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Z_2^k-actions with a special fixed point set

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Abstract. Let F^n be a connected, smooth and closed n-dimensional manifold satisfying the following property: if N^m is any smooth and closed m-dimensional manifold with $m > n$ and $T : N^m \to N^m$ is a smooth involution whose fixed point set is F^n, then $m = 2n$. We describe the equivariant cobordism classification of smooth actions $(M^m; \Phi)$ of the group $G = Z_2^k$ on closed smooth m-dimensional manifolds M^m for which the fixed point set of the action is a submanifold F^n with the above property. This generalizes a result of F. L. Capobianco, who obtained this classification for $F^n = \mathbb{R}P^{2r}$ (P. E. Conner and E. E. Floyd had previously shown that $\mathbb{R}P^{2r}$ has the property in question). In addition, we establish some properties concerning these F^n and give some new examples of these special manifolds.

1. Introduction. Given a connected, smooth and closed n-dimensional manifold F^n, one has the twist involution $t : F^n \times F^n \to F^n \times F^n$, $t(x, y) = (y, x)$, and the identity involution $\mathrm{Id} : F^n \to F^n$, $\mathrm{Id}(x) = x$. The fixed point set of each of these involutions is F^n. We call F^n a manifold with *property* $\mathcal{H}$ if the only equivariant cobordism classes of involutions fixing F^n are $[F^n \times F^n, t]$ and $[F^n, \mathrm{Id}]$.

In [4], C. Kosniowski and R. E. Stong showed that, if (W^{2n}, T) is an involution fixing F^n, then (W^{2n}, T) is equivariantly cobordant to $(F^n \times F^n, t)$. In this way, the above concept can be restated in terms of dimensions: F^n has property $\mathcal{H}$ if every involution (M^m, T) with fixed point set F^n has $m = n$ or $m = 2n$. The real, complex and quaternionic even-dimensional projective spaces $\mathbb{R}P^{2n}$, $\mathbb{C}P^{2n}$ and $\mathbb{H}P^{2n}$ are examples of manifolds with property $\mathcal{H}$; also the Cayley projective plane $\mathbb{Q}P^2$ has this property (see [9]).

Now consider the group $G = Z_2^k = Z_2 \oplus \cdots \oplus Z_2$ (k copies). For each r with $1 \le r \le k$ one may construct a twist action of G on the prod-

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uct $(F^n)^{2^r} = F^n \times \cdots \times F^n$ (2^r factors), which we denote by t_r^k, in the following inductive way: considering G as the group generated by k commuting involutions $T_1, \ldots, T_k$, first set $t_1^1 = t : F^n \times F^n \to F^n \times F^n$. Supposing one has constructed t_{k-1}^{k-1}, on $(F^n)^{2^k} = (F^n)^{2^{k-1}} \times (F^n)^{2^{k-1}}$ let $T_1, \ldots, T_{k-1}$ act as $t_{k-1}^{k-1} \times t_{k-1}^{k-1}$ and T_k act by switching $(F^n)^{2^{k-1}} \times (F^n)^{2^{k-1}}$. This defines t_k^k for any $k \geq 1$. Finally, we define t_r^k by letting $T_1, \ldots, T_r$ act as t_r^r and $T_{r+1}, \ldots, T_r$ act trivially; also we extend this definition to $r = 0$ by letting t_0^k be the trivial action on F^n. The fixed point set of t_r^k is the diagonal copy of F^n. On the other hand, given any G-action $(M; \Phi)$, $\Phi = (T_1, \ldots, T_k)$, each automorphism $\sigma : G \to G$ yields a new G-action given by $(M; \sigma(T_1), \ldots, \sigma(T_k))$; we denote this action by $\sigma(M; \Phi)$. In [1], F. Capobianco proved that every G-action $(M; \Phi)$ fixing $\mathbb{R}P^{2n}$ is cobordant to $\sigma((\mathbb{R}P^{2n})^{2^r}; t_r^k)$ for some $\sigma : G \to G$ and $1 \leq r \leq k$. The main goal of this paper is to generalize this result by showing that it is true for any manifold with property $\mathcal{H}$.

The crucial point of Capobianco's argument was the following special property of $\mathbb{R}P^{2n}$, proved by Stong in [9]: if $\eta^r \to \mathbb{R}P^{2n}$ is an r-dimensional vector bundle over $\mathbb{R}P^{2n}$ which is the fixed data of an involution, then $r = 2n$ and the Stiefel–Whitney class of η^r is $W(\eta^r) = (1 + \alpha)^{2n+1}$, where $\alpha \in H^1(\mathbb{R}P^{2n}, Z_2)$ is the generator. In particular, this fact implies that $\mathbb{R}P^{2n}$ has property $\mathcal{H}$ (the same type of argument works for $\mathbb{C}P^{2n}$, $\mathbb{H}P^{2n}$ and $\mathbb{Q}P^2$). The subtle point of our method is that property $\mathcal{H}$, and not the above special property of $\mathbb{R}P^{2n}$, $\mathbb{C}P^{2n}$, $\mathbb{H}P^{2n}$ and $\mathbb{Q}P^2$, determines the desired result.

Section 2 will study manifolds with property $\mathcal{H}$ and give examples of such manifolds. In Section 3, the proof of the main result will be given; this result is the following

THEOREM. *Suppose* $(M; \Phi)$, $\Phi = (T_1, \ldots, T_k)$, *is a* G*-action fixing* F^n, *where* F^n *has property* $\mathcal{H}$. *Then* $(M; \Phi)$ *is equivariantly cobordant to the action* $\sigma((F^n)^{2^r}; t_r^k)$ *for some automorphism* $\sigma : G \to G$ *and some* $1 \leq r \leq k$.

