

Two-to-one continuous images of \mathbb{N}^*

by

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Abstract. A function is two-to-one if every point in the image has exactly two inverse points. We show that every two-to-one continuous image of \mathbb{N}^* is homeomorphic to \mathbb{N}^* when the continuum hypothesis is assumed. We also prove that there is no irreducible two-to-one continuous function whose domain is \mathbb{N}^* under the same assumption.

1. Introduction. A function $f : X \rightarrow Y$ is *two-to-one* if for each $y \in Y$, there are exactly two points of X that map to y . All spaces considered are Tikhonov. For some spaces X , there does not exist a two-to-one continuous function $f : X \rightarrow Y$ for any choice of Y . For example, Harrold [Ha39] showed that there is no two-to-one continuous function $f : [0, 1] \rightarrow Y$ and Mioduszewski [Mi61] proved that there is no two-to-one continuous function $f : \mathbb{R} \rightarrow Y$. In fact, Heath [He86] later showed that every two-to-one function $f : \mathbb{R} \rightarrow Y$ has infinitely many discontinuities.

Another situation is when there are two-to-one continuous functions $f : X \rightarrow Y$ defined on a space X , but given any such function the image space Y is determined up to a homeomorphism. For example, if $f : [0, 1) \rightarrow S^1$, where S^1 is the unit circle $\{p \in \mathbb{R}^2 : \|p\| = 1\}$, is defined by $f(x) = \exp(4\pi ix)$, then f is a two-to-one continuous function. Mioduszewski [Mi61] proved that if $f : [0, 1) \rightarrow Y$ is a two-to-one continuous function, then Y is homeomorphic to S^1 .

In this paper we investigate the behavior of two-to-one continuous maps defined on \mathbb{N}^* , the remainder $\beta\mathbb{N} \setminus \mathbb{N}$ of the Stone-Ćech compactification of the space \mathbb{N} of natural numbers. We give partial answers to questions recently raised by R. Levy [Le04]. In particular, we show that every two-to-one continuous image of \mathbb{N}^* is homeomorphic to \mathbb{N}^* when the continuum hypothesis (CH) is assumed.

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There are two-to-one continuous functions defined on \mathbb{N}^* . For example, the space \mathbb{N}^* is homeomorphic to $\mathbb{N}^* \times 2$, so the projection map $f : \mathbb{N}^* \times 2 \rightarrow \mathbb{N}^*$ given by $f(x, k) = x$ is a two-to-one continuous function on \mathbb{N}^* and the image is \mathbb{N}^* . Such an example would be called trivial. A continuous two-to-one function $f : \mathbb{N}^* \rightarrow \mathbb{N}^*$ is *trivial* if there is a clopen subset C of \mathbb{N}^* such that the restrictions $f|_C$ and $f|_{\mathbb{N}^* \setminus C}$ are homeomorphisms. In [Do04], the first author proved that all maps on \mathbb{N}^* that are two-to-one are trivial, in the above sense, under the presence of the Proper Forcing Axiom (PFA). Eric van Douwen [vD93] has also produced a surprising answer to a similar question raised by R. Levy. He showed that the space \mathbb{N}^* , which is a compact space and very far from being separable, can be mapped onto a compact separable space by a \leq two-to-one continuous function. We are concerned with the question of whether every exactly two-to-one continuous image of \mathbb{N}^* is homeomorphic to \mathbb{N}^* .

It is well known that if a space Y is homeomorphic to \mathbb{N}^* , then Y is a *Parovichenko space*, that is, a compact zero-dimensional F -space of weight \mathfrak{c} which has no isolated points and with the property that every nonempty G_δ -set has infinite interior. Therefore, if we are interested in whether or not two-to-one continuous images of \mathbb{N}^* are homeomorphic to \mathbb{N}^* , we should investigate which of these six properties are satisfied by the spaces which are two-to-one continuous images of \mathbb{N}^* .

If Y is a two-to-one continuous image of \mathbb{N}^* , obviously Y is compact since \mathbb{N}^* is compact, and Y has no isolated points since two-to-one continuous maps preserve the property of having no isolated points. Ronnie Levy [Le04] showed that Y has the property that countable discrete subsets are C^* -embedded and Y contains a copy of \mathbb{N}^* and so Y has weight \mathfrak{c} . We include his proof for the reader's convenience.

