the familiar families (see [1, 19] for example).

LEMMA 2.4.

(1) $\mathcal{F}_{\mathrm{cen}}$, $\mathcal{F}_{\mathrm{ip}}$, $\mathcal{F}_{\mathrm{ps}}$, $\mathcal{F}_{\mathrm{pud}}$ *and* $\mathcal{F}_{\mathrm{pubd}}$ *are filterduals.*
(2) $\mathcal{F}_{\mathrm{ps}}$, $\mathcal{F}_{\mathrm{pud}}$, $\mathcal{F}_{\mathrm{pubd}}$ *and* $\mathcal{F}_{\mathrm{s}}$ *are translation invariant.*
(3) $b\mathcal{F}_{\mathrm{cf}} = \mathcal{F}_{\mathrm{t}}$, $b\mathcal{F}_{\mathrm{s}} = \mathcal{F}_{\mathrm{ps}}$ *and* $b\mathcal{F}_{\mathrm{pud}} = \mathcal{F}_{\mathrm{pubd}}$.

We now introduce the notion of $\mathcal{F}$-limit. Let $\mathcal{F}$ be a family and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a topological space. We say that x is an *$\mathcal{F}$-limit* of $\{x_n\}$ if for every open neighborhood U of x the set $\{n\in\mathbb{N} : x_n\in U\}$ is in $\mathcal{F}$. The $\mathcal{F}_{\mathrm{cf}}$-limit is just the ordinary limit. It is easy to check that if $\mathcal{F}$ is a filter then $\mathcal{F}\text{-}\lim x_n$ exists and is unique in every compact Hausdorff space.

3. $\beta\mathbb{N}$: the Stone–Čech compactification of $\mathbb{N}$. Endowing $\mathbb{N}$ with the discrete topology, we take the points of the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$, the principal ultrafilters being identified with the points of $\mathbb{N}$. For $A\subset\mathbb{N}$, let $\overline{A}=\{p\in\beta\mathbb{N} : A\in p\}$. Then the sets $\{\overline{A} : A\subset\mathbb{N}\}$ form a basis for the open sets (and a basis for the closed sets) of $\beta\mathbb{N}$.

Since $(\mathbb{N},+)$ is a semigroup, we can extend the operation $+$ to $\beta\mathbb{N}$ by

$$p+q=\{F\subset\mathbb{N} : \{n\in\mathbb{N} : -n+F\in q\}\in p\}.$$

Then $(\beta\mathbb{N},+)$ is a compact Hausdorff right topological semigroup with $\mathbb{N}$ contained in the topological center of $\beta\mathbb{N}$. That is, for each $p\in\beta\mathbb{N}$ the map $\rho_p:\beta\mathbb{N}\to\beta\mathbb{N}$, $q\mapsto q+p$, is continuous, and for each $n\in\mathbb{N}$ the map $\lambda_n:\beta\mathbb{N}\to\beta\mathbb{N}$, $q\mapsto n+q$, is continuous. It is well known that $\beta\mathbb{N}$ has a smallest ideal $K(\beta\mathbb{N})=\bigcup\{L : L \text{ is a minimal left ideal of } \beta\mathbb{N}\}=\bigcup\{R : R$ is a minimal right ideal of $\beta\mathbb{N}\}$ ([19, Theorem 2.8]).

Lemma 3.1. *Let $\mathcal{F}$ be a filter. If for every $F\in\mathcal{F}$ there exists some $F'\in F$ such that $-n+F\in\mathcal{F}$ for every $n\in F'$, then $\bigcap_{F\in\mathcal{F}}\overline{F}$ is a closed subsemigroup of $\beta\mathbb{N}$.*

Proof. Since $\mathcal{F}$ has the finite intersection property, $\bigcap_{F\in\mathcal{F}}\overline{F}$ is nonempty. Let $p,q\in\bigcap_{F\in\mathcal{F}}\overline{F}$. We want to show that $p+q\in\bigcap_{F\in\mathcal{F}}\overline{F}$. Let $F\in\mathcal{F}$. It suffices to show that $F\in p+q$. For this F, there exists some $F'\in\mathcal{F}$ such that $-n+F\in\mathcal{F}$ for every $n\in F'$. Then $F'\subset\{n\in\mathbb{N} : -n+F\in q\}$ and $\{n\in\mathbb{N} : -n+F\in q\}\in p$. By the definition of "+" in $\beta\mathbb{N}$ we have $F\in p+q$. ∎

Lemma 3.2 ([19, Theorem 4.20]). *Let $\mathcal{A}$ be a collection of subsets of $\mathbb{N}$. If $\mathcal{A}$ has the finite intersection property and for every $F\in\mathcal{A}$ and $n\in F$ there exists $F'\in\mathcal{A}$ such that $n+F'\subset F$, then $\bigcap_{F\in\mathcal{A}}\overline{F}$ is a closed subsemigroup of $\beta\mathbb{N}$.*

For a filterdual $\mathcal{F}$, the *hull* of $\mathcal{F}$ is defined by

$$h(\mathcal{F})=\{p\in\beta\mathbb{N} : p\subset\mathcal{F}\}.$$

It is a nonempty closed subset of $\beta\mathbb{N}$, and $F\in\mathcal{F}$ if and only if $\overline{F}\cap h(\mathcal{F})\neq\emptyset$. Conversely, for a nonempty closed subset Z of $\beta\mathbb{N}$, the *kernel* of Z is defined by

$$k(Z)=\{F\subset\mathbb{N} : \overline{F}\cap Z\neq\emptyset\}.$$

It is a filterdual, and $h(k(Z)) = Z$ and $k(h(\mathcal{F})) = \mathcal{F}$. Thus, we obtain a one-to-one correspondence between the set of filterduals on $\mathbb{N}$ and the set of nonempty closed subsets of $\beta\mathbb{N}$ ([10, 15]).

LEMMA 3.3 ([15, 19]). *We have the following correspondences:*

(1) $h(\mathcal{F}_{\mathrm{ps}}) = \overline{K(\beta\mathbb{N})}$.
(2) $h(\mathcal{F}_{\mathrm{ip}}) = \overline{\{p \in \beta\mathbb{N} : p \textit{ is an idempotent}\}}$.
(3) $h(\mathcal{F}_{\mathrm{cen}}) = \overline{\{p \in \beta\mathbb{N} : p \textit{ is a minimal idempotent}\}}$.
(4) $h(\mathcal{F}_{\mathrm{pubd}}) = \overline{\bigcup\{\mathrm{supp}(\mu) : \mu \in \mathcal{M}\}}$, *where* $\mathcal{M}$ *is the set of all* $\mathbb{N}$-*invariant probability measures on* $\beta\mathbb{N}$.

LEMMA 3.4. *Let* $\mathcal{F}$ *be a filterdual. Then* $\mathcal{F}$ *is translation* + *invariant if and only if* $h(\mathcal{F})$ *is a closed left ideal of* $\beta\mathbb{N}$.

Proof. Assume that $\mathcal{F}$ is translation + invariant. In order to show that $h(\mathcal{F})$ is a closed left ideal, it suffices to show that $m+h(\mathcal{F}) \subset h(\mathcal{F})$ for every $m \in \mathbb{N}$. Let $m \in \mathbb{N}$, $p \in h(\mathcal{F})$ and $F \in m+p$. Then $m \in \{n \in \mathbb{N} : -n+F \in p\}$ and $-m+F \in p \subset \mathcal{F}$. Since $\mathcal{F}$ is translation + invariant, $m+(-m+F) \subset F$, so $F \in \mathcal{F}$ and $m+p \subset \mathcal{F}$, i.e., $m+p \in h(\mathcal{F})$.

Conversely, assume that $h(\mathcal{F})$ is a closed left ideal of $\beta\mathbb{N}$. Let $F \in \mathcal{F}$ and $n \in \mathbb{N}$. We want to show that $n+F \in \mathcal{F}$. By Lemma 2.2, there exists some $p \in h(\mathcal{F})$ with $F \in p$. Clearly, $n \in \{m \in \mathbb{N} : -m+(n+F) \in p\}$, so $n+F \in n+p \in h(\mathcal{F})$ and $n+F \in \mathcal{F}$. ∎

LEMMA 3.5. *Let* $\mathcal{F}$ *be a filterdual and* $b\mathcal{F} = \mathcal{F}$. *Then* $h(\mathcal{F})$ *is a closed two-sided ideal of* $\beta\mathbb{N}$.

Proof. Since $b\mathcal{F}$ is translation + invariant, by Lemma 3.4, $h(\mathcal{F})$ is a closed left ideal of $\beta\mathbb{N}$. Thus it suffices to show that $h(\mathcal{F})$ is also a right ideal.

Let $p \in h(\mathcal{F})$, $q \in \beta\mathbb{N}$ and $A \in p+q$. We need to show that $A \in \mathcal{F}$. Let $F = \{n \in \mathbb{N} : -n+A \in q\}$. Then $F \in p \subset \mathcal{F}$. For every finite subset E of F, $\bigcap_{n\in E}(-n+A) \in q$ is not empty; choose $n_E \in \bigcap_{n\in E}(-n+A)$; then $n_E + E \subset A$. This implies $A \in b\mathcal{F} = \mathcal{F}$. ∎

Let $\mathcal{F}$ be a filterdual. We call $F \subset \mathbb{N}$ an *essential* $\mathcal{F}$-*set* if there is an idempotent $p \in h(\mathcal{F})$ such that $F \in p$. Denote by $\widetilde{\mathcal{F}}$ the collection of all essential $\mathcal{F}$-sets. Then $\widetilde{\mathcal{F}}$ is also a filterdual since

$$h(\widetilde{\mathcal{F}}) = \overline{\{p \in \beta\mathbb{N} : p \text{ is an idempotent in } h(\mathcal{F})\}}.$$

Let F be a subset of $\mathbb{N}$. Then

(1) F is an IP set if and only if it is an essential $b\mathcal{F}_{\mathrm{ip}}$-set.
(2) F is a quasi-central set if and only if it is an essential $\mathcal{F}_{\mathrm{ps}}$-set.
(3) F is a D-set if and only if it is an essential $\mathcal{F}_{\mathrm{pubd}}$-set.

(4) F is a C-set if and only if it is an essential $\mathcal{J}$-set, where $\mathcal{J}$ is the collection of all J-sets.

THEOREM 3.6. *Let $\mathcal{F}$ be a translation invariant filterdual and $\{x_n\}_{n=1}^\infty$ be a sequence in* $\mathbb{N}$. *If* $\mathrm{FS}(\{x_n\}_{n=1}^\infty) \in \mathcal{F}$, *then for every* $m \in \mathbb{N}$, $\mathrm{FS}(\{x_n\}_{n=m}^\infty)$ *is an essential $\mathcal{F}$-set.*

Proof. We first prove the following claim.

CLAIM. *For each $m \in \mathbb{N}$, $h(\mathcal{F}) \cap \overline{\mathrm{FS}(\{x_n\}_{n=m}^\infty)} \neq \emptyset$.*

Proof of the Claim. Clearly, the claim holds for $m = 1$. Now assume that $m \geq 2$. Then

$$\begin{aligned}\mathrm{FS}(\{x_n\}_{n=1}^\infty) = \mathrm{FS}(\{x_n\}_{n=1}^{m-1}) &\cup \mathrm{FS}(\{x_n\}_{n=m}^\infty)\\ &\cup \{t + \mathrm{FS}(\{x_n\}_{n=m}^\infty) : t \in \mathrm{FS}(\{x_n\}_{n=1}^{m-1})\}.\end{aligned}$$

Since $\mathcal{F}$ is translation invariant, p cannot be a principal ultrafilter, so the finite set $\mathrm{FS}(\{x_n\}_{n=1}^{m-1})$ is not in p. If $\mathrm{FS}(\{x_n\}_{n=m}^\infty) \in p$, then the claim holds. Now assume that we have some $t \in \mathrm{FS}(\{x_n\}_{n=1}^{m-1})$ such that $t + \mathrm{FS}(\{x_n\}_{n=m}^\infty) \in p$. Choose $q \in \overline{\mathrm{FS}(\{x_n\}_{n=m}^\infty)}$ such that $t + q = p$. For every $F \in q$, $t \in \{n \in \mathbb{N} : -n + (t + F) \in q\}$, so $t + F \in p \subset \mathcal{F}$. Since $\mathcal{F}$ is translation invariant, we have $F \in \mathcal{F}$ and $q \in h(\mathcal{F})$. This ends the proof of the claim.