2. On manifolds with property $\mathcal{H}$. We begin with some general facts concerning property $\mathcal{H}$.

PROPOSITION 2.1. *If* F^n *has property* $\mathcal{H}$, *then* F^n *is nonbounding.*

Proof. This follows from the fact that if F^n bounds, then there are involutions of every dimension fixing F^n. In fact, consider an $(n+1)$-dimensional manifold W^{n+1} whose boundary $\partial(W^{n+1})$ is F^n. For any natural number $m > 0$, form $W^{n+1} \times D^m$ with the involution $(w, x) \mapsto (w, -x)$, where D^m is the m-dimensional disc. Then $\partial(W^{n+1} \times D^m)$ is a closed $(n+m)$-dimensional manifold equipped with an involution whose fixed point set is F^n. ∎

PROPOSITION 2.2. *If F^n has property $\mathcal{H}$, then n is even.*

Proof. If n is odd it is known that, since the Euler characteristic of F^n is zero, the tangent bundle $\tau^n \to F^n$ has a section, that is, there is an $(n-1)$-dimensional vector bundle $\mu^{n-1} \to F^n$ so that τ^n is equivalent to $\mu^{n-1} \oplus R \to F^n$, where $R \to F^n$ is the one-dimensional trivial bundle (see, for example, [7]). Since $\tau^n \to F^n$ is the fixed data of the twist involution $(F^n \times F^n, t)$, one deduces from the stability property of [3, Theorem 26.4] that also $\mu^{n-1} \to F^n$ is the fixed data of an involution. ∎

REMARK. Obviously the above argument also serves to show that if F^n has property $\mathcal{H}$, then the tangent bundle $\tau^n \to F^n$ does not have a section. It is interesting to note that the converse is not true. The essential point is that $\tau^n \to F^n$, while not having a section itself, may be cobordant to a bundle over F^n with a section. For example, consider the connected sum $\mathbb{R}P^4 \# \mathbb{C}P^2 = F^4$. The Euler characteristic of F^4 is $\chi(F^4) = \chi(\mathbb{R}P^4) + \chi(\mathbb{C}P^2) - 2 = 1 + 3 - 2 = 2$, thus the tangent bundle $\tau^4 \to F^4$ does not have a section. We know that $H^*(F^4, Z_2)$ is generated by $\alpha \in H^1(F^4, Z_2)$ and $\beta \in H^2(F^4, Z_2)$, with relations $\alpha^5 = 0$, $\beta^3 = 0$, $\alpha\beta = 0$ and $\alpha^4 = \beta^2$. The Stiefel–Whitney class of F^4 is $W(F^4) = 1 + \alpha + \beta$. Over F^4 there is a real line bundle $\lambda \to F^4$ with $W(\lambda) = 1 + \alpha$ and a complex line bundle $\xi \to F^4$ with $W(\xi) = 1 + \beta$, and one has

$$W(\lambda \oplus \xi) = (1 + \alpha)(1 + \beta) = 1 + \alpha + \beta$$

(since $\alpha\beta = 0$). Thus $W(\lambda \oplus \xi) = W(\tau^4)$, and τ^4 is cobordant to $\lambda \oplus \xi \oplus R$. Then one has an involution (M^8, T) cobordant to $(F^4 \times F^4, t)$ with fixed data $\lambda \oplus \xi \oplus R \to F^4$. It follows that $\lambda \oplus \xi \to F^4$ is the fixed data of an involution and F^4 does not have property $\mathcal{H}$.

PROPOSITION 2.3. *Suppose that F^n is the total space of a differentiable fibering of closed manifolds, where V^r is the base space, K^s is the fiber and $\pi : F^n \to V^r$ is the projection map, with $r, s > 0$. Then F^n does not have property $\mathcal{H}$.*

Proof. Consider $E \subset F^n \times F^n$, $E = \{(x, y) \mid \pi(x) = \pi(y)\}$. Then E is a closed $(n + s)$-dimensional submanifold of $F^n \times F^n$. On E one has the fiberwise twist involution fixing a diagonal copy of F^n, and since $s < n$ the result follows. ∎

A consequence of Proposition 2.3 is that property $\mathcal{H}$ is not a cobordism invariant. In fact, $\mathbb{C}P^2$ has property $\mathcal{H}$ and is cobordant to $\mathbb{R}P^2 \times \mathbb{R}P^2$ which does not, since it fibers. As another example, consider the 3-dimensional vector bundle $\lambda \oplus R^2 \to \mathbb{R}P^2$, where λ is the canonical line bundle and R^2 is the trivial 2-dimensional bundle over $\mathbb{R}P^2$. Then the projective space bundle $\mathbb{R}P(\lambda \oplus R^2)$ is cobordant to $\mathbb{R}P^4$ (this can be checked by computing characteristic numbers) and does not have property $\mathcal{H}$, since it is a fibering. However, one has

Proposition 2.4. *Property $\mathcal{H}$ is a homotopy invariant.*

Proof. Suppose F^n does not have property $\mathcal{H}$ and is homotopy equivalent to V^n. Then there exists a vector bundle $\mu^r \to F^n$ which is the fixed data of an involution with $r < n$. Take a homotopy equivalence $f : V^n \to F^n$. Then the pullback $f * (\mu^r) \to V^n$ is a vector bundle having the same Whitney numbers as μ^r, which implies that $f * (\mu^r)$ and μ^r are cobordant as elements of the bordism group of closed n-dimensional manifolds with r-dimensional vector bundles, $\mathcal{N}_n(BO(r))$. It follows that $f * (\mu^r)$ is also the fixed data of an involution and V^n does not have property $\mathcal{H}$. ∎

The following proposition gives the first new examples of manifolds with property $\mathcal{H}$.

Proposition 2.5. *If F^2 is nonbounding then F^2 has property $\mathcal{H}$.*

Proof. First we recall the following result of Kosniowski and Stong of [4]: if an involution (M^m, T) fixes F^n and $m > 2n$, then (M^m, T) bounds equivariantly. Now if (M^m, T) fixes the nonbounding F^2, the fixed data of (M^m, T) is nonbounding and hence (M^m, T) is nonbounding. It follows that $m \leq 4$. But from [3] one knows that an involution with codimension one fixed point set bounds, hence $m = 3$ is impossible and the result is proved. ∎

Observe that Proposition 2.5 gives examples not homotopy equivalent to $\mathbb{R}P^2$. Other new examples will be obtained with the following

Proposition 2.6. *Suppose F^{2n} is nonbounding and $H^j(F^{2n}, Z_2) = 0$ for $0 < j < n$. Then F^{2n} has property $\mathcal{H}$.*