THEOREM 1 (Levy). *Let X be a space such that every countable discrete subset of X is C^* -embedded in X . If $f : X \rightarrow Y$ is a two-to-one continuous function, then every countable discrete subset of Y is C^* -embedded in Y .*

Proof. Let C be a countable discrete subset of Y . Since f is two-to-one and C is countable discrete, $f^{-1}(C)$ is a countable discrete subset of X . Therefore, $f^{-1}(C)$ is C^* -embedded in X . We must show that disjoint subsets of C have disjoint closures in Y . Let $A, B \subseteq C$ be such that $A \cap B = \emptyset$. Assume that there exists $p \in \text{Cl}_Y A \cap \text{Cl}_Y B$.

For each $a \in A$ let $f^{-1}(\{a\}) = \{a_1, a_2\}$, and for each $b \in B$, let $f^{-1}(\{b\}) = \{b_1, b_2\}$. Let $A_1 = \{a_1 : a \in A\}$, $A_2 = \{a_2 : a \in A\}$, $B_1 = \{b_1 : b \in B\}$, $B_2 = \{b_2 : b \in B\}$. These four sets are pairwise disjoint subsets of $f^{-1}(C)$ and therefore their closures are also pairwise disjoint since $f^{-1}(C)$ is C^* -embedded. By the continuity of f , each of the four sets $\text{Cl}_X A_1$, $\text{Cl}_X A_2$, $\text{Cl}_X B_1$, $\text{Cl}_X B_2$ contains an element of the fiber $f^{-1}(\{p\})$. Since these sets

are pairwise disjoint, $|f^{-1}(\{p\})| \geq 4$. This contradicts the fact that f is two-to-one. ■

Then Levy asked in the same paper whether Y has the remaining three properties. We show that Y actually has these three properties under CH, that is, Y is a zero-dimensional F -space in which every nonempty G_δ -set has infinite interior. A consequence then is that two-to-one continuous images of \mathbb{N}^* are homeomorphic to \mathbb{N}^* when CH is assumed since \mathbb{N}^* is the only Parovichenko space under CH [vM84].

2. Irreducible maps. A mapping f of X onto Y is *irreducible* if no proper closed subset of X maps onto Y . Thus, the image of an open set by a closed irreducible mapping will have interior. It follows easily from Zorn's lemma [Wa74] that if X and Y are compact spaces and f is a continuous function from X onto Y , then there is a closed subspace F of X such that $f|_F$ is an irreducible map from F onto Y . Levy [Le04] asked if there exists an irreducible two-to-one continuous function whose domain is \mathbb{N}^* . Under CH, we will show that there is no such function.

NOTATION. For a map $f : X \rightarrow Y$ and $A \subseteq X$, let $J_A = f^{-1}(f[X \setminus A]) \cap A$, and for a point x , let x' denote a point $\neq x$ such that $f(x) = f(x')$.

LEMMA 2. *Let $f : X \rightarrow Y$ be an irreducible continuous closed map. If A is an open subset of X , then J_A is nowhere dense in X .*

Proof. Suppose that J_A is not nowhere dense in X . Then $\text{Int } \bar{J}_A$ is a nonempty open subset of X . Then clearly $f[A \cap \text{Int } \bar{J}_A] \subseteq f[A]$ and $\bar{J}_A \subseteq f^{-1}(f[X \setminus A])$ since $J_A \subseteq f^{-1}(f[X \setminus A])$ and $f[X \setminus A]$ is closed. Thus $f[\bar{J}_A] \subseteq f[X \setminus A]$ and in particular $f[A \cap \text{Int } \bar{J}_A] \subseteq f[X \setminus A]$. Therefore, $X \setminus (A \cap \text{Int } \bar{J}_A)$ is a proper closed subset of X since $A \cap \text{Int } \bar{J}_A$ is a nonempty open set contained in A , and $f[X \setminus (A \cap \text{Int } \bar{J}_A)] = Y$ since $f[A \cap \text{Int } \bar{J}_A] \subseteq f[A]$ and $f[A \cap \text{Int } \bar{J}_A] \subseteq f[X \setminus A]$. This is a contradiction since f is irreducible. ■

The following result is the main ingredient in our analysis of the structure of two-to-one continuous functions.