By Lemma 3.2, $\bigcap_{m=1}^\infty \overline{\mathrm{FS}(\{x_n\}_{n=m}^\infty)}$ is a closed subsemigroup of $\beta\mathbb{N}$, and by Lemma 3.4, $h(\mathcal{F})$ is a closed left ideal of $\beta\mathbb{N}$. Then by the above claim $h(\mathcal{F}) \cap \bigcap_{m=1}^\infty \overline{\mathrm{FS}(\{x_n\}_{n=m}^\infty)}$ is a nonempty subsemigroup of $\beta\mathbb{N}$. By the well known Ellis–Namakura Theorem, there exists some idempotent in this semigroup. Thus for every $m \in \mathbb{N}$, $\mathrm{FS}(\{x_n\}_{n=m}^\infty)$ is an essential $\mathcal{F}$-set. ■

For convenience, we also consider $\beta\mathbb{Z}_+$, the Stone–Čech compactification of $\mathbb{Z}_+$. There is a natural imbedding map $i : \beta\mathbb{N} \to \beta\mathbb{Z}_+$ defined by $i(p) = p \cup \{A \cup \{0\} : A \in p\}$. Thus we can regard $\beta\mathbb{N}$ as a subset of $\beta\mathbb{Z}_+$ and $\beta\mathbb{Z}_+ = \beta\mathbb{N} \cup \{0\}$. The advantage of $\beta\mathbb{Z}_+$ is that it contains the identity element 0, but we do not want to take 0 into account when considering multiplication.

4. Relationships between algebraic properties of $\beta\mathbb{N}$ and dynamical properties. A *topological dynamical system* (or just *system*) is a pair (X, T), where X is a nonempty compact Hausdorff space and T is a continuous map from X to itself. When X is metrizable or T is a homeomorphism, we call (X, T) a *metrizable* or *invertible dynamical system* respectively.

Let (X, T) be a dynamical system and $x \in X$. The *orbit* of x is $\mathrm{Orb}(x, T) = \{T^n x : n \in \mathbb{Z}_+\}$. Let $\omega(x, T)$ be the *ω-limit set* of x, i.e., the limit set of $\mathrm{Orb}(x, T)$. A point $x \in X$ is *recurrent* if $x \in \omega(x, T)$. We call the system (X, T) *minimal* if it contains no proper subsystems, and $x \in X$ is a *minimal point* if it belongs to some minimal subsystem of X.

A *factor map* $\pi : (X, T) \to (Y, S)$ is a continuous surjective map from X to Y such that $S \circ \pi = \pi \circ T$. In this situation (X, T) is said to be an *extension* of (Y, S), and (Y, S) is a *factor* of (X, T).

Let $\mathcal{F}$ be a family and (X, T) be a dynamical system. A point $x \in X$ is called *$\mathcal{F}$-recurrent* if for every open neighborhood U of x the entering time set $N(x, U) = \{n \in \mathbb{N} : T^n x \in U\}$ is in $\mathcal{F}$. If x is $\mathcal{F}$-recurrent, then so is Tx. Let $\pi : (X, T) \to (Y, S)$ be a factor map. If $x \in X$ is $\mathcal{F}$-recurrent, then so is $\pi(x)$. It is well known that x is recurrent if and only if it is $\mathcal{F}_{\mathrm{ip}}$-recurrent, and x is a minimal point if and only if it is $\mathcal{F}_{\mathrm{s}}$-recurrent. If $\mathcal{F}$ is a filter, then x is $\mathcal{F}$-recurrent if and only if $\mathcal{F}\text{-}\lim T^n x = x$.

Now we generalize the notion of ω-limit set. Let $\mathcal{F}$ be a family, (X, T) be a dynamical system and $x \in X$. A point $y \in X$ is called an *$\mathcal{F}$-ω-limit point* of x if for every neighborhood U of y the entering time set $N(x, U) \in \mathcal{F}$. Denote by $\omega_{\mathcal{F}}(x, T)$ the set of all $\mathcal{F}$-ω-limit points. Then x is $\mathcal{F}$-recurrent if and only if $x \in \omega_{\mathcal{F}}(x, T)$.

An *invariant measure* for a dynamical system (X, T) is a regular Borel probability measure μ on X such that $\mu(T^{-1}A) = \mu(A)$ for all Borel subsets A of X.

LEMMA 4.1. *Let (X, T) be a dynamical system and $x \in X$. If x is a recurrent point with $\overline{\mathrm{Orb}(x, T)} = X$, then*

(1) *x is $\mathcal{F}_{\mathrm{ps}}$-recurrent if and only if (X, T) has dense minimal points* ([24]).
(2) *x is $\mathcal{F}_{\mathrm{pubd}}$-recurrent if and only if for every open neighborhood U of x there exists an invariant measure μ on (X, T) such that $\mu(U) > 0$* ([23, 4]).

LEMMA 4.2. *Let $\mathcal{F}$ be a family and $p \in \beta\mathbb{N}$.*

(1) *If p is an idempotent and $p \subset \mathcal{F}$, then p is $\mathcal{F}$-recurrent in $(\beta\mathbb{Z}_+, \lambda_1)$.*
(2) *If p is $\mathcal{F}$-recurrent in $(\beta\mathbb{Z}_+, \lambda_1)$, then $p \subset b\mathcal{F}$.*

Proof. (1) For every neighborhood U of p, there exists some $F \in p$ such that $\overline{F} \subset U$. Then $N(p, \overline{F}) = \{n \in \mathbb{N} : (\lambda_1)^n p \in \overline{F}\} = \{n \in \mathbb{N} : n+p \in \overline{F}\} = \{n \in \mathbb{N} : -n+F \in p\}$. Since $F \in p = p+p$, we have $\{n \in \mathbb{N} : -n+F \in p\} \in p$. Thus $N(p, \overline{F}) \in \mathcal{F}$ and p is $\mathcal{F}$-recurrent.

(2) For every $F \in p$, $\overline{F}$ is an open neighborhood of p and $N(0, \overline{F}) = F$. Let $F' = N(p, \overline{F})$. Since p is $\mathcal{F}$-recurrent, $F' \in \mathcal{F}$. For every finite subset W of F', by the continuity of λ_1, there exists an open neighborhood U of p such that $(\lambda_1)^n U \subset \overline{F}$ for every $n \in W$. Since $p \in \overline{\mathrm{Orb}(0, \lambda_1)}$, there exists some $m \in \mathbb{Z}_+$ such that $(\lambda_1)^m 0 \in U$. Then $m + W \subset N(0, \overline{F})$. Thus, $F \in b\mathcal{F}$. ∎

Let (X, T) be a dynamical system. Then (X^X, T) also forms a dynamical system, where X^X is endowed with its compact, pointwise convergence topology and T acts on X^X as composition. The *enveloping semigroup* of

(X,T), denoted by $E(X,T)$, is defined as the closure of the set $\{T^n : n \in \mathbb{Z}_+\}$ in X^X.

From the algebraic point of view, $E(X,T)$ is a compact Hausdorff right topological semigroup. On the other hand, $(E(X,T),T)$ is a subsystem of (X^X,T). Those two structures are closely related. A subset $L \subset E(X,T)$ is a closed left ideal of $E(X,T)$ if and only if (L,T) is a subsystem of $(E(X,T),T)$, and L is a minimal left ideal of $E(X,T)$ if and only if (L,T) is a minimal subsystem of $(E(X,T),T)$.

If $\pi : (X,T) \to (Y,S)$ is a factor map, then there is a unique continuous semigroup homomorphism $\tilde{\pi} : E(X,T) \to E(Y,S)$ such that $\pi(px) = \tilde{\pi}(p)\pi(x)$.

Let (X,T) be a dynamical system and I be any nonempty set. Let X^I be the product space and define $T^{(I)} : X^I \to X^I$ by $T^{(I)}((x_i)_{i\in I}) = (Tx_i)_{i\in I}$. Then there is a natural isomorphism between $E(X,T)$ and $E(X^I,T^{(I)})$. For convenience, we regard $E(X,T)$ acting on factors of (X,T) and on product systems of (X,T).

For each $x \in X$, there is a canonical factor map

$$\varphi_x : E(X,T) \to (\overline{\mathrm{Orb}(x,T)},T), \quad q \mapsto qx.$$

Let (X,T) be a dynamical system. $\mathbb{Z}_+$ acts on X as

$$\Phi : \mathbb{Z}_+ \times X \to X, \quad (n,x) \mapsto T^n x.$$

Since $\beta\mathbb{Z}_+$ is the Stone–Čech compactification of $\mathbb{Z}_+$, we can extend Φ to

$$\beta\mathbb{Z}_+ \times X \to X, \quad (p,x) \mapsto px.$$

For each $x \in X$, the map $\Phi_x : (\beta\mathbb{Z}_+,\lambda_1) \to (\overline{\mathrm{Orb}(x,T)},T)$, $p \mapsto px$, is a factor map and $\Phi_x(\beta\mathbb{N}^*) = \omega(x,T)$, where $\beta\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$.

LEMMA 4.3. *Let (X,T) be a dynamical system, $x \in X$ and $p \in \beta\mathbb{N}$. Then $px = p\text{-}\lim T^n x$.*

Proof. Clearly, the result holds for principal ultrafilters. Now we assume that p is a non-principal ultrafilter. Consider the factor map

$$\Phi_x : (\beta\mathbb{Z}_+,\lambda_1) \to (\overline{\mathrm{Orb}(x,T)},T), \qquad p \mapsto px.$$

For every neighborhood U of px, let $V = \Phi_x^{-1}(U)$. Then V is a neighborhood p. There exists a subset F of $\mathbb{N}$ such that $p \in \overline{F} \subset V$. Hence $F \subset N(0,V) \subset N(x,U)$. Thus, $N(x,U) \in p$. ∎

We can also extend $\Psi : \mathbb{Z}_+ \to X^X$, $n \mapsto T^n$, to $\beta\mathbb{Z}_+ \to E(X,T)$. It is easy to see that Ψ is a semigroup homomorphism and $\Psi : (\beta\mathbb{Z}_+,\lambda_1) \to E(X,T)$ is also a factor map. For every $x \in X$, Φ_x and $\varphi_x \circ \Psi$ agree on $\mathbb{Z}_+$ which is dense in $\beta\mathbb{Z}_+$, so $\Phi_x = \varphi_x \circ \Psi$, i.e., the following diagram commutes:

$$\begin{tikzcd}(\beta\mathbb{Z}_+, \lambda_1) \arrow[r, "\Psi"] \arrow[d, "\Phi_x"] & (E(X,T),T) \arrow[dl, "\varphi_x"] \\ (\overline{\mathrm{Orb}(x,T)}, T) & \end{tikzcd}$$

Before continuing, we need some preparation about symbolic dynamics. Let $\Sigma_2 = \{0,1\}^{\mathbb{Z}_+}$ and $\sigma : \Sigma_2 \to \Sigma_2$ be the shift map

$$(x(0), x(1), x(2), \ldots) \mapsto (x(1), x(2), x(3), \ldots).$$

Let $[i_0 i_1 \ldots i_n] = \{x \in \Sigma_2 : x(0) = i_0, x(1) = i_1, \ldots, x(n) = i_n\}$ for $i_j = 0, 1$ and $j = 0, 1, \ldots, n$. For any $F \subset \mathbb{Z}_+$, we denote by $\mathbf{1}_F$ the indicator function from $\mathbb{Z}_+$ to $\{0,1\}$, i.e., $\mathbf{1}_F(n) = 1$ if $n \in F$ and $\mathbf{1}_F(n) = 0$ if $n \notin F$. In a natural way, each indicator function can be regarded as an element of Σ_2. It should be noticed that the enveloping semigroup of $(\{0,1\}^{\mathbb{Z}_+}, \sigma)$ is topologically and algebraically isomorphic to $\beta\mathbb{Z}_+$ ([10, 15]). Similarly, we can define the two-sided symbolic dynamics $(\{0,1\}^{\mathbb{Z}}, \sigma)$.