Proof. If (M^m, T) fixes F^{2n} and $\eta^r \to F^{2n}$ is the normal bundle of F^{2n} in M^m, then by the argument used in the proof of Proposition 2.5 one has $r \leq 2n$. Hence we must show that $0 < r < 2n$ is impossible. We need the following fact, which follows from Conner–Floyd's exact sequence of [3, 28.1]: if $\mu^r \to W$ is an r-dimensional vector bundle over a nonbounding and closed manifold W which is cobordant to the trivial r-dimensional bundle over W, then μ^r cannot be the fixed data of an involution. If $0 < r < n$, the hypothesis says that the Stiefel–Whitney class of η^r is $W(\eta^r) = 1$ and thus η^r is cobordant to the trivial r-dimensional bundle over F^{2n}. Hence we can assume $n \leq r < 2n$. By Poincaré duality, $H^j(F^{2n}, Z_2) = 0$ for $n < j < 2n$, thus the Stiefel–Whitney classes of η^r and F^{2n} can be written as $W(\eta^r) = 1 + u_n$, $W(F^{2n}) = 1 + w_n + w_{2n}$. Also the Wu class of F^{2n} can be written as $V(F^{2n}) = 1 + v_n$, and if Sq is the Steenrod operation one has

$$\mathrm{Sq}(V(F^{2n})) = 1 + v_n + \sum_{i=1}^{n} \mathrm{Sq}^i(v_n) = 1 + v_n + \mathrm{Sq}^n(v_n) = 1 + v_n + v_n^2.$$

From $\mathrm{Sq}(V(F^{2n})) = W(F^{2n})$ one then gets $v_n = w_n$ and $w_{2n} = v_n^2 = w_n^2$. Since F^{2n} does not bound, in particular one has $w_{2n} = w_n^2 \neq 0$. Our objective is to prove that η^r is cobordant to the r-dimensional trivial vector bundle over F^{2n}. To do this, it suffices to show that every Whitney number of η^r involving classes of η^r vanishes. But the only such number which can be nonzero is

$$w_n u_n[F^{2n}] = v_n u_n[F^{2n}] = \mathrm{Sq}^n(u_n)[F^{2n}] = u_n^2[F^{2n}].$$

Denote by $\lambda \to \mathbb{R}P(\eta^r)$ the usual line bundle over the projective space bundle $\mathbb{R}P(\eta^r)$, and write $W(\lambda) = 1 + c$. Since $\eta^r \to F^{2n}$ is the fixed data of an involution, $\lambda \to \mathbb{R}P(\eta^r)$ bounds as an element of the bordism group of manifolds with line bundles, $\mathcal{N}_{2n+r-1}(BO(1))$. It follows that $c^{2n+r-1}[\mathbb{R}P(\eta^r)] = 0$. Denoting by

$$\overline{W}(\eta^r) = \frac{1}{W(\eta^r)} = 1 + \overline{u}_n + \overline{u}_{2n}$$

the dual Stiefel–Whitney class of η^r, one infers from [2] that $c^{2n+r-1}[\mathbb{R}P(\eta^r)] = \overline{u}_{2n}[F^{2n}]$. But

$$\frac{1}{W(\eta^r)} = \frac{1}{1 + u_n} = 1 + u_n + u_n^2,$$

which means that $\overline{u}_{2n} = u_n^2$ and $u_n^2 = 0$. ∎

For example, simply connected nonbounding 4-dimensional manifolds satisfy the hypotheses of Proposition 2.6. In particular, for every $k \geq 1$, the connected sum of $\mathbb{C}P^2$ and k copies of $S^2 \times S^2$,

$$\mathbb{C}P^2 \,\#\, (S^2 \times S^2) \,\#\, \cdots \,\#\, (S^2 \times S^2),$$

has this property, and the same is valid for $\mathbb{H}P^2 \,\#\, (S^4 \times S^4) \,\#\cdots\#\, (S^4 \times S^4)$ and $\mathbb{Q}P^2 \,\#\, (S^8 \times S^8) \,\#\cdots\#\, (S^8 \times S^8)$. Observe that these examples are not homotopy equivalent to the known examples $\mathbb{C}P^2$, $\mathbb{H}P^2$ and $\mathbb{Q}P^2$. For an 8-dimensional nonbounding M^8 with $H^1(M^8, Z_2) = 0$ and $H^2(M^8, Z_2) = 0$, the result is also valid. In fact, in this case any vector bundle $\nu \to M^8$ has $w_3(\nu) = 0$ and $w_5(\nu) = 0$, so the argument is the same. For example, besides $S^4 \times S^4$, we may add handles $S^3 \times S^5$ to $\mathbb{H}P^2$. In the same way, a 16-dimensional nonbounding M^{16} with $H^j(M^{16}, Z_2) = 0$ for $1 \leq j \leq 4$ has property $\mathcal{H}$, which allows one to add handles $S^5 \times S^{11}$, $S^6 \times S^{10}$ and $S^7 \times S^9$ to $\mathbb{Q}P^2$.

Remark. Proposition 2.6 does not give examples of manifolds with property $\mathcal{H}$ in dimensions different from 2, 4, 8 and 16. This follows from the fact that if F^{2n} has $H^j(F^{2n}, Z_2) = 0$ for $0 < j < n$ and $n \neq 1, 2, 4$ and 8, then F^{2n} bounds. In fact, from the proof of Proposition 2.6 one sees that F^{2n} is nonbounding if and only if $w_{2n}(F^{2n})$, which is equal to $w_n^2(F^{2n}) = \mathrm{Sq}^n(w_n)(F^{2n})$, is different from zero. But Sq^n is decomposable through the Adem relations for $n \neq 2^s$, and is decomposable in terms of secondary operations for $n = 2^s \geq 16$.

Remark. Proposition 2.5 says that any 2-dimensional nonbounding manifold has property $\mathcal{H}$. Now every nonbounding M^2 is an $\mathbb{R}P^2$ with handles, that is, $\mathbb{R}P^2 \# (S^1 \times S^1) \# \cdots \# (S^1 \times S^1)$. This suggests considering, as in the above examples,

$$F^{2n} = \mathbb{R}P^{2n} \# (S^n \times S^n) \# \cdots \# (S^n \times S^n) \quad (k \text{ copies of } S^n \times S^n).$$