THEOREM 3 (CH). *Let X be a compact space of weight \mathfrak{c} and suppose that countable discrete subsets of X are C^* -embedded. If $f : X \rightarrow K$ is a two-to-one continuous function and Z is a closed subset of X such that $f|_Z$ is irreducible and maps Z onto K , then for every nonempty open set $W \subseteq K$ there exists an open set B in X such that $\bar{B} \cap Z = \emptyset$ and $\text{Int}(f[B]) \cap W \neq \emptyset$. Furthermore, if X is zero-dimensional, then B can be chosen to be clopen.*

Proof. Let W be a nonempty open subset of K . Seeking a contradiction, suppose that $f[B] \cap W$ is nowhere dense in K for each open set $B \subseteq X$ with closure disjoint from Z . For each open subset $B \subseteq X \setminus Z$,

let $I_B = f^{-1}(\overline{f[B] \cap W}) \cap Z$. Then I_B is nowhere dense in Z since f is closed irreducible and $f[I_B] \subseteq \overline{f[B] \cap W}$, which is nowhere dense.

Since f is continuous, $f^{-1}(W) \cap Z$ has nonempty interior in Z . For each open subset $A \subseteq f^{-1}(W) \cap Z$, J_A is nowhere dense in Z by Lemma 2.

We construct, by induction, a family $\{A_\alpha : \alpha < \omega_1\}$ which is a filter base of cozero subsets of $f^{-1}(W) \cap Z$ such that $\bigcap_{\alpha < \omega_1} A_\alpha$ is a singleton $\{x\}$ so that $f^{-1}(f(x)) = \{x\}$. This yields a contradiction as f is an exactly two-to-one function.

Let $\{B_\alpha : \alpha < \omega_1\}$ enumerate all the cozero sets B in X such that $\overline{B} \cap Z = \emptyset$, and let $\{C_\alpha^0, C_\alpha^1 : \alpha < \omega_1\}$ enumerate all pairs of cozero sets in Z such that $Z = C_\alpha^0 \cup C_\alpha^1$. We construct $\{A_\alpha : \alpha < \omega_1\}$ such that for each $\alpha < \omega_1$:

- (1) $\bigcap_{\beta \leq \alpha} A_\beta$ is nonempty;
- (2) $\overline{A}_\alpha \subseteq C_\alpha^0$ or $\overline{A}_\alpha \subseteq C_\alpha^1$;
- (3) if $\alpha = \beta + 1$, then $\overline{A}_\alpha \subseteq A_\beta \setminus (J_{A_\beta} \cup I_{B_\beta})$.

We show how to define the first two cozero sets A_0 and A_1 : Let $x \in f^{-1}(W) \cap Z$. Then choose a cozero set neighborhood A_0 of x such that $\overline{A}_0 \subseteq f^{-1}(W) \cap Z$ and either $\overline{A}_0 \subseteq C_0^0$ or $\overline{A}_0 \subseteq C_0^1$.

Then J_{A_0} and I_{B_0} are nowhere dense in Z , hence $A_0 \setminus (J_{A_0} \cup I_{B_0}) \neq \emptyset$. Let x , possibly different from the previous x , be a point in $A_0 \setminus (J_{A_0} \cup I_{B_0})$. Then choose a cozero set neighborhood A_1 of x with $\overline{A}_1 \subseteq A_0 \setminus (J_{A_0} \cup I_{B_0})$ and either $\overline{A}_1 \subseteq C_1^0$ or $\overline{A}_1 \subseteq C_1^1$. For each $n \in \omega$ we can define A_n in the same manner.

Suppose that $\alpha \geq \omega$ and we have constructed the family $\{A_\beta : \beta < \gamma\}$ for all $\gamma < \alpha$. If $\alpha = \beta + 1$, then $\bigcap_{\gamma \leq \beta} A_\gamma \neq \emptyset$ by the induction assumption. If α is a limit, then the induction hypothesis (1) ensures that $G_\alpha = \bigcap_{\beta < \alpha} A_\beta$ is not empty since Z is compact and $A_\beta \subseteq \overline{A}_{\beta+1}$ for each $\beta < \alpha$ so that $\bigcap_{\beta < \alpha} A_\beta \supseteq \bigcap_{\beta < \alpha} \overline{A}_{\beta+1}$.

If α is a limit and $x \in G_\alpha = \bigcap_{\beta < \alpha} A_\beta$, we choose a cozero set neighborhood A_α of x such that $\overline{A}_\alpha \subseteq C_\alpha^0$ if $x \in C_\alpha^0$ or $\overline{A}_\alpha \subseteq C_\alpha^1$ otherwise. If $\alpha = \beta + 1$ with $\beta \geq \omega$, we must define a cozero set A_α so that $\overline{A}_\alpha \subseteq A_\beta \setminus [J_{A_\beta} \cup I_{B_\beta}]$. Let λ be the largest limit ordinal less than α . We enumerate $\lambda \cup \{\beta\}$ as $\beta = \beta_0, \beta_1, \beta_2, \beta_3, \dots$. We now consider the cases when G_α has nonempty interior and G_α is nowhere dense.