Theorem 4.4. *Let $\mathcal{F}$ be a filterdual. Suppose that $h(\mathcal{F})$ is a subsemigroup of $\beta\mathbb{N}$. Let (X,T) be a dynamical system and $x \in X$. Then the following conditions are equivalent:*

(1) *x is $\mathcal{F}$-recurrent;*
(2) *there exists an idempotent $u \in h(\mathcal{F})$ such that $ux = x$;*
(3) *there exists an $\mathcal{F}$-recurrent idempotent $v \in E(X,T)$ with $vx = x$;*
(4) *x is $\widetilde{\mathcal{F}}$-recurrent, where $\widetilde{\mathcal{F}}$ is the collection of all essential $\mathcal{F}$-sets.*

Proof. (1)⇒(2). Let

$$\mathcal{A} = \{N(x,U) : U \text{ is an open neighborhood of } x\}.$$

Then $\mathcal{A} \subset \mathcal{F}$ and the intersection of any finite collection of elements of $\mathcal{A}$ is also in $\mathcal{A}$. By Lemma 2.2 there exists some $p \in h(\mathcal{F})$ such that $\mathcal{A} \subset p$, thus $px = x$.

Let $L = \{q \in \beta\mathbb{N} : qx = x\}$. Then L is a closed subsemigroup of $\beta\mathbb{N}$ and so is $L \cap h(\mathcal{F})$ since $p \in L \cap h(\mathcal{F})$. By the Ellis–Namakura Theorem there exists an idempotent $u \in L \cap h(\mathcal{F})$.

(2)⇒(3). Let $v = \Psi(u)$. Since u is $\mathcal{F}$-recurrent, so is v. Since Ψ is a semigroup homomorphism, $vv = \Psi(u)\Psi(u) = \Psi(uu) = \Psi(u) = v$. As $\Phi_x = \varphi_x \circ \Psi$, we have $x = ux = \Phi_x(u) = \varphi_x(\Psi(u)) = \varphi_x(v) = vx$.

(2)⇒(4), (3)⇒(1) and (4)⇒(1) are obvious. ∎

Proposition 4.5. *Let $\mathcal{F}$ be a filterdual. Suppose that $h(\mathcal{F})$ is a subsemigroup of $\beta\mathbb{N}$. Let $\pi : (X,T) \to (Y,S)$ be a factor map. If $y \in Y$ is $\mathcal{F}$-recurrent, then there is an $\mathcal{F}$-recurrent point x in $\pi^{-1}(y)$.*

Proof. By Theorem 4.4 there exists an idempotent $u \in h(\mathcal{F})$ such that $uy = y$. Choose $z \in \pi^{-1}(y)$ and let $x = uz$. Then $\pi(x) = \pi(uz) = u\pi(z) = uy = y$ and $ux = uuz = uz = x$, so x is $\mathcal{F}$-recurrent and $x \in \pi^{-1}(y)$. ■

REMARK 4.6. Recall that a point $x \in X$ is minimal if and only if it is $\mathcal{F}_{\mathrm{s}}$-recurrent. Unfortunately, $\mathcal{F}_{\mathrm{s}}$ is not a filterdual. Can we use some filterdual instead of $\mathcal{F}_{\mathrm{s}}$ to characterize minimal points? Intuitively, $\mathcal{F}_{\mathrm{cen}}$ may be a good choice. But this is not true: it is shown in [25] that there exist $\mathcal{F}_{\mathrm{cen}}$-recurrent points which are not minimal.

Let (X, T) be a dynamical system and $x, y \in X$. We call x, y are *proximal* if there exists $z \in X$ such that $(z, z) \in \omega((x, y), T \times T)$.

PROPOSITION 4.7 ([10, 12, 5]). *Let (X, T) be a dynamical system and $x, y \in X$. Then the following conditions are equivalent:*

(1) *x, y are proximal and y is a minimal point;*
(2) *there exists a minimal idempotent $u \in \beta\mathbb{N}$ such that $ux = uy = y$;*
(3) *there exists a minimal idempotent $v \in E(X, T)$ with $vx = vy = y$;*
(4) *$(y, y) \in \omega_{\mathcal{F}_{\mathrm{cen}}}((x, y), T \times T)$.*

Let (X, T) be a dynamical system and $x, y \in X$. We call x *strongly proximal* to y if $(y, y) \in \omega((x, y), T \times T)$. It is easy to see that if y is a minimal point then x, y are proximal if and only if x is strongly proximal to y.

LEMMA 4.8. *Let (X, T) be a dynamical system and $x, y \in X$. Then the following conditions are equivalent:*

(1) *x is strongly proximal to y;*
(2) *$(y, y) \in \omega_{\mathcal{F}_{\mathrm{ip}}}((x, y), T \times T)$;*
(3) *for every $n \in \mathbb{N}$, x is strongly proximal to y in (X, T^n).*

Proof. (2)$\Rightarrow$(1) and (3)$\Rightarrow$(1) are obvious.

(2)$\Rightarrow$(3) follows from the fact that if F is an IP set then for every $n \in \mathbb{N}$ the set $\{m \in \mathbb{N} : mn \in F\}$ is also an IP set.

(1)$\Rightarrow$(2). Consider the factor map

$$\Phi_{(x,y)} : (\beta\mathbb{Z}_+, \lambda_1) \to (\overline{\mathrm{Orb}((x, y), T \times T)}, T \times T), \quad q \mapsto q(x, y).$$

Let $L = \{p \in \beta\mathbb{N} : p(x, y) = (y, y)\} = \Phi_{(x,y)}^{-1}(y, y) \cap \beta\mathbb{N}$. Then L is a nonempty closed subset of $\beta\mathbb{N}$, since $(y, y) \in \omega((x, y), T \times T)$. We show that L is a subsemigroup of $\beta\mathbb{N}$. Let $p, q \in L$. Then $p(x, y) = (px, py) = (y, y)$ and $q(x, y) = (qx, qy) = (y, y)$, so $pq(x, y) = (pqx, pqy) = (py, py) = (y, y)$. By the Ellis–Namakura Theorem there exists an idempotent p in L. Then by Lemma 4.3 and $p \subset \mathcal{F}_{\mathrm{ip}}$ one has $(y, y) \in \omega_{\mathcal{F}_{\mathrm{ip}}}((x, y), T \times T)$. ■

Let $\mathcal{F}$ be a family, (X, T) be a dynamical system and $x, y \in X$. We say that x is *$\mathcal{F}$-strongly proximal* to y if $(y, y) \in \omega_{\mathcal{F}}((x, y), T \times T)$ ([1]).

THEOREM 4.9. *Let $\mathcal{F}$ be a filterdual. Suppose that $h(\mathcal{F})$ is a subsemigroup of $\beta\mathbb{N}$. Let (X,T) be a dynamical system and $x,y \in X$. Then the following conditions are equivalent:*

(1) *x is $\mathcal{F}$-strongly proximal to y;*
(2) *there exists an idempotent $u \in h(\mathcal{F})$ such that $ux = uy = y$;*
(3) *there exists an $\mathcal{F}$-recurrent idempotent $v \in E(X,T)$ such that $vx = vy = y$;*
(4) *x is $\widetilde{\mathcal{F}}$-strongly proximal to y.*

Proof. (1)⇒(2). Let

$$\mathcal{A} = \{N((x,y), U \times U) : U \text{ is an open neighborhood of } y\}.$$

By the definition of $\mathcal{F}$-strong-proximity, we have $\mathcal{A} \subset \mathcal{F}$ and the intersection of finitely many elements of $\mathcal{A}$ is also in $\mathcal{A}$. Hence by Lemma 2.2 there exists some $p \in h(\mathcal{F})$ such that $\mathcal{A} \subset p$, so $p(x,y) = (y,y)$. Let $L = \{q \in \beta\mathbb{N} : qx = qy = y\}$. Then $L \cap h(\mathcal{F})$ is a nonempty closed subsemigroup of $\beta\mathbb{N}$. By the Ellis–Namakura Theorem there exists an idempotent $u \in L \cap h(\mathcal{F})$.

(2)⇒(3). Let $v = \Psi(u)$. Since u is $\mathcal{F}$-recurrent, so is v. Then from $\Phi_{(x,y)} = \varphi_{(x,y)} \circ \Psi$ we have $vx = vy = y$.

(3)⇒(2). By Theorem 4.4 there exists an idempotent $u \in h(\mathcal{F})$ such that $v = uv = \Psi(u)$. Then $\Phi_{(x,y)} = \varphi_{(x,y)} \circ \Psi$ yields $ux = uy = y$.

(2)⇒(4). Since $u(x,y) = (y,y)$ and u is an idempotent in $h(\mathcal{F})$, by Lemma 4.3, $(y,y) \in \omega_{\widetilde{\mathcal{F}}}((x,y), T \times T)$.

(4)⇒(1) is obvious. ∎

PROPOSITION 4.10. *Let $\mathcal{F}$ be a filterdual. Suppose that $b\mathcal{F} = \mathcal{F}$. Let (X,T) be a dynamical system and $x,y \in X$. Then x is $\mathcal{F}$-strongly proximal to y if and only if y is an $\mathcal{F}$-recurrent point and x is strongly proximal to y.*

Proof. By definition, if x is $\mathcal{F}$-strongly proximal to y, then y is $\mathcal{F}$-recurrent and x is strongly proximal to y.

Conversely, assume that y is $\mathcal{F}$-recurrent and x is strongly proximal to y. Consider the factor map

$$\Phi_{(x,y)} : (\beta\mathbb{Z}_+, \lambda_1) \to (\overline{\mathrm{Orb}((x,y), T \times T)}, T \times T), \qquad p \mapsto p(x,y).$$

Since $(y,y) \in \overline{\mathrm{Orb}((x,y), T \times T)}$ and (y,y) is $\mathcal{F}$-recurrent, by Proposition 4.5 there exists an $\mathcal{F}$-recurrent point q in $\beta\mathbb{N}$ with $q(x,y) = (y,y)$. By Lemma 4.2 we have $q \subset b\mathcal{F} = \mathcal{F}$, so $(y,y) \in \omega_{\mathcal{F}}((x,y), T \times T)$. ∎

Now we can set up a general correspondence between essential $\mathcal{F}$-sets and sets defined by $\mathcal{F}$-strong proximity.

THEOREM 4.11. *Let $\mathcal{F}$ be a filterdual. Suppose that $h(\mathcal{F})$ is a subsemigroup of $\beta\mathbb{N}$. Then a subset F of $\mathbb{N}$ is an essential $\mathcal{F}$-set if and only if there exists a dynamical system (X,T), a pair of points $x,y \in X$ where x*

is $\mathcal{F}$-strongly proximal to y, and an open neighborhood U of y such that $F = N(x, U)$.