Then $H^*(F^{2n}, Z_2)$ is isomorphic to $H^*(\mathbb{R}P^{2n}, Z_2)$ except in dimension n, where we have added cohomology classes $a_i, b_i \in H^n(F^{2n}, Z_2)$, $1 \leq i \leq k$, with each $a_i b_i$ being the nonzero element of $H^{2n}(F^{2n}, Z_2)$. The Stiefel–Whitney class of F^{2n} is still $W(F^{2n}) = (1+\alpha)^{2n+1}$, where $\alpha \in H^1(F^{2n}, Z_2)$ is the generator coming from $\mathbb{R}P^{2n}$. If (M^m, T) fixes F^{2n} and $\eta^{m-2n} \to F^{2n}$ is the normal bundle, with $W(\eta^{m-2n}) = 1 + u_1 + u_2 + \cdots$, then u_{2^s} is a multiple of α^{2^s} except possibly in dimension n. If n is not a power of 2, then this shows that u_{2^s} is a multiple of α^{2^s} for each $s \geq 0$ and so $W(\eta^{m-2n}) = (1+\alpha)^t$ for some t. Summarizing, one has an involution (M^m, T) fixing $\eta^{m-2n} \to F^{2n}$ with $W(F^{2n}) = (1+\alpha)^{2n+1}$ and $W(\eta^{m-2n}) = (1+\alpha)^t$; this is exactly the situation used in the proof of [3, 27.7] (which established that $\mathbb{R}P^{2r}$ has property $\mathcal{H}$), which gives $m - 2n = 0$ or $2n$. Thus F^{2n} has property $\mathcal{H}$ if n is not a power of 2.

3. Proof of the main theorem. First we need some preliminaries about G-actions, where $G = Z_2^k$. Given a G-action $(M; \Phi)$, $\Phi = (T_1, \ldots, T_k)$, the fixed point set of Φ, F, is a disjoint union of closed submanifolds of M. The normal bundle of F in M, η, decomposes under the action of G into the Whitney sum of subbundles on which G acts as one of the irreducible (nontrivial) real representations. These irreducible and nontrivial representations of G are all one-dimensional and can be described by homomorphisms $\varrho : G \to Z_2 = \{+1, -1\}$ which are onto, and G acts on the reals so that $g \in G$ acts as multiplication by $\varrho(g)$. In other words,

$$\eta = \bigoplus_{\varrho} \varepsilon_{\varrho},$$

where ε_{ϱ} is the subbundle of η on which G acts in the fibers as ϱ, that is, where each T_j acts as multiplication by $\varrho(T_j)$, and where the sum excludes the trivial homomorphism $1 \in \mathrm{Hom}(G, Z_2)$. Alternatively, ε_{ϱ} is the normal bundle of F in the fixed point set F_H of the subgroup $H = \ker(\varrho)$. Set

$\mathcal{P} = \mathrm{Hom}(G, Z_2) - \{1\}$; then the *fixed data* of $(M;\Phi)$ is $(F, \{\varepsilon_\varrho\}_{\varrho\in\mathcal{P}})$, a closed manifold (the fixed point set) and a list of $2^k - 1$ vector bundles over it indexed by $\mathcal{P}$. Each s-dimensional component of $(F, \{\varepsilon_\varrho\}_{\varrho\in\mathcal{P}})$ can be considered as an element of $\mathcal{N}_s(\prod_{\varrho\in\mathcal{P}} BO(n_\varrho))$, the bordism of s-dimensional manifolds with a map into a product of classifying spaces $BO(n_\varrho)$ for n_ϱ-dimensional vector bundles, where n_ϱ denotes the dimension of ε_ϱ over the component (this is the *simultaneous cobordism* between lists of vector bundles: if $\mathcal{P}$ is any finite set, two lists (indexed by $\mathcal{P}$) of vector bundles over closed n-dimensional manifolds, $(F^n, \{\varepsilon_\varrho\}_{\varrho\in\mathcal{P}})$ and $(V^n, \{\mu_\varrho\}_{\varrho\in\mathcal{P}})$, are *simultaneously cobordant* if there exists an $(n+1)$-dimensional manifold W^{n+1} with boundary $\partial(W^{n+1}) = F^n \cup V^n$ (disjoint union) and a list of vector bundles over W^{n+1}, $(W^{n+1}, \{\eta_\varrho\}_{\varrho\in\mathcal{P}})$, so that each η_ϱ restricted to $F^n \cup V^n$ is equivalent to $\varepsilon_\varrho \cup \mu_\varrho$). According to [8], the equivariant cobordism class of $(M;\Phi)$ is determined by the cobordism class of $(F, \{\varepsilon_\varrho\}_{\varrho\in\mathcal{P}})$.

For example, the fixed data of the twist G-action $((F^n)^{2^r}; t_r^k)$ described in Section 1 is $(F, \{\varepsilon_\varrho\}_{\varrho\in\mathcal{P}})$, where $F = F^n$ and $\{\varepsilon_\varrho\}_{\varrho\in\mathcal{P}}$ consists of $2^r - 1$ copies of the tangent bundle $\tau \to F^n$ and $2^k - 2^r$ copies of the zero bundle $0 \to F^n$. More precisely, $\varepsilon_\varrho = \tau$ when $\varrho(T_i) = 1$ for all $i \ge r+1$, and $\varepsilon_\varrho = 0$ for the remaining $\varrho \in \mathcal{P}$.

Remark. For every automorphism $\sigma : G \to G$, $\sigma((F^n)^{2^k}; t_k^k)$ is cobordant to $((F^n)^{2^k}; t_k^k)$. However, if $r < k$, then $\sigma((F^n)^{2^r}; t_r^k)$ may or not be cobordant to $((F^n)^{2^r}; t_r^k)$. If $(F^n, \{\varepsilon_\varrho\})$ and $(F^n, \{\mu_\varrho\})$ are, respectively, the fixed data of $((F^n)^{2^r}; t_r^k)$ and $\sigma((F^n)^{2^r}; t_r^k)$, these actions are cobordant if and only if $\varepsilon_\varrho = \mu_\varrho$ for every $\varrho \in \mathcal{P}$. Let $H \subset G$ be the subgroup of G generated by $T_{r+1}, \dots, T_k$. By the above description of the fixed data of $((F^n)^{2^r}; t_r^k)$ and the fact that $\sigma(H) = H$ if and only if

$$\begin{aligned}\{\varrho \in \mathcal{P} \mid \varrho(T_i) = 1 \text{ for all } i \ge r+1\}&\\ = \{\varrho \in \mathcal{P} \mid \varrho(\sigma(T_i)) = 1 \text{ for all } i \ge r+1\},&\end{aligned}$$

one concludes that $((F^n)^{2^r}; t_r^k)$ and $\sigma((F^n)^{2^r}; t_r^k)$ are cobordant if and only if $\sigma(H) = H$. For example, write $((F^n)^4; t_2^4) = ((F^n)^4; T_1, T_2, T_3, T_4)$. Then $((F^n)^4; t_2^4)$ and $((F^n)^4; T_2T_3, T_1T_2, T_4, T_3)$ are cobordant, while $((F^n)^4; t_2^4)$ and $((F^n)^4; T_2, T_1T_3, T_3, T_1T_4)$ are not cobordant.