Proof. The sufficiency follows from Theorem 4.9 and $N((x,y), U \times U) \subset N(x, U)$.

Now we show the necessity. If F is an essential $\mathcal{F}$-set, there exists an idempotent $u \in h(\mathcal{F})$ such that $F \in u$. Let $x = \mathbf{1}_F \in \{0,1\}^{\mathbb{Z}_+}$ and $y = ux$. Then $ux = y = y$, so x is $\mathcal{F}$-strongly proximal to y. Clearly, $N(x, [1]) = F$. Then it suffices to show that $y \in [1]$. If not, then $y \in [0]$. Thus, $N(x, [0]) \in p$ and $N(x, [0]) \cap N(x, [1]) \neq \emptyset$, This is a contradiction. ■

REMARK 4.12. (1) In the proof of Theorem 4.11, if we use $\{0,1\}^{\mathbb{Z}}$ instead of $\{0,1\}^{\mathbb{Z}_+}$, then the proof shows that every essential $\mathcal{F}$-set can be realized by an invertible metrizable system.

(2) Since $\mathcal{F}_{\mathrm{ps}}$ and $\mathcal{F}_{\mathrm{pubd}}$ are filterduals, and $b\mathcal{F}_{\mathrm{ps}} = \mathcal{F}_{\mathrm{ps}}$, $b\mathcal{F}_{\mathrm{pubd}} = \mathcal{F}_{\mathrm{pubd}}$, Theorems 1.5 and 1.6 are special cases of Theorem 4.11.

We now give a combinatorial characterization of essential $\mathcal{F}$-sets.

PROPOSITION 4.13. *Let $\mathcal{F}$ be a filterdual. Suppose that $h(\mathcal{F})$ is a subsemigroup of $\beta\mathbb{N}$. Then a subset F of $\mathbb{N}$ is an essential $\mathcal{F}$-set if and only if there is a decreasing sequence $\{C_n\}_{n=1}^{\infty}$ of subsets of F such that $C_n \in \mathcal{F}$ for every $n \in \mathbb{N}$, and for every $r \in C_n$ there exists $m \in \mathbb{N}$ such that $r + C_m \subset C_n$.*

Proof. If F is an essential $\mathcal{F}$-set, there exists an idempotent $u \in h(\mathcal{F})$ such that $F \in u$. Let $x = \mathbf{1}_F \in \{0,1\}^{\mathbb{Z}_+}$ and $y = ux$. Then $u(x,y) = (y,y)$, $y \in [1]$ and $N(x, [1]) = F$. For each $n \in \mathbb{N}$, let $U_n = [y(0)y(1)\ldots y(n)]$ and $C_n = N((x,y), U_n \times U_n)$. Then by Theorem 4.9 each C_n is an essential $\mathcal{F}$-set. For every $r \in C_n$, we have $(\sigma \times \sigma)^r(y,y) \in U_n \times U_n$. By the continuity of σ, there exists $m \in \mathbb{N}$ such that $(\sigma \times \sigma)^r(U_m \times U_m) \subset U_n \times U_n$, so $r + C_m \subset C_n$.

Conversely, assume that there is a sequence $\{C_n\}_{n=1}^{\infty}$ as in the statement. By Lemma 2.2 there exists $p \in h(\mathcal{F})$ such that $\{C_n : n \in \mathbb{N}\} \subset p$. Let $L = \bigcap_{n=1}^{\infty} \overline{C_n}$. By Lemma 3.2, L is a closed subsemigroup of $\beta\mathbb{N}$. Then $p \in L \cap h(\mathcal{F})$ and $L \cap h(\mathcal{F})$ is a nonempty closed subsemigroup of $\beta\mathbb{N}$. By the Ellis–Namakura Theorem there exists an idempotent in $L \cap h(\mathcal{F})$. Thus, each C_n is an essential $\mathcal{F}$-set. In particular, F is an essential $\mathcal{F}$-set. ■

COROLLARY 4.14. *Let p be an idempotent in $\beta\mathbb{N}$ and $F \subset \mathbb{N}$. Then $F \in p$ if and only if there is a decreasing sequence $\{C_n\}_{n=1}^{\infty}$ of subsets of F such that $C_n \in p$ for every $n \in \mathbb{N}$, and for every $r \in C_n$ there exists $m \in \mathbb{N}$ such that $r + C_m \subset C_n$.*

5. The set's forcing. In this section, we discuss the set's forcing. This terminology was first introduced in [7]; the idea goes back at least to [11] and [15]. We say that a subset F of $\mathbb{N}$ *forces $\mathcal{F}$-recurrence* if for every dynamical system (X,T) and $x\in X$ there exists some $\mathcal{F}$-recurrent point in $\overline{T^F x}$, where $T^F x=\{T^n x: n\in F\}$.

In [11] and [15], the authors call a subset F of $\mathbb{N}$ *big* if there exists a minimal point in $\overline{\mathrm{Orb}(x,\sigma)}\cap[1]$, where $x=\mathbf{1}_F\in\Sigma$.

PROPOSITION 5.1 ([15, 7]). *Let $F\subset\mathbb{N}$. Then the following conditions are equivalent:*

(1) *F is big;*
(2) *F is piecewise syndetic;*
(3) *F forces $\mathcal{F}_{\mathrm{s}}$-recurrence;*
(4) *there exists a minimal left ideal L of $\beta\mathbb{N}$ such that $\overline{F}\cap L\neq\emptyset$.*

Let $\mathcal{F}$ be a family. Denote by $\mathrm{Force}(\mathcal{F})$ the collection of all sets that force $\mathcal{F}$-recurrence. Clearly, $\mathrm{Force}(\mathcal{F})$ is a family. It is easy to see that $\mathrm{Force}(\mathcal{F})$ is not empty if and only if there exists an $\mathcal{F}$-recurrent point in $(\beta\mathbb{Z}_+,\lambda_1)$.

THEOREM 5.2. *Let $\mathcal{F}$ be a family and $F\subset\mathbb{N}$. Then $F\in\mathrm{Force}(\mathcal{F})$ if and only if there exists an $\mathcal{F}$-recurrent point $p\in\beta\mathbb{N}$ such that $F\in p$.*

Proof. Let $F\in\mathrm{Force}(\mathcal{F})$. Consider the system $(\beta\mathbb{Z}_+,\lambda_1)$ and $0\in\beta\mathbb{Z}_+$. Since F forces $\mathcal{F}$-recurrence, there exists an $\mathcal{F}$-recurrent point $p\in\overline{(\lambda_1)^F 0}=\overline{\{(\lambda_1)^n 0: n\in F\}}=\overline{F}$. Thus, $F\in p$.

Conversely, assume that there exists an $\mathcal{F}$-recurrent point $p\in\beta\mathbb{N}$ such that $F\in p$. For every dynamical system (X,T) and $x\in X$, consider the factor map $\Phi_x:(\beta\mathbb{Z}_+,\lambda_1)\to(\overline{\mathrm{Orb}(x,T)},T)$. Let $y=px$. Then y is $\mathcal{F}$-recurrent. Thus it suffices to show that $y\in\overline{T^F x}$. For every open neighborhood U of y, we have $N(x,U)\in p$. Since $F\in p$, we have $N(x,U)\cap F\neq\emptyset$, thus $y\in\overline{T^F x}$. ∎

COROLLARY 5.3. *Let $\mathcal{F}$ be a family. Then*

$$h(\mathrm{Force}(\mathcal{F}))=\overline{\bigcup\{\beta\mathbb{Z}_+ + p: p \text{ is an } \mathcal{F}\text{-recurrent point}\}}.$$

PROPOSITION 5.4. *Let $\mathcal{F}$ be a family. If $\mathrm{Force}(\mathcal{F})$ is not empty, then $\mathrm{Force}(\mathcal{F})$ is a filterdual and $\mathrm{Force}(\mathcal{F})=b(\mathrm{Force}(\mathcal{F}))\subset b\mathcal{F}$.*

Proof. Let $F\in\mathrm{Force}(\mathcal{F})$ and $F=F_1\cup F_2$. If neither F_1 nor F_2 is in $\mathrm{Force}(\mathcal{F})$, then there exist dynamical systems (X,T) , (Y,S) and points $x\in X$, $y\in Y$ such that neither $\overline{T^{F_1}x}$ nor $\overline{S^{F_2}x}$ contains $\mathcal{F}$-recurrent points. Consider the system $(X\times Y,T\times S)$ and $(x,y)\in X\times Y$. Since F forces $\mathcal{F}$-recurrence, there exists an $\mathcal{F}$-recurrent point

$$(z_1,z_2)\in\overline{(T\times S)^F(x,y)}=\overline{(T\times S)^{F_1}(x,y)}\cup\overline{(T\times S)^{F_2}(x,y)}.$$

Without loss of generality, assume that $(z_1,z_2)\in\overline{(T\times S)^{F_1}(x,y)}$. Then $z_1\in\overline{T^{F_1}x}$ and z_1 is $\mathcal{F}$-recurrent, a contradiction. Thus, $\mathrm{Force}(\mathcal{F})$ is a filterdual.

Let $F\in b(\mathrm{Force}(\mathcal{F}))$. Then there exists a sequence $\{a_n\}$ in $\mathbb{Z}_+$ and $F'\in\mathrm{Force}(\mathcal{F})$ such that $\bigcup_{n=1}^{\infty}(a_n+F'\cap[1,n])\subset F$. Let (X,T) be a dynamical system and $x \in X$. Since X is compact, there is a subnet $\{a_{n_i}\}$ of $\{a_n\}$ such that $\lim T^{a_{n_i}} x = y$. Since F forces $\mathcal{F}$-recurrence, there exists an $\mathcal{F}$-recurrent point $z \in \overline{T^{F'} y}$. It suffices to show that $z \in \overline{T^F x}$. For every open neighborhood U of z, there exists $k \in F'$ such that $T^k y \in U$. By the continuity of T, choose an open neighborhood V of y such that $T^k V \subset U$. Since $\lim T^{a_{n_i}} x = y$ and $\{a_{n_i}\}$ is a subnet of $\{a_n\}$, there exists $n \geq k$ such that $T^{a_n} x \in V$. Then $a_n + k \in F$ and $T^{a_n + k} x \in U$, so $z \in \overline{T^F x}$.

Let $F \in \mathrm{Force}(\mathcal{F})$. We will show that $F \in b\mathcal{F}$. Let $x = \mathbf{1}_F \in \{0,1\}^{\mathbb{Z}_+}$. Since F forces $\mathcal{F}$-recurrence, there exists an $\mathcal{F}$-recurrent point $y \in \overline{T^F x}$. Clearly, $y \in [1]$ and $N(x,[1]) = F$. Let $N(y,[1]) = F'$. Then $F' \in \mathcal{F}$. For every finite subset W of F', by the continuity of σ, there exists an open neighborhood U of y such that $\sigma^n(U) \subset [1]$ for every $n \in W$. Since $y \in \overline{\mathrm{Orb}(x,\sigma)}$, choose $m \in \mathbb{Z}_+$ such that $\sigma^m x \in U$; then $m + W \subset N(x,[1])$. So $F \in b\mathcal{F}$. ∎

Theorem 5.5. *Let $\mathcal{F}$ be a filterdual and $F \subset \mathbb{N}$. Suppose that $h(\mathcal{F})$ is a subsemigroup of $\beta\mathbb{N}$. Then the following conditions are equivalent:*

(1) *F forces $\mathcal{F}$-recurrence;*
(2) *for $x = \mathbf{1}_F \in \{0,1\}^{\mathbb{Z}_+}$, there exists an $\mathcal{F}$-recurrent point in $\overline{\mathrm{Orb}(x,\sigma)} \cap [1]$;*
(3) *F is a block essential $\mathcal{F}$-set, i.e., $F \in b\widetilde{\mathcal{F}}$.*

Proof. (1)⇒(2). Let $x = \mathbf{1}_F \in \{0,1\}^{\mathbb{Z}_+}$. Since F forces $\mathcal{F}$-recurrence, there exists an $\mathcal{F}$-recurrent point y in $\overline{\sigma^F x} \subset [1]$.

(2)⇒(3). Choose an $\mathcal{F}$-recurrent point y in $\overline{\mathrm{Orb}(x,\sigma)} \cap [1]$. By Theorem 4.4, y is also $\widetilde{\mathcal{F}}$-recurrent. Since $N(x,[1]) = F$ and $N(y,[1]) \in \widetilde{\mathcal{F}}$, by the continuity of σ we have $F \in b\widetilde{\mathcal{F}}$.

(3)⇒(1). By Proposition 5.4, it suffices to show that every essential $\mathcal{F}$-set forces $\mathcal{F}$-recurrence. Let $F \in \widetilde{\mathcal{F}}$. Then there exists an idempotent $u \in h(\mathcal{F})$ such that $F \in u$. Let (X,T) be a dynamical system and $x \in X$. Let $y = ux$. Then $uy = y$, so y is $\mathcal{F}$-recurrent. For every open neighborhood U of y, $N(x,U) \in u$. Since $F \in u$, we have $F \cap N(x,U) \neq \emptyset$, thus $y \in \overline{T^F x}$. ∎

Corollary 5.6. *Let $\mathcal{F}$ be a filterdual and $F \subset \mathbb{N}$. Suppose that $h(\mathcal{F})$ is a subsemigroup of $\beta\mathbb{N}$. Let (X,T) be a dynamical system and $x \in X$. Then x is a unique $\mathcal{F}$-recurrent point in (X,T) if and only if for every $y \in X$, $\kappa(b\widetilde{\mathcal{F}})$-$\lim T^n y = x$.*

Proof. Since $\mathcal{F}$ is a filterdual, $\kappa(b\widetilde{\mathcal{F}})$ is a filter. If x is a unique $\mathcal{F}$-recurrent point, then by Theorem 5.5 for every $y \in X$ and every $F \in b\widetilde{\mathcal{F}}$ we have $x \in \overline{T^F y}$, so $\kappa(b\widetilde{\mathcal{F}})$-$\lim T^n y = x$.

Conversely, assume that there exists another $\mathcal{F}$-recurrent point $y \in X$. Choose open subsets U, V of X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Then

$N(y,U) \in \kappa(b\widetilde{\mathcal{F}})$ and $N(y,V) \in \widetilde{\mathcal{F}} \subset b\widetilde{\mathcal{F}}$. Thus, $N(y,U) \cap N(y,V) \neq \emptyset$. This is a contradiction. ∎

REMARK 5.7. (1) Since $\mathcal{F}_{\mathrm{ip}} = \widetilde{\mathcal{F}_{\mathrm{ip}}}$, we have $b\mathcal{F}_{\mathrm{ip}} = b\widetilde{\mathcal{F}_{\mathrm{ip}}}$. Hence a subset F of $\mathbb{N}$ forces recurrence if and only if $F \in b\mathcal{F}_{\mathrm{ip}}$ ([7]).

(2) It is shown in [27] that a subset F of $\mathbb{N}$ forces $\mathcal{F}_{\mathrm{pubd}}$-recurrence if and only if $F \in \mathcal{F}_{\mathrm{pubd}}$, i.e., $b\widetilde{\mathcal{F}_{\mathrm{pubd}}} = \mathcal{F}_{\mathrm{pubd}}$. For completeness, we include a proof. Let $F \in \mathcal{F}_{\mathrm{pubd}}$ and $x = \mathbf{1}_F \in \{0,1\}^{\mathbb{Z}_+}$. By [12, Lemma 3.17], there exists a σ-invariant measure μ such that $\mu(\overline{\mathrm{Orb}(x,\sigma)} \cap [1]) > 0$. By the ergodic decomposition theorem, choose an ergodic σ-invariant measure ν such that $\nu(\overline{\mathrm{Orb}(x,\sigma)} \cap [1]) > 0$. Then a generic point y in $\overline{\mathrm{Orb}(x,\sigma)} \cap [1]$ for ν is $\mathcal{F}_{\mathrm{pubd}}$-recurrent ([12, pp. 62–64]). Thus, F forces $\mathcal{F}_{\mathrm{pubd}}$-recurrence.

It is interesting that central sets also have some kind of forcing.

PROPOSITION 5.8. *Let $F \subset \mathbb{N}$. Then the following conditions are equivalent:*

(1) *F is central;*
(2) *for $x = \mathbf{1}_F \in \{0,1\}^{\mathbb{Z}_+}$, there exists a minimal point $y \in \overline{\mathrm{Orb}(x,\sigma)} \cap [1]$ such that x, y are proximal;*
(3) *for every dynamical system (X,T) and $x \in X$ there exists a minimal point $y \in \overline{T^F x}$ such that x, y are proximal.*

Proof. (2)⇒(1) follows from the definition of central sets and $N(x,[1]) = F$.

(3)⇒(2) follows from $\overline{T^F x} \subset [1]$.

(1)⇒(3). If F is central, then there exists a minimal idempotent $u \in \beta\mathbb{N}$ such that $F \in u$. Let (X,T) be a dynamical system and $x \in X$. Let $y = ux$. Then $ux = uy = y$, so y is a minimal point and x, y are proximal. Thus it suffices to show that $y \in \overline{T^F x}$. For every open neighborhood U of y, $N(x,U) \in u$. Since $F \in u$, we have $F \cap N(x,U) \neq \emptyset$, so $y \in \overline{T^F x}$. ∎

We say a subset F of $\mathbb{N}$ *forces $\mathcal{F}$-strong proximity* if for every dynamical system (X,T) and $x \in X$ there exists y in $\overline{T^F x}$ such that x is $\mathcal{F}$-strongly proximal to y.

PROPOSITION 5.9. *Let $\mathcal{F}$ be a filterdual. Suppose that $h(\mathcal{F})$ is a subsemigroup of $\beta\mathbb{N}$. Let $F \subset \mathbb{N}$. Then the following conditions are equivalent:*

(1) *F is an essential $\mathcal{F}$-set;*
(2) *for $x = \mathbf{1}_F \in \{0,1\}^{\mathbb{Z}_+}$, there exists $y \in \overline{\mathrm{Orb}(x,\sigma)} \cap [1]$ such that x is $\mathcal{F}$-strongly proximal to y;*
(3) *F forces $\mathcal{F}$-strong proximity.*

Proof. (2)⇒(1) follows from Theorem 4.11 and $N(x,[1]) = F$.

(3)⇒(2) follows from $\overline{T^F x} \subset [1]$.

(1)$\Rightarrow$(3). If F is an essential $\mathcal{F}$-set, then there exists an idempotent $u \in h(\mathcal{F})$ such that $F \in u$. Let (X,T) be a dynamical system and $x \in X$. Let $y = ux$. Then $ux = uy = y$ and by Theorem 4.9, x is $\mathcal{F}$-strongly proximal to y. Thus it suffices to show that $y \in \overline{T^F x}$. For every open neighborhood U of y, $N(x,U) \in u$. Since $F \in u$, we have $F \cap N(x,U) \neq \emptyset$, so $y \in \overline{T^F x}$. ∎

6. Multiplication in $\mathbb{N}$ and $\beta\mathbb{N}$. In this section, we consider both addition and multiplication in $\mathbb{N}$ and $\beta\mathbb{N}$. For $n \in \mathbb{N}$ and $F \subset \mathbb{N}$, let $nF = \{nm : m \in F\}$ and $n^{-1}F = \{m \in \mathbb{N} : nm \in F\}$. For $p, q \in \beta\mathbb{N}$, the product $p \cdot q$ in $\beta\mathbb{N}$ is

$$\{A \subset \mathbb{N} : \{n \in \mathbb{N} : n^{-1}A \in q\} \in p\}.$$

A family $\mathcal{F}$ is called *multiplication invariant* if for each $n \in \mathbb{N}$ and $F \in \mathcal{F}$ one has $nF \in \mathcal{F}$. It is easy to see that $\mathcal{F}_{\mathrm{ip}}$, $\mathcal{F}_{\mathrm{s}}$ and $\mathcal{F}_{\mathrm{pubd}}$ are multiplication invariant. Similarly to Lemma 3.4, we have

LEMMA 6.1. *Let $\mathcal{F}$ be a filterdual. Then $\mathcal{F}$ is multiplication invariant if and only if $h(\mathcal{F})$ is a left ideal of $(\beta\mathbb{N}, \cdot)$.*

PROPOSITION 6.2 ([12, 6]). *Let $F \subset \mathbb{N}$. If F is a central set, then for each $n \in \mathbb{N}$ both nF and $n^{-1}F$ are also central.*

The main purpose of this section is to extend Proposition 6.2 to more general settings. In particular, similar results hold for quasi-center sets and D-sets.

THEOREM 6.3. *Let $\mathcal{F}$ be a filterdual and $F \subset \mathbb{N}$. Suppose that $\mathcal{F}$ is multiplication invariant and $h(\mathcal{F})$ is a subsemigroup of $(\beta\mathbb{N}, +)$. If F is an essential $\mathcal{F}$-set, then for each $n \in \mathbb{N}$, nF is also an essential $\mathcal{F}$-set.*

Proof. Let $x = \mathbf{1}_F \in \{0,1\}^{\mathbb{Z}_+}$. Then by Proposition 5.9 there exists $y \in \overline{\sigma^F x} \subset [1]$ such that $\mathcal{F}\text{-}\lim(\sigma \times \sigma)^m(x,y) = (y,y)$. Fix $n \in \mathbb{N}$ and let $Y = \{1, \ldots, n\}$ be endowed with the discrete topology and $X = \{0,1\}^{\mathbb{Z}_+} \times Y$. Define $T : X \to X$ by $T(z,i) = (z, i+1)$ for $i \leq n-1$ and $T(z,n) = (\sigma z, 1)$.

For every neighborhood U of y, we have

$$N((x,1,y,1), U \times \{1\} \times U \times \{1\}) = nN((x,y), U \times U).$$

Since $\mathcal{F}$ is multiplication invariant, $\mathcal{F}\text{-}\lim(T \times T)^m(x,1,y,1) = (y,1,y,1)$. Thus, $nF = N((x,1), [1] \times \{1\})$ is also an essential $\mathcal{F}$-set. ∎

We say that $\mathcal{F}$-recurrence is *iteratively invariant* if for every dynamical system (X,T) and every $\mathcal{F}$-recurrent point x in (X,T), x is also an $\mathcal{F}$-recurrent point in (X,T^n) for each $n \in \mathbb{N}$. It is well known that $\mathcal{F}_{\mathrm{ip}}$-recurrence and $\mathcal{F}_{\mathrm{s}}$-recurrence are iteratively invariant. We show

THEOREM 6.4. *Let $\mathcal{F}$ be a filterdual and $F \subset \mathbb{N}$. Suppose that $b\mathcal{F} = \mathcal{F}$ and $\mathcal{F}$-recurrence is iteratively invariant. If F is an essential $\mathcal{F}$-set, then for each $n \in \mathbb{N}$, $n^{-1}F$ is also an essential $\mathcal{F}$-set.*

Proof. Let $x = \mathbf{1}_F \in \{0,1\}^{\mathbb{Z}_+}$. Then by Proposition 5.9 there exists an $\mathcal{F}$-recurrent point $y \in \overline{\sigma^F x} \subset [1]$ such that x is strongly proximal to y. For each $n \in \mathbb{N}$, since $\mathcal{F}$-recurrence is iteratively invariant, y is also an $\mathcal{F}$-recurrent point in $(\{0,1\}^{\mathbb{Z}_+}, \sigma^n)$. By Lemma 4.8, x is also strongly proximal to y in $(\{0,1\}^{\mathbb{Z}_+}, \sigma^n)$. Then by Proposition 4.10 and Theorem 4.9, $n^{-1}F = \{m \in \mathbb{N} : (\sigma^n)^m x \in [1]\}$ is an essential $\mathcal{F}$-set. ∎

A dynamical system (X,T) is called *topologically transitive* if for any two nonempty open subsets U, V of X there exists some $n \in \mathbb{N}$ such that $T^n U \cap V \neq \emptyset$. A point $x \in X$ is called *transitive* if the orbit of x is dense in X. The system (X,T) is called *point transitive* if there exists a transitive point in X. In general, there is no implication between topological transitivity and point transitivity. For example, $(\beta\mathbb{Z}_+, \lambda_1)$ is point transitive but not topologically transitive. The system (X,T) is called *recurrent transitive* if there exists a recurrent transitive point, i.e., $x \in X$ whose ω-limit set is X. It is easy to see that every recurrent transitive system is topologically transitive.