Let $(M;\Phi)$ be a G-action with fixed data $(F, \{\varepsilon_\varrho\}_{\varrho\in\mathcal{P}})$, and let Ω be a subgroup of $\mathrm{Hom}(G, Z_2)$. Our first step will be to show that the part of the fixed data of $(M;\Phi)$ given by $(F, \{\varepsilon_\varrho\}_{\varrho\in\Omega\cap\mathcal{P}})$ can be realized as the fixed data of some subgroup $H \subset G$ acting (by restriction) on the fixed point set of the restriction of Φ to some appropriate subgroup $K \subset G$.

First note that there exists a subgroup $H \subset G$ so that the restriction $\mathrm{Hom}(G, Z_2) \to \mathrm{Hom}(H, Z_2)$ maps Ω isomorphically onto $\mathrm{Hom}(H, Z_2)$. In fact, consider the natural isomorphism $G \to \mathrm{Hom}(\mathrm{Hom}(G, Z_2), Z_2)$ given by $T \mapsto \varphi_T$, where $\varphi_T(\varrho) = \varrho(T)$ for any $\varrho \in \mathrm{Hom}(G, Z_2)$. Choose a basis $(\varrho_1, \ldots, \varrho_t, \xi_1, \ldots, \xi_{k-t})$ for $\mathrm{Hom}(G, Z_2)$ so that $(\varrho_1, \ldots, \varrho_t)$ is a basis for Ω, and consider the basis $(T_1, \ldots, T_t, S_1, \ldots, S_{k-t})$ for G which corresponds to the dual basis $(\varrho_1^*, \ldots, \varrho_t^*, \xi_1^*, \ldots, \xi_{k-t}^*)$ of $\mathrm{Hom}(\mathrm{Hom}(G, Z_2), Z_2)$ under the above isomorphism. Evidently, the basis $(\varrho_1, \ldots, \varrho_t, \xi_1, \ldots, \xi_{k-t})$ is dual to $(T_1, \ldots, T_t, S_1, \ldots, S_{k-t})$. Set $H = \langle T_1, \ldots, T_t \rangle$. Since $\varrho_i(T_j) = -1$ if $i = j$ and $\varrho_i(T_j) = 1$ if $i \neq j$, it follows that $(\varrho_{1|H}, \ldots, \varrho_{t|H})$ is a basis for $\mathrm{Hom}(H, Z_2)$, and thus the restriction maps Ω isomorphically onto $\mathrm{Hom}(H, Z_2)$. Now set $K = \langle S_1, \ldots, S_{k-t} \rangle$, $F_K =$ the fixed point set of K and $\Psi =$ the restriction of Φ to $H \times F_K$. One then has the following

Lemma 3.1. *The fixed data of the H-action $(F_K; \Psi)$ is $(F, \{\mu_{\varrho'}\}_{\varrho' \in \mathcal{P}'})$, where for each $\varrho' \in \mathcal{P}' = \mathrm{Hom}(H, Z_2) - \{1\}$ one has $\mu_{\varrho'} = \varepsilon_\varrho$, where ϱ is the unique element of $\Omega \cap \mathcal{P}$ with $\varrho_{|H} = \varrho'$. In other words, the fixed data of H acting on the fixed point set of K is F with the subbundles ε_ϱ, $\varrho \in \Omega \cap \mathcal{P}$, and in terms of $\mathcal{P}' = \mathrm{Hom}(H, Z_2) - \{1\}$, these subbundles are indexed under the restriction $\Omega \cap \mathcal{P} \to \mathcal{P}'$.*

Proof. The inverse of the restriction $\Omega \to \mathrm{Hom}(H, Z_2)$ is the isomorphism $\mathrm{Hom}(H, Z_2) \to \Omega$ given by $\varrho' \mapsto \varrho$, where $\varrho_{|H} = \varrho'$ and ϱ is the trivial homomorphism on K. The fixed point set of $(F_K; \Psi)$ is F, and if $(F, \{\mu_{\varrho'}\}_{\varrho' \in \mathcal{P}'})$ is the fixed data, each $\mu_{\varrho'}$ is equal to ε_ϱ for some $\varrho \in \mathcal{P}$. Thus, to complete the argument, it suffices to show that, if $\mu_{\varrho'} = \varepsilon_\varrho$, then $\varrho_{|H} = \varrho'$ and ϱ is the trivial homomorphism on K.

Take $T \in H$. Then T acts on $\mu_{\varrho'}$ as $\varrho'(T)$, and since $\mu_{\varrho'} = \varepsilon_\varrho$, also T acts on $\mu_{\varrho'}$ as $\varrho(T)$. Hence $\varrho(T) = \varrho'(T)$ and $\varrho_{|H} = \varrho'$. Now take $T \in K$. Note that $\mu_{\varrho'}$ is a subbundle of the normal bundle of F in F_K, which is the fixed point set of K. Thus T acts on $\mu_{\varrho'}$ identically. Since $\mu_{\varrho'} = \varepsilon_\varrho$ and T acts on ε_ϱ as $\varrho(T)$, we conclude that $\varrho(T) = 1$, which gives the result. ∎

Remark. Suppose $(F, \{\varepsilon_\varrho\}_{\varrho \in \mathcal{P}})$ is the fixed data of a G-action $(M; \Phi)$. Denote by $\mathcal{A}$ the set of vector bundles over F which lie in $\{\varepsilon_\varrho\}$. Then $(F, \{\varepsilon_\varrho\})$ gives a map $\theta : \mathcal{P} \to \mathcal{A}$, and if $\sigma : G \to G$ is an automorphism, $\sigma(M; \Phi)$ gives rise to a new map $\mathcal{P} \to \mathcal{A}$ which is θ composed with some bijection $\mathcal{P} \to \mathcal{P}$. We note that not every bijection $\mathcal{P} \to \mathcal{P}$ gives a map $\mathcal{P} \to \mathcal{A}$ derived from some automorphism $G \to G$, since the number of such bijections may be greater than the number of bases of G. In particular, we cannot in principle guarantee that all such maps $\mathcal{P} \to \mathcal{A}$ come from G-actions. This is not the case, however, when $G = Z_2^2$; if $(F; \{\varepsilon_{\varrho_1}, \varepsilon_{\varrho_2}, \varepsilon_{\varrho_3}\})$ is a fixed data with a map $\mathcal{P} = \{\varrho_1, \varrho_2, \varrho_3\} \to \mathcal{A}$, then all the other possible maps $\mathcal{P} \to \mathcal{A}$ come from Z_2^2-actions, since they are derived from automorphisms $Z_2^2 \to Z_2^2$. Therefore the next lemma is independent of the map $\mathcal{P} \to \mathcal{A}$.

LEMMA 3.2. *Suppose F^n is a nonbounding, connected and closed n-dimensional manifold. Let η be any p-dimensional vector bundle over F^n ($p > 0$) and μ an n-dimensional vector bundle over F^n cobordant to the tangent bundle $\tau \to F^n$. Denote by 0 the zero bundle over F^n. Then $(F^n; \{\eta, \mu, 0\})$ cannot be the fixed data of a Z_2^2-action.*

Proof. If $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are three vector bundles over F^n and $(F^n; \{\varepsilon_1, \varepsilon_2, \varepsilon_3\})$ is the fixed data of a Z_2^2-action, then the argument outlined in [5, Section 3; pp. 88–90] (or in [6, Section 2; pp. 107–108]) shows that $(\mathbb{R}P(\varepsilon_1); \lambda, \varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda))$, the projective space bundle of ε_1 with its standard line bundle $\lambda \to \mathbb{R}P(\varepsilon_1)$ and the bundle $\varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda) \to \mathbb{R}P(\varepsilon_1)$, bounds as an element of the bordism group $\mathcal{N}_{n+s_1-1}(BO(1) \times BO(s_2 + s_3))$; here, $s_i = \dim(\varepsilon_i)$ and we are suppressing bundle maps. In particular, if we suppose by contradiction that $(F^n; \{\eta, \mu, 0\})$ is the fixed data of a Z_2^2-action, by taking $\varepsilon_1 = \eta$, $\varepsilon_2 = \mu$ and $\varepsilon_3 = 0$, one sees that $(\mathbb{R}P(\eta); \lambda, \mu)$ bounds as an element of $\mathcal{N}_{n+p-1}(BO(1) \times BO(n))$. Since F^n is nonbounding, there is a Stiefel–Whitney number $w_{i_1}(\tau) \cdots w_{i_r}(\tau)[F^n]$ which is nonzero. Since μ is cobordant to τ, $w_{i_1}(\mu) \cdots w_{i_r}(\mu)[F^n]$ is also nonzero. Set $W(\lambda) = 1 + c$; it is known that $H^*(\mathbb{R}P(\eta), Z_2)$ is the free $H^*(F^n, Z_2)$-module on $1, c, c^2, \ldots, c^{p-1}$. Therefore $c^{p-1} w_{i_1}(\mu) \cdots w_{i_r}(\mu)$ is the nonzero class of $H^{n+p-1}(\mathbb{R}P(\eta), Z_2)$. Since $c^{p-1} w_{i_1}(\mu) \cdots w_{i_r}(\mu)[\mathbb{R}P(\eta)]$ is a characteristic number of $(\mathbb{R}P(\eta); \lambda, \mu)$, this gives a contradiction. ∎

LEMMA 3.3. *Let $(M^m; \Phi)$ be a G-action ($G = Z_2^k$) with fixed point set F^n being a connected n-dimensional submanifold, and with $m = 2^k n$. Then $(M^m; \Phi)$ is equivariantly cobordant to $((F^n)^{2^k}; t_k^k)$.*

Proof. This is the main result of [6]. ∎

Now we have in hand the necessary tools to prove our main result. Suppose that $(M^m; \Phi)$, $\Phi = (T_1, \ldots, T_k)$, is a G-action fixing F^n, where F^n has property $\mathcal{H}$. As stated in the introduction, our aim is to prove that $(M^m; \Phi)$ is equivariantly cobordant to $\sigma((F^n)^{2^r}; t_r^k)$ for some automorphism $\sigma : G \to G$ and some $1 \le r \le k$. The essential point is that Lemmas 3.1 and 3.2 allow us to find a special subgroup $H \subset G$ so that Lemma 3.3 can be applied to the restriction of Φ to $H \times M^m$. Let $(F^n, \{\varepsilon_\varrho\}_{\varrho \in \mathcal{P}})$ be the fixed data of Φ.

LEMMA 3.4. *For each $\varrho \in \mathcal{P}$, ε_ϱ is either cobordant to the tangent bundle $\tau \to F^n$ or cobordant to the zero bundle $0 \to F^n$; in particular, $m = pn$ for some $1 \le p \le 2^k$ (note that $p - 1$ is the number of bundles cobordant to $\tau \to F^n$).*

Proof. For each $\varrho \in \mathcal{P}$, let V_ϱ be the component of the fixed point set of the subgroup $\ker(\varrho)$ containing F^n, and set $n_\varrho = \dim(V_\varrho)$. Taking $T \in G - \ker(\varrho)$, one finds that (V_ϱ, T) is an involution fixing F^n. Since F^n has property $\mathcal{H}$, $n_\varrho = 0$ or $2n$. If $n_\varrho = 0$, then $(V_\varrho, T) = (F^n, \mathrm{Id})$ and ε_ϱ is the zero bundle $0 \to F^n$. If $n_\varrho = 2n$, then by the result of C. Kosniowski and R. E. Stong of [4] cited in the introduction, (V_ϱ, T) is cobordant to $(F^n \times F^n, t)$ and $\varepsilon_\varrho \to F^n$ is cobordant to the tangent bundle $\tau \to F^n$. ∎

If $p = 1$, Φ is the trivial G-action, that is, $\Phi = t_0^k$; in this case one has only zero bundles. If $p = 2^k$, then $m = 2^k n$ and Lemma 3.3 says in this case that $(M^m; \Phi)$ is cobordant to $((F^n)^{2^k}; t_k^k)$ (which is cobordant to $\sigma((F^n)^{2^k}; t_k^k)$ for any automorphism $\sigma : G \to G$); in this case, one has only bundles cobordant to the tangent bundle $\tau \to F^n$. Therefore we can assume $1 < p < 2^k$, which means that the two possible cobordism types of bundles occur. To ease the notation, write $\varepsilon_\varrho \equiv \tau$ when ε_ϱ is cobordant to τ.