The following is a "folklore" result; for similar results, see [2] for example.

LEMMA 6.5. *Let (X,T) be a recurrent transitive system. Then for every $n \in \mathbb{N}$ there is $k \in \mathbb{N}$ with $k \mid n$ and a decomposition $X = X_0 \cup X_1 \cup \cdots \cup X_{k-1}$ satisfying*

(1) *$X_i \neq X_j$, $0 \le i < j \le k-1$,*
(2) *$TX_i = X_{i+1\,(\mathrm{mod}\,k)}$,*
(3) *(X_i, T^n) is recurrent transitive, $i = 0, \ldots, k-1$,*
(4) *the interior of X_i is dense in X_i, $i = 0, \ldots, k-1$.*

Proof. Let $x \in X$ with $\omega(x,T) = X$. Let $Y_i = \overline{\mathrm{Orb}(T^i x, T^n)}$ for $i = 0, 1, \ldots, n-1$. Then $X = Y_0 \cup Y_1 \cup \cdots \cup Y_{n-1}$ and $TY_i = Y_{i+1\,(\mathrm{mod}\,n)}$. Since x is recurrent in (X,T), $T^i x$ is also recurrent in (X, T^n). Then (Y_i, T^n) is recurrent transitive for $i = 0, 1, \ldots, n-1$. Let k be the smallest positive integer such that $T^k Y_0 = Y_0$. Let $X_i = Y_i$ for $i = 0, 1, \ldots, k-1$. Now we show that those X_i satisfy the requirements.

Clearly, $k \le n$. Let $n = lk + r$ with $l > 0$ and $0 \le r < k$. Then $X_0 = T^n(X_0) = T^r(T^{lk} X_0) = T^r(X_0)$; by the minimality of k, we have $r = 0$, so $k \mid n$.

If there existed $0 \le i < j \le k-1$ such that $X_i = X_j$, then $T^{j-i} X_0 = T^{j-i}(T^n X_0) = T^{n-i}(T^j X_0) = T^{n-i}(T^i X_0) = T^n X_0 = X_0$. This contradicts the minimality of k. So $X_i \neq X_j$ for $0 \le i < j \le k-1$.

For $0 \le i \neq j \le k-1$ let $Z_{ij} = X_i \cap X_j$. Then Z_{ij} is a T^n-invariant closed subset of X_i. Since (X_i, T^n) is topologically transitive, Z_{ij} either equals X_i

or is nowhere dense in X_i. If $Z_{ij} = X_i$, then $X_i \subset X_j$. Without loss of generality, assume $i < j$; then $X_0 = T^{k-i}X_i \subset T^{k-i}X_j = X_{j-i}$. Thus,

$$X_0 \subset X_{j-i} \subset X_{2(j-i)\,(\mathrm{mod}\,k)} \subset \cdots \subset X_{k(j-i)\,(\mathrm{mod}\,k)} = X_0.$$

This contradicts the minimality of k. So Z_{ij} is nowhere dense in X_i.

Now fix $i \in \{0, 1, \dots, k-1\}$ and let $Z_i = \bigcup_{j\neq i} Z_{ij}$. Then Z_i is also nowhere dense in X_i. The boundary of X_i in X is

$$\partial X_i = X_i \cap (\overline{X \setminus X_i}) \subset X_i \cap \bigcup_{j\neq i} X_j = \bigcup_{j\neq i} (X_i \cap X_j) = Z_i.$$

As $X_i = \mathrm{int}(X_i) \cup Z_i$, the interior of X_i is dense in X_i. ∎

LEMMA 6.6. *$\mathcal{F}_{\mathrm{ps}}$-recurrence and $\mathcal{F}_{\mathrm{pubd}}$-recurrence are iteratively invariant.*

Proof. Let (X,T) be a dynamical system and $x \in X$ be an $\mathcal{F}_{\mathrm{ps}}$-recurrent point. Without loss of generality, assume that $\overline{\mathrm{Orb}(x,T)} = X$. By Lemma 4.1, (X,T) has dense minimal points. For every $n \in \mathbb{N}$, (X,T^n) also has dense minimal points. By Lemma 6.5, the interior of $\overline{\mathrm{Orb}(x,T^n)}$ is dense in $\overline{\mathrm{Orb}(x,T^n)}$, so $(\overline{\mathrm{Orb}(x,T^n)}, T^n)$ also has dense minimal points. Thus x is $\mathcal{F}_{\mathrm{ps}}$-recurrent in (X,T^n).

Let (X,T) be a dynamical system and $x \in X$ be an $\mathcal{F}_{\mathrm{pubd}}$-recurrent point. Without loss of generality, assume that $\overline{\mathrm{Orb}(x,T)} = X$. By Lemma 4.1 and since (X,T) is transitive, for every nonempty open subset U of X there exists a T-invariant measure μ on X such that $\mu(U) > 0$. For every $n \in \mathbb{N}$, by Lemma 6.5, the interior of $\overline{\mathrm{Orb}(x,T^n)}$ is dense in $\overline{\mathrm{Orb}(x,T^n)}$. Then for every nonempty open subset V of $\overline{\mathrm{Orb}(x,T^n)}$ there exists an open subset U of X such that $U \subset V$. So there exists a T-invariant measure μ on X such that $\mu(U) > 0$. Clearly, μ is also T^n-invariant. Define a measure ν on $\overline{\mathrm{Orb}(x,T^n)}$ by $\nu(A) = \mu(A)/\mu(\overline{\mathrm{Orb}(x,T^n)})$ for every Borel subset A of $\overline{\mathrm{Orb}(x,T^n)}$. Then ν is T^n-invariant with $\nu(V) > 0$. Thus x is $\mathcal{F}_{\mathrm{pubd}}$-recurrent in (X,T^n). ∎

PROPOSITION 6.7. *Let $F \subset \mathbb{N}$ and $n \in \mathbb{N}$.*

(1) *If F is a quasi-central set, then both nF and $n^{-1}F$ are also quasi-central.*
(2) *If F is a D-set, then both nF and $n^{-1}F$ are also D-sets.*

Proof. This follows from Theorems 6.4 and 6.3, Lemma 6.6 and the fact that $\mathcal{F}_{\mathrm{ps}}$ and $\mathcal{F}_{\mathrm{pubd}}$ are multiplication invariant. ∎

7. Dynamical characterization of C-sets. In this section, we show the following dynamical characterization of C-sets.

Theorem 7.1. *Let $F \subset \mathbb{N}$. Then F is a C-set if and only if there exists a dynamical system (X,T), a pair of points $x,y \in X$ where y is $\mathcal{J}$-recurrent and x is strongly proximal to y, and an open neighborhood U of y such that $N(x,U)=F$.*

By Proposition 4.10 and Theorem 4.11, it suffices to show the following two lemmas.

Lemma 7.2. *$\mathcal{J}$ is a filterdual.*

Lemma 7.3. *$\mathcal{J}=b\mathcal{J}$ and it is multiplication invariant. Then $h(\mathcal{J})$ is a closed two-sided ideal in $(\beta\mathbb{N},+)$ and a left ideal in $(\beta\mathbb{N},\cdot)$.*

Proof of Lemma 7.2. Let F be a J-set and $F=F_1\cup F_2$. Using an argument from [21, Theorem 2.14], we first show the following claim.

Claim. *For every IP-system $\{s_\alpha=(s_\alpha^{(1)},\dots,s_\alpha^{(m)})\}$ in $\mathbb{Z}^m$, there exist $i\in\{1,2\}$, $r\in\mathbb{Z}$ and $\alpha\in\mathcal{P}_\mathrm{f}(\mathbb{N})$ such that $\bar{r}^{(m)}+s_\alpha\in F_i^m$.*

Proof of the Claim. For $j=1,\dots,m$, define $f_j:\mathbb{N}\to\mathbb{Z}$ by $f_j(n)=s_{\{n\}}^{(j)}$. Then $s_\alpha^{(j)}=\sum_{n\in\alpha} f_j(n)$ for $\alpha\in\mathcal{P}_\mathrm{f}(\mathbb{N})$.

By the Hales–Jewett Theorem [16] pick $n\in\mathbb{N}$ such that whenever the length n words over the alphabet $\{1,\dots,m\}$ are 2-colored, there exists a variable word $w(v)$ such that $\{w(j):j=1,\dots,m\}$ is monochromatic.

Let W be the set of length n words over $\{1,\dots,m\}$. For $w=b_1\cdots b_n\in W$ define $g_w:\mathbb{N}\to\mathbb{Z}$ by $g_w(ln+i)=f_{b_i}(ln+i)$ for $l\in\mathbb{Z}_+$ and $i=1,\dots,n$. For $l\in\mathbb{Z}_+$, let $H_l=\{ln+1,\dots,ln+n\}$. For every $w\in W$ and $\alpha\in\mathcal{P}_\mathrm{f}(\mathbb{N})$, let $h_\alpha^{(w)}=\sum_{l\in\alpha}\sum_{t\in H_l} g_w(t)$. Then $(h_\alpha)=(h_\alpha^{(w)}:w\in W)$ is an IP-system in $\mathbb{Z}^{|W|}$. Hence there exist $r\in\mathbb{Z}$ and $\alpha\in\mathcal{P}_\mathrm{f}(\mathbb{N})$ such that $r+h_\alpha^{(w)}\in F$ for every $w\in W$. Define $\phi:W\to\{0,1\}$ by $\phi(w)=1$ if $r+h_\alpha^{(w)}\in F_1$. Pick a variable word $w(v)$ such that $\{w(j):j=1,\dots,m\}$ is monochromatic with respect to ϕ. Without loss of generality assume that $\phi(w(j))=1$ for $j=1,\dots,k$. Let $w(v)=c_1\cdots c_n$ where each $c_i\in\{1,\dots,m\}\cup\{v\}$. Let $A=\{i\in\{1,\dots,n\}:c_i=v\}\neq\emptyset$ and $B=\{1,\dots,n\}\setminus A$. For $l\in\mathbb{Z}_+$, let $H_l^A=H_l\cap(ln+A)$ and $H_l^B=H_l\cap(ln+B)$. For $j=1,\dots,m$, rewrite $h_\alpha^{(w(j))}$ as

$$h_\alpha^{(w(j))}=\sum_{l\in\alpha}\sum_{t\in H_l} g_{w(j)}(t)=\sum_{l\in\alpha}\sum_{t\in H_l^A} g_{w(j)}(t)+\sum_{l\in\alpha}\sum_{t\in H_l^B} g_{w(j)}(t).$$

Then $\sum_{t\in H_l^A} g_{w(j)}(t)=\sum_{t\in H_l^A} f_j(t)$ and $\sum_{t\in H_l^B} g_{w(j)}(t)$ does not depend on j. Let $\alpha'=\bigcup_{l\in\alpha} H_l^A$ and $r'=r+\sum_{l\in\alpha}\sum_{t\in H_l^B} g_{w(j)}(t)$. Then $r+h_\alpha^{(w(j))}=r'+s_{\alpha'}^{(j)}$. So $\bar{r}'^{(m)}+s_{\alpha'}\in F_1^m$. This ends the proof of the Claim.

We now show that in the Claim we can pick $r \in \mathbb{N}$ instead of $r \in \mathbb{Z}$. For every IP-system $\{s_\alpha = (s_\alpha^{(1)}, \dots, s_\alpha^{(m)})\}$ in $\mathbb{Z}^m$, let $s_\alpha^{(0)} = -|\alpha|$ for each $\alpha \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})$ and $\{s'_\alpha = (s_\alpha^{(0)}, s_\alpha^{(1)}, \dots, s_\alpha^{(m)})\}$. Applying the Claim to $\{s'_\alpha\}$ yields $i \in \{1,2\}$, $r \in \mathbb{Z}$ and $\alpha \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})$ such that $\bar{r}^{(m+1)} + s'_\alpha \in F_i^{m+1}$. Since $r + s_\alpha^{(0)} \in F_i$ and $s_\alpha^{(0)}$ is negative, r must be positive.

If neither F_1 nor F_2 is a J-set, let $\{s_\alpha = (s_\alpha^{(1)}, \dots, s_\alpha^{(m)})\}$ and $\{s'_\alpha = (s_\alpha'^{(1)}, \dots, s_\alpha'^{(m')})\}$ be witnesses to this fact. Let $s''_\alpha = (s_\alpha^{(1)}, \dots, s_\alpha^{(m)}, s_\alpha'^{(1)}, \dots, s_\alpha'^{(m')})$. Applying the Claim to $\{s''_\alpha\}$, we get a contradiction. ∎

Proof of Lemma 7.3. If F is a block J-set, then there exists a sequence $\{a_n\}$ in $\mathbb{Z}_+$ and $F' \in \mathcal{J}$ such that $\bigcup_{n=1}^\infty (a_n + F' \cap [1,n]) \subset F$. For every IP-system $\{s_\alpha\}$ in $\mathbb{Z}^m$, there exist $r \in \mathbb{N}$ and $\alpha \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})$ such that $\bar{r}^{(m)} + s_\alpha \in F'^{(m)}$. Choose n large enough such that $\bar{r}^{(m)} + s_\alpha \in (F' \cap [1,n])^{(m)}$ and let $r' = r + a_n$. Then $\bar{r}'^{(m)} + s_\alpha \in F^m$. Hence, F is also a J-set.

Let F be a J-set and $n \in \mathbb{N}$; we want to show that nF is also a J-set. Let $\{s_\alpha\}$ be an IP-system in $\mathbb{Z}^m$. Without loss of generality, assume that $\{s_\alpha\} \subset n\mathbb{Z}^m$. Let $s'_\alpha = n^{-1} s_\alpha$. Then $\{s'_\alpha\}$ is also an IP-system in $\mathbb{Z}^m$. Since F is a J-set, there exist $r \in \mathbb{N}$ and $\alpha \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})$ such that $\bar{r}^{(m)} + s'_\alpha \in F^{(m)}$. Then $n\bar{r}^{(m)} + s_\alpha \in nF^{(m)}$. Hence, nF is also a J-set. ∎

Remark 7.4. It is shown in [17] that there exists a C-set with upper Banach density 0. Thus there exists a dynamical system (X,T) and $x \in X$ such that x is $\mathcal{J}$-recurrent but not $\mathcal{F}_{\mathrm{pubd}}$-recurrent.

8. Solvability of Rado systems in C-sets. In order to show that Rado systems are solvable in C-sets, by the method developed in [12, pp. 169–174], it suffices to show the following two results.

Lemma 8.1. *If F is a C-set, then for each $n \in \mathbb{N}$, nF and $n^{-1}F$ are also C-sets.*

Theorem 8.2. *Let F be a C-set. Then for every $m \in \mathbb{N}$ and every IP-system $\{s_\alpha\}$ in $\mathbb{Z}^m$ there exists an IP-system $\{r_\alpha\}$ in $\mathbb{N}$ and an IP-subsystem $\{s_{\phi(\alpha)}\}$ such that for every $\alpha \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})$, $\bar{r}_\alpha^{(m)} + s_{\phi(\alpha)} \in F^m$.*

To discuss $\mathcal{J}$-recurrence, we first introduce a new kind of dynamical system. Let (X,T) be an invertible dynamical system. We say that (X,T) has the *multiple IP-recurrence property* if for every IP-system $\{s_\alpha = (s_\alpha^{(1)}, \dots, s_\alpha^{(m)})\}$ in $\mathbb{Z}^m$ and every open subset U of X, there exists $\alpha \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})$ such that

$$\bigcap_{i=1}^m T^{-s_\alpha^{(i)}} U \neq \emptyset.$$

If an invertible dynamical system is a minimal system, or if there exists an invariant measure with full support, then the system has the multiple IP-recurrence property ([12, 13]).

Lemma 8.3. *Let (X,T) be an invertible dynamical system and $n \in \mathbb{N}$. Then the following conditions are equivalent:*

(1) *(X,T) has the multiple IP-recurrence property;*
(2) *for every IP-system $\{s_\alpha = (s_\alpha^{(1)}, \dots, s_\alpha^{(m)})\}$ in $\mathbb{Z}^m$, every open subset U of X and $k \in \mathbb{N}$, there exists $\alpha \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})$ with $\min \alpha > k$ such that*

$$U \cap \bigcap_{i=1}^{m} T^{-s_\alpha^{(i)}} U \neq \emptyset;$$

(3) *(X,T^n) has the multiple IP-recurrence property.*

Proof. (1)⇒(3) and (2)⇒(1) are obvious.

(1)⇒(2). Let $\{s_\alpha = (s_\alpha^{(1)}, \dots, s_\alpha^{(m)})\}$ be an IP-system $\mathbb{Z}^m$ and $k \in \mathbb{N}$. Define a homomorphism $\phi : \mathcal{P}_{\mathrm{f}}(\mathbb{N}) \to \mathcal{P}_{\mathrm{f}}(\mathbb{N})$ by $\phi(\{i\}) = \{i+k\}$ for any $i \in \mathbb{N}$. Let $s_\alpha^{(0)} = 0$ for any $\alpha \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})$. Then $\{s'_\alpha = (s_\alpha^{(0)}, s_{\phi(\alpha)}^{(1)}, \dots, s_{\phi(\alpha)}^{(m)})\}$ is an IP-system in $\mathbb{Z}^{m+1}$. Now (2) follows by applying (1) to $\{s'_\alpha\}$.

(3)⇒(1). Let $\{s_\alpha\}$ be an IP-system in $\mathbb{Z}^m$. Without loss of generality, assume that $\{s_\alpha\} \subset n\mathbb{Z}^m$. Let $s'_\alpha = n^{-1} s_\alpha$. Then $\{s'_\alpha\}$ is also an IP-system in $\mathbb{Z}^m$. Then (1) follows by applying (3) to $\{s'_\alpha\}$ in (X,T^n). ∎

Let $\{x_\alpha\}_{\alpha \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})}$ be a sequence in a topological space X and $x \in X$. We say that $x_\alpha \to x$ *as a $\mathcal{P}_{\mathrm{f}}(\mathbb{N})$-sequence* if for every neighborhood U of x there exists $\alpha_U \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})$ such that $x_\alpha \in U$ for all $\alpha > \alpha_U$. If $\{x_\alpha\}$ is a $\mathcal{P}_{\mathrm{f}}(\mathbb{N})$-sequence in a compact metric space, then there exists a $\mathcal{P}_{\mathrm{f}}(\mathbb{N})$-subsequence $\{x_{\phi(\alpha)}\}$ which converges as a $\mathcal{P}_{\mathrm{f}}(\mathbb{N})$-sequence ([12, Theorem 8.14]).

Proposition 8.4. *Let (X,T) be an invertible metrizable dynamical system. Then (X,T) has the multiple IP-recurrent property if and only if for every IP-system $\{s_\alpha\}$ in $\mathbb{Z}^m$ and every open subset U of X there exists $x \in U$ and an IP-subsystem $\{s_{\phi(\alpha)}\}$ such that $T^{s_{\phi(\alpha)}^{(i)}} x \to x$ for $i = 1, \dots, m$.*

Proof. The sufficiency is obvious.

We now show the necessity. Let $\{s_\alpha = (s_\alpha^{(1)}, s_\alpha^{(2)}, \dots, s_\alpha^{(m)})\}$ be an IP-system in $\mathbb{Z}^m$ and U be an open subset of X. Let $U_0 = U$. By Lemma 8.3, there exists $\alpha_1 \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})$ such that

$$U_0 \cap \bigcap_{i=1}^{m} T^{-s_{\alpha_1}^{(i)}} U_0 \neq \emptyset.$$

Then choose an open subset U_1 with $\overline{U}_1 \subset U_0$ and $\operatorname{diam}(U_1) < 1$ such that

$$\bigcup_{i=1}^{m} T^{s_{\alpha_1}^{(i)}} U_1 \subset U_0.$$

Proceeding inductively, we define a sequence of open subsets $U_1, U_2, \ldots$ in X and a sequence $\alpha_1 < \alpha_2 < \cdots$ in $\mathcal{P}_{\mathrm{f}}(\mathbb{N})$ such that

$$\overline{U}_{n+1} \subset U_n, \quad \operatorname{diam}(U_n) < \frac{1}{n}, \quad \bigcup_{i=1}^{m} T^{s^{(i)}_{\alpha_n}} U_n \subset U_{n-1}.$$

Then there is a unique point x in $\bigcap_{n=1}^{\infty} \overline{U}_n$. Now set $\phi(\{n\}) = \alpha_n$ for each $n \in \mathbb{N}$. For every $\beta = \{r_1 < \cdots < r_k\}$, if $\min \beta > n+1$ then

$$\bigcup_{i=1}^{m} T^{s^{(i)}_{\phi(\beta)}} U_{r_k} = \bigcup_{i=1}^{m} T^{s^{(i)}_{\alpha_{r_1}}} \cdots T^{s^{(i)}_{\alpha_{r_k}}} U_{r_k} \subset U_{r_1 - 1} \subset U_n.$$

Hence, for $i = 1, \ldots, m$, $T^{s^{(i)}_{\phi(\beta)}} x \in U_n$ if $\min \beta > n+1$. It follows that $T^{s^{(i)}_{\phi(\alpha)}} x \to x$ for $i = 1, \ldots, m$. ∎

THEOREM 8.5. *Let (X,T) be an invertible dynamical system and $x \in X$. Then x is $\mathcal{J}$-recurrent if and only if $(\overline{\mathrm{Orb}(x,T)}, T)$ has the multiple IP-recurrence property.*

Proof. Without loss of generality, assume that $\overline{\mathrm{Orb}(x,T)} = X$. If x is $\mathcal{J}$-recurrent, then for every open subset U of X there exists $k \in \mathbb{N}$ and an open neighborhood V of x such that $T^k V \subset U$. Since x is $\mathcal{J}$-recurrent, $N(x,V)$ is a J-set. Then for every IP-system $\{s_\alpha = (s^{(1)}_\alpha, \ldots, s^{(m)}_\alpha)\}$ in $\mathbb{Z}^m$ there exist $r \in \mathbb{N}$ and $\alpha \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})$ such that $T^{r+s^{(i)}_\alpha} x \in V$ for $i = 1, \ldots, m$. Let $y = T^{r+k} x$. Then $T^{s^{(i)}_\alpha} y = T^k (T^{r+s^{(i)}_\alpha} x) \in T^k V \subset U$ for $i = 1, \ldots, m$. So $y \in \bigcap_{i=1}^{m} T^{-s^{(i)}_\alpha} U$.