LEMMA 3.5. *Let Ω be the subset of* $\mathrm{Hom}(G, Z_2)$ *given by*

$$\Omega = \{1\} \cup \{\varrho \in \mathcal{P} \mid \varepsilon_\varrho \equiv \tau\}.$$

Then Ω is a subgroup of $\mathrm{Hom}(G, Z_2)$. *In particular, the number of bundles ε_ϱ with $\varepsilon_\varrho \equiv \tau$ is $2^r - 1$ for some $1 \le r \le k-1$ (r is the dimension of Ω as Z_2-vector space); that is, $p = 2^r$ and $m = 2^r n$.*

Proof. Take $\varrho_1, \varrho_2 \in \Omega$. Then $\{1, \varrho_1, \varrho_2, \varrho_1\varrho_2\}$ is a subgroup of $\mathrm{Hom}(G, Z_2)$. By Lemma 3.1, there exist subgroups $H, K \subset G$ with H isomorphic to Z_2^2 and $G = H \oplus K$, so that the fixed data of the Z_2^2-action obtained by letting H act on the fixed set of K is $(F^n; \{\varepsilon_{\varrho_1}, \varepsilon_{\varrho_2}, \varepsilon_{\varrho_1\varrho_2}\})$. Since $\varepsilon_{\varrho_1} \equiv \tau$ and $\varepsilon_{\varrho_2} \equiv \tau$, Lemma 3.2 shows that $\varepsilon_{\varrho_1\varrho_2} \equiv \tau$, and Ω is a subgroup of $\mathrm{Hom}(G, Z_2)$. ∎

Now we can complete the argument. The desired special subgroup $H \subset G$ to which Lemma 3.3 will be applied is any subgroup of G which corresponds to $\Omega = \{1\} \cup \{\varrho \in \mathcal{P} \mid \varepsilon_\varrho \equiv \tau\}$ through Lemma 3.1. First note that Lemma 3.5 indicates a similarity between the fixed data of Φ and an action of type $\sigma((F^n)^{2^r}; t_r^k)$. We call attention, however, to the following subtle point: one has a list $\{\varepsilon_\varrho\}_{\varrho \in \Omega \cap \mathcal{P}}$ with each ε_ϱ being individually cobordant to τ, but this list might not be simultaneously cobordant to the list $\{\mu_\varrho\}_{\varrho \in \Omega \cap \mathcal{P}}$ with each μ_ϱ being equal to τ, and the desired result requires simultaneous cobordism (this obstacle does not appear when $F^n = \mathbb{R}P^{2s}$, since $W(\varepsilon_\varrho) = (1 + \alpha)^{2s+1}$ automatically yields the simultaneous cobordism in this case). In the final remark of the paper we present an example illustrating this situation.

Fortunately, the essence of this point is bypassed already by Lemma 3.3. In fact, by Lemma 3.1, there exist subgroups $H, K \subset G$, with H isomorphic to Z_2^r and $G = H \oplus K$, so that the fixed data of H acting on the fixed set F_K of K is $(F^n, \{\varepsilon_\varrho\}_{\varrho \in \Omega \cap \mathcal{P}})$. More precisely, and in terms of $\mathcal{P}' = \mathrm{Hom}(H, Z_2) - \{1\}$, this fixed data is $(F^n, \{\mu_{\varrho'}\}_{\varrho' \in \mathcal{P}'})$, where for each $\varrho' \in \mathcal{P}'$ one has $\mu_{\varrho'} = \varepsilon_\varrho$, with $\varrho_{|H} = \varrho'$ and $\varrho_{|K} =$ the trivial homomorphism. In particular, $\dim(F_K) = 2^r n = \dim(M^m)$, and thus F_K is the component of M^m containing F^n (note that each $T \in K$ acts on M^m trivially). Since $(M^m - F_K, \Phi)$ is a G-action without fixed points, the main result of [8] says that $(M^m - F_K, \Phi)$ bounds as a manifold with G-action. Thus we can suppose, without loss of generality, that $F_K = M^m$. Then one has an action $(M^m; \Phi_{|H \times M^m})$ with H isomorphic to Z_2^r and $m = 2^r n$; by Lemma 3.3, this action is equivariantly cobordant to $((F^n)^{2^r}; t_r^r)$. In particular, the list $\{\varepsilon_\varrho\}_{\varrho \in \Omega \cap \mathcal{P}}$ is simultaneously cobordant to the list $\{\mu_\varrho\}_{\varrho \in \Omega \cap \mathcal{P}}$ with each μ_ϱ being equal to τ.

Now choose a basis $(T'_1, \ldots, T'_r, T''_{r+1}, \ldots, T''_k)$ for G so that $(T'_1, \ldots, T'_r)$ is a basis for H and $(T''_{r+1}, \ldots, T''_k)$ is a basis for K. Consider the automorphism $\varphi : G \to G$ where $\varphi(T_i) = T'_i$ if $1 \le i \le r$ and $\varphi(T_i) = T''_i$ if $r < i \le k$, and the G-action $\varphi(M^m; \Phi)$. To describe the fixed data of this action, note that if $\varrho \in \mathcal{P}$ is the trivial homomorphism on K, then $\varrho \in \Omega$ and thus $\varepsilon_\varrho \equiv \tau$; otherwise, $\varrho \notin \Omega$, which means that ε_ϱ is the zero bundle. Since the list $\{\varepsilon_\varrho\}_{\varrho \in \Omega \cap \mathcal{P}}$ is simultaneously cobordant to the list $\{\mu_\varrho\}_{\varrho \in \Omega \cap \mathcal{P}}$, the fixed data of $\varphi(M^m; \Phi)$ is then simultaneously cobordant to the list $\{\varepsilon_\varrho\}_{\varrho \in \mathcal{P}}$ given by $\varepsilon_\varrho = \tau$ when ϱ is the trivial homomorphism on K and $\varepsilon_\varrho = 0$ otherwise. But then $\varphi(M^m; \Phi)$ is equivariantly cobordant to $((F^n)^{2^r}; t_r^k)$ and $(M^m; \Phi)$ is equivariantly cobordant to $\sigma((F^n)^{2^r}; t_r^k)$, where $\sigma = \varphi^{-1}$.

Remark. Let (M, T) be an involution. For each r with $1 \le r \le k$, one may form a Z_2^k-action $\Phi = (T_1, T_2, \ldots, T_k)$ on the product $M^{2^{r-1}}$ by letting $T_1 = T$ and letting $(T_2, \ldots, T_k)$ be the twist Z_2^{k-1}-action t_{r-1}^{k-1}. Denote this action by $\Gamma_r^k(M, T)$. Note that if $(F \times F, t)$ is the twist involution, then $\Gamma_r^k(F \times F, t) = (F^{2^r}; t_r^k)$. Also if (M, T) and (V, S) are Z_2-cobordant, then $\Gamma_r^k(M, T)$ and $\Gamma_r^k(V, S)$ are Z_2^k-cobordant (it suffices to look at the fixed data of $\Gamma_r^k(M, T)$: if $\eta \to F$ is the fixed data of (M, T), then the fixed point set of $\Gamma_r^k(M, T)$ is also F and the fixed data consists of 2^{r-1} copies of η, $2^{r-1} - 1$ copies of the tangent bundle over F and $2^k - 2^r$ zero bundles).

Now suppose that F^n has property $\mathcal{H}$, and let (M^{2n}, T) be any involution fixing F^n. Then (M^{2n}, T) is equivariantly cobordant to the twist involution $(F^n \times F^n, t)$, and thus $\sigma\Gamma_r^k(M^{2n}, T)$ is equivariantly cobordant to

$$\sigma\Gamma_r^k(F^n \times F^n, t) = \sigma((F^n)^{2^r}; t_r^k)$$

for every automorphism $\sigma : Z_2^k \to Z_2^k$, $k \ge 1$ and $1 \le r \le k$. In this way, every Z_2^k-action fixing F^n is equivariantly cobordant to $\sigma\Gamma_r^k(M^{2n}, T)$ for some automorphism $\sigma : G \to G$ and some $1 \le r \le k$. For example, every Z_2^k-action fixing $\mathbb{R}P^{2n}$ is equivariantly cobordant to $\sigma\Gamma_r^k(\mathbb{C}P^{2n}, c)$ for some automorphism $\sigma : G \to G$ and some $1 \le r \le k$, where c means complex conjugation on homogeneous coordinates.

Still in this context, let $F^n \subset \mathbb{R}^m$ be a smooth and closed n-dimensional manifold F^n which is a real algebraic variety, and where $\mathbb{R}^m$ is a suitable euclidean real m-dimensional space. Let $M^{2n} \subset \mathbb{C}^m$ be the corresponding complex algebraic variety, with real dimension $2n$. Then M^{2n} is invariant under the complex conjugation c on $\mathbb{C}^m$, and the fixed point set of the involution (M^{2n}, c) is F^n. Thus if F^n has property $\mathcal{H}$, then every Z_2^k-action fixing F^n is equivariantly cobordant to $\sigma\Gamma_r^k(M^{2n}, c)$ for some automorphism $\sigma : G \to G$ and some $1 \leq r \leq k$.

We thank the referee for having inspired this remark.

Remark. Consider $F^4 = \mathbb{C}P^2 \# (S^2 \times S^2) = \mathbb{C}P^2 \# (\mathbb{C}P^1 \times \mathbb{C}P^1)$. We have seen that F^4 has property $\mathcal{H}$. The algebra $H^*(F^4, Z_2)$ is generated by $\alpha, \beta, \gamma \in H^2(F^4, Z_2)$ with $\alpha^2 = \beta\gamma$ being the nonzero element of $H^4(F^4, Z_2)$ and $\alpha\beta = \alpha\gamma = \beta^2 = \gamma^2 = 0$. The Stiefel–Whitney class of F^4 is $W(F^4) = 1 + \alpha + \alpha^2$. Denote by $\xi_1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$ the pullback of the usual complex line bundle over $\mathbb{C}P^1$ under the first projection, and by $\mathbb{R}^2 \to \mathbb{C}P^1 \times \mathbb{C}P^1$ the trivial 2-dimensional bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1$. Then over F^4 one has a 4-dimensional bundle $\mu^4 \to F^4$ given by forming the connected sum of the tangent bundle $\tau(\mathbb{C}P^2) \to \mathbb{C}P^2$ and $\xi_1 \oplus R^2 \to \mathbb{C}P^1 \times \mathbb{C}P^1$. One has $W(\mu^4) = 1 + (\alpha + \beta) + \alpha^2$, and computing characteristic numbers it is easy to see that μ^4 is cobordant to the tangent bundle $\tau(F^4) \to F^4$.

Similarly one has a 4-dimensional bundle $\eta^4 \to F^4$ given by forming the connected sum of $\tau(\mathbb{C}P^2)$ and $R^2 \oplus \xi_2 \to \mathbb{C}P^1 \times \mathbb{C}P^1$, where ξ_2 is the pullback of the usual complex line bundle over $\mathbb{C}P^1$ under the second projection. One has $W(\eta^4) = 1 + (\alpha + \gamma) + \alpha^2$, and in the same way η^4 is cobordant to $\tau(F^4)$. However, there is no simultaneous cobordism of $(F^4; \mu^4, \eta^4)$ with $(F^4; \tau(F^4), \tau(F^4))$. In fact, $w_2(\mu^4)w_2(\eta^4) = (\alpha + \beta)(\alpha + \gamma) = \alpha^2 + \beta\gamma = 0$ and $w_2(\tau(F^4))w_2(\tau(F^4)) = \alpha\alpha = \alpha^2$. This obviously extends to $\mathbb{C}P^2 \# (\mathbb{C}P^1 \times \mathbb{C}P^1) \# \cdots \# (\mathbb{C}P^1 \times \mathbb{C}P^1)$ to get any number of bundles. It also works for $\mathbb{R}P^2 \# (\mathbb{R}P^1 \times \mathbb{R}P^1) \# \cdots \# (\mathbb{R}P^1 \times \mathbb{R}P^1)$, $\mathbb{H}P^2 \# (\mathbb{H}P^1 \times \mathbb{H}P^1) \# \cdots \# (\mathbb{H}P^1 \times \mathbb{H}P^1)$ and $\mathbb{Q}P^2 \# (\mathbb{Q}P^1 \times \mathbb{Q}P^1) \# \cdots \# (\mathbb{Q}P^1 \times \mathbb{Q}P^1)$.

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