Conversely, assume that (X,T) has the multiple IP-recurrence property. It is easy to see that x is recurrent. For every open neighborhood U of x and every IP-system $\{s_\alpha = (s^{(1)}_\alpha, \ldots, s^{(m)}_\alpha)\}$ in $\mathbb{Z}^m$, there exists $\alpha \in \mathcal{P}_{\mathrm{f}}(\mathbb{N})$ such that $\bigcap_{i=1}^{m} T^{-s^{(i)}_\alpha} U \neq \emptyset$. Choose $y \in \bigcap_{i=1}^{m} T^{-s^{(i)}_\alpha} U$; then $T^{s^{(i)}_\alpha} y \in U$ for $i = 1, \ldots, m$. By the continuity of T, choose an open neighborhood V of y such that $T^{s^{(i)}_\alpha} V \in U$ for $i = 1, \ldots, m$. Since $y \in \omega(x,T)$, there exists $r \in \mathbb{N}$ such that $T^r x \in U$ and $\bar{r}^{(m)} + s_\alpha \in \mathbb{N}^m$. Then $\bar{r}^{(m)} + s_\alpha \in N(x,U)^m$. Therefore, $N(x,U)$ is a J-set. ∎

PROPOSITION 8.6. *Let (X,T) be an invertible dynamical system, $x \in X$ and $n \in \mathbb{N}$. Then x is $\mathcal{J}$-recurrent in (X,T) if and only if it is $\mathcal{J}$-recurrent in (X,T^n).*

Proof. Without loss of generality, assume that $\overline{\mathrm{Orb}(x,T)} = X$. Since $\mathcal{J}$ is multiplication invariant, if x is $\mathcal{J}$-recurrent in (X,T^n), then it is so in (X,T).

Conversely, if x is $\mathcal{J}$-recurrent in (X,T), then (X,T) has the multiple IP-recurrence property, and so does (X,T^n). Since the interior of $\overline{\mathrm{Orb}(x,T^n)}$

is dense in $\overline{\mathrm{Orb}(x,T^n)}$, it is easy to see that $(\overline{\mathrm{Orb}(x,T^n)},T^n)$ also has the multiple IP-recurrence property. Thus x is $\mathcal{J}$-recurrent in (X,T^n). ∎

Proof of Lemma 8.1. This follows from Theorems 6.4 and 6.3, Lemma 7.3 and Proposition 8.6. ∎

Proof of Theorem 8.2. Since F is a C-set, there exists an idempotent $p \in h(\mathcal{J})$ such that $F \in p$. Let $x = \mathbf{1}_F \in \{0,1\}^{\mathbb{Z}}$ and $y = px \in [1]$. Then y is $\mathcal{J}$-recurrent, x is strongly proximal to y and $N(x,[1]) = F$.

Let $\{s_{\phi(\alpha)} = (s_\alpha^{(1)},\dots,s_\alpha^{(m)})\}$ be an IP-system in $\mathbb{Z}^m$. Let $U_1 = [1]$. Since $N((x,y),U_1\times U_1)$ is a J-set, there exist $r_1\in\mathbb{N}$ and $\alpha_1\in\mathcal{P}_{\mathrm{f}}(\mathbb{N})$ such that $\sigma\times\sigma^{r_1+s_{\alpha_1}^{(i)}}(x,y)\in U_1\times U_1$ for $i=1,\dots,m$. By continuity of σ, choose a neighborhood U_2 of y such that $U_2\subset U_1$ and

$$\bigcup_{i=1}^{m}\sigma^{r_1+s_{\alpha_1}^{(i)}}U_2\subset U_1.$$

Now suppose that we have chosen neighborhoods $U_1,\dots U_n,U_{n+1}$ of y, $r_1,\dots,r_n$ in $\mathbb{N}$ and $\alpha_1<\dots<\alpha_n$ in $\mathcal{P}_{\mathrm{f}}(\mathbb{N})$ satisfying the following conditions: for every $\beta\subset\{1,\dots,n\}$, letting $r_\beta=\sum_{j\in\beta}r_j$, $\phi(\beta)=\bigcup_{j\in\beta}\alpha_j$ and $U_\beta=U_{\min\beta}$, we have

(1) $\sigma^{r_\beta+s_{\phi(\beta)}^{(i)}}x\in U_\beta$ for $i=1,\dots,m$,

(2) $\sigma^{r_\beta+s_{\phi(\beta)}^{(i)}}U_{n+1}\subset U_\beta$ for $i=1,\dots,m$.

Since $N((x,y),U_{n+1}\times U_{n+1})$ is a J-set, there exist $r_{n+1}\in\mathbb{N}$ and $\alpha_{n+1}>\alpha_n$ such that $\sigma\times\sigma^{r_{n+1}+s_{\alpha_{n+1}}^{(i)}}(x,y)\in U_{n+1}\times U_{n+1}$ for $i=1,\dots,m$. Choose a neighborhood U_{n+2} of y such that $U_{n+2}\subset U_{n+1}$ and

$$\bigcup_{i=1}^{m}\sigma^{r_{n+1}+s_{\alpha_{n+1}}^{(i)}}U_{n+2}\subset U_{n+1}.$$

Now we show that (1) and (2) are satisfied with β replaced by $\beta'=\beta\cup\{n+1\}$ and $n+1$ replaced by $n+2$. This in fact follows from

$$\sigma^{r_{\beta'}+s_{\phi(\beta')}^{(i)}}x\in\sigma^{r_\beta+s_{\phi(\beta)}^{(i)}}(\sigma^{r_{n+1}+s_{\alpha_{n+1}}^{(i)}}x)\in\sigma^{r_\beta+s_{\phi(\beta)}^{(i)}}U_{n+1}\subset U_\beta$$

and

$$\sigma^{r_{\beta'}+s_{\phi(\beta')}^{(i)}}U_{n+2}\subset\sigma^{r_\beta+s_{\phi(\beta)}^{(i)}}(\sigma^{r_{n+1}+s_{\alpha_{n+1}}^{(i)}}U_{n+2})\subset\sigma^{r_\beta+s_{\phi(\beta)}^{(i)}}U_{n+1}\subset U_\beta.$$

Then by induction, $\sigma^{r_\beta+s_{\phi(\beta)}^{(i)}}x\in[1]$ for every $\beta\in\mathcal{P}_{\mathrm{f}}(\mathbb{N})$ and $i=1,\dots,m$. Thus, $\bar{r}_\alpha^{(m)}+s_{\phi(\alpha)}\in F^m$ for every $\alpha\in\mathcal{P}_{\mathrm{f}}(\mathbb{N})$. ∎

REMARK 8.7. One can use the algebraic properties of $\beta\mathbb{N}$ to prove Theorem 8.2 ([3, 20]). It is of interest whether one can deduce Lemma 8.1 from algebraic properties of $\beta\mathbb{N}$.

Acknowledgments. The author would like to thank Prof. Xiangdong Ye for helpful suggestions which significantly improved the presentation. This work was partly supported by the National Natural Science Foundation of China (Nos. 11171320, 11001071, 11071231).

References

[1] E. Akin, *Recurrence in Topological Dynamical Systems: Families and Ellis Actions*, Univ. Ser. Math. Plenum Press, New York, 1997.

[2] J. Banks, *Regular periodic decompositions for topologically transitive maps*, Ergodic Theory Dynam. Systems 17 (1997), 505–529.

[3] M. Beiglbock, V. Bergelson, T. Downarowicz, and A. Fish, *Solvability of Rado systems in D-sets*, Topology Appl. 156 (2009), 2565–2571.

[4] V. Bergelson and T. Downarowicz, *Large sets of integers and hierarchy of mixing properties of measure preserving systems*, Colloq. Math. 110 (2008), 117–150.

[5] V. Bergelson and N. Hindman, *Nonmetrizable topological dynamics and Ramsey theory*, Trans. Amer. Math. Soc. 320 (1990), 293–320.

[6] V. Bergelson, N. Hindman and B. Kra, *Iterated spectra of numbers—elementary, dynamical, and algebraic approaches*, ibid. 348 (1996), 893–912.

[7] A. Blokh and A. Fieldsteel, *Sets that force recurrence*, Proc. Amer. Math. Soc. 130 (2002), 3571–3578.

[8] S. Burns and N. Hindman, *Quasi-central sets and their dynamical characterization*, Topology Proc. 31 (2007), 445–455.

[9] D. De, N. Hindman and D. Strauss, *A new and stronger central sets theorem*, Fund. Math. 199 (2008), 155–175.

[10] R. Ellis, *Lectures on Topological Dynamics*, Benjamin, New York, 1969.

[11] R. Ellis and H. Keynes, *Bohr compactifications and a result of Følner*, Israel J. Math. 12 (1972), 314–330.

[12] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton Univ. Press, Princeton, NJ, 1981.

[13] H. Furstenberg and Y. Katznelson, *An ergodic Szemerédi theorem for IP-systems and combinatorial theory*, J. Anal. Math. 45 (1985), 117–168.

[14] H. Furstenberg and B. Weiss, *Topological dynamics and combinatorial number theory*, ibid. 34 (1978), 61–85.

[15] S. Glasner, *Divisible properties and the Stone–Čech compactification*, Canad. J. Math. 34 (1980), 993–1007.

[16] A. Hales and R. Jewett, *Regularity and positional games*, Trans. Amer. Math. Soc. 106 (1963), 222–229.

[17] N. Hindman, *Small sets satisfying the central sets theorem*, in: Combinatorial Number Theory, B. Landman et al. (eds.), de Gruyter, Berlin, 2009, 57–63.

[18] N. Hindman, A. Maleki, and D. Strauss, *Central sets and their combinatorial characterization*, J. Combin. Theory Ser. A 74 (1996), 188–208.

[19] N. Hindman and D. Strauss, *Algebra in the Stone–Čech Compactification: Theory and Applications*, de Gruyter, Berlin, 1998.

[20] —, —, *Sets satisfying the central sets theorem*, Semigroup Forum 79 (2009), 480–506.

[21] —, —, *Cartesian products of sets satisfying the Central Sets Theorem*, Topology Proc. 35 (2010), 203–223.

[22] W. Huang, H. Li and X. Ye, *Family-independence for topological and measurable dynamics*, Trans. Amer. Math. Soc., in press; arXiv:0908.0574
[23] W. Huang, K. K. Park amd X. Ye, *Dynamical systems disjoint from all minimal systems with zero entropy*, Bull. Soc. Math. France 135 (2007), 259–282.
[24] W. Huang and X. Ye, *Dynamical systems disjoint from any minimal system*, Trans. Amer. Math. Soc. 357 (2005), 669–694.
[25] J. Li, *Transitive points via Furstenberg family*, Topology Appl. 158 (2011), 2221–2231.
[26] R. Rado, *Studien zur Kombinatorik*, Math. Z. 36 (1933), 424–470.
[27] S. Shao, *Dynamical systems and Furstenberg families*, doctoral dissertation, Univ. of Science and Technology of China, 2004.

Jian Li
Department of Mathematics
University of Science and Technology of China
Hefei, Anhui, 230026, P.R. China
E-mail: lijian09@mail.ustc.edu.cn

Received 8 September 2011;
in revised form 18 November 2011