If G_α has nonempty interior, then $G_\alpha \setminus [J_{A_\beta} \cup I_{B_\beta}] \neq \emptyset$ since $J_{A_\beta} \cup I_{B_\beta}$ is nowhere dense. Then pick a point $x \in G_\alpha \setminus [J_{A_\beta} \cup I_{B_\beta}]$ and a cozero set A_α containing x such that $\overline{A}_\alpha \subseteq G_\alpha$ and either $\overline{A}_\alpha \subseteq C_\alpha^0$ or $\overline{A}_\alpha \subseteq C_\alpha^1$.

If G_α is nowhere dense, $J_{A_\beta} \cup I_{B_\beta} \cup G_\alpha$ is nowhere dense and $A_\beta \setminus [J_{A_\beta} \cup I_{B_\beta} \cup G_\alpha] \neq \emptyset$. Let $x_0 \in A_\beta \setminus [J_{A_\beta} \cup I_{B_\beta} \cup G_\alpha]$. Then choose $i_1 > 1$ large enough such that $x_0 \notin A_{\beta_{i_1}}$, which we may do since $x_0 \notin G_\alpha$. Then either $x'_0 \in A_{\beta_{i_1}}$, $x'_0 \in J_{A_{\beta_{i_1}}}$, or $x'_0 \notin Z$. In any case, $x'_0 \notin \overline{A}_{\beta_{i_1}+1}$ since $\overline{A}_{\beta_{i_1}+1} \subseteq A_{\beta_{i_1}} \setminus J_{A_{\beta_{i_1}}}$.

by construction. So there is an $i'_1 > 1$ such that $\beta_{i'_1} = \beta_{i_1} + 1$ and $x'_0 \notin \bar{A}_{\beta_{i'_1}}$. Thus $x_0, x'_0 \notin G_\alpha$.

Similarly, pick $x_1 \in [A_\beta \cap A_{\beta_1} \cap \cdots \cap A_{\beta_{i_1}}] \setminus [J_{A_\beta} \cup I_{B_\beta} \cup G_\alpha]$ and choose $i_2 > i'_1, i_1 > 1$ large enough so that $x_1 \notin A_{\beta_{i_2}}$. Then $x'_1 \notin \bar{A}_{\beta_{i_2+1}}$ by construction. So there is an $i'_2 > i'_1, i_1 > 1$ such that $\beta_{i'_2} = \beta_{i_2} + 1$ and $x'_1 \notin \bar{A}_{\beta_{i'_2}}$. Thus $x_1, x'_1 \notin G_\alpha$. Continuing this process, for every integer n , pick $x_n \in [A_\beta \cap A_{\beta_1} \cap \cdots \cap A_{\beta_{i_n}}] \setminus [J_{A_\beta} \cup I_{B_\beta} \cup G_\alpha]$ and choose an integer $i_{n+1} > i_n, i'_n > \cdots > i_2, i'_2 > i'_1, i_1 > 1$ large enough so that $x_n \notin A_{\beta_{i_{n+1}}}$. Then there is an integer $i'_{n+1} > i'_n, i_n$ such that $x'_n \notin \bar{A}_{\beta_{i'_{n+1}}}$. Thus $x_n, x'_n \notin G_\alpha$.

Hence, we get a countable set $\{x_n\}_n \cup \{x'_n\}_n$. We remark that the x_n 's and x'_n 's can be chosen from some dense subset of $f^{-1}(W) \cap Z$. From the construction of the x_n 's and x'_n 's:

$$\begin{aligned} \forall i > n \quad x_i, x'_i &\in f^{-1}(f[\bar{A}_\beta \cap \bar{A}_{\beta_1} \cap \cdots \cap \bar{A}_{\beta_{i_n}}]), \\ \forall j \leq n \quad x_j, x'_j &\notin f^{-1}(f[\bar{A}_\beta \cap \bar{A}_{\beta_1} \cap \cdots \cap \bar{A}_{\beta_{i_{n+2}}}]), \end{aligned}$$

and since $f^{-1}(f[\bar{A}_\beta \cap \bar{A}_{\beta_1} \cap \cdots \cap \bar{A}_{\beta_k}])$ is closed for all k , the set $\{x_n\}_n \cup \{x'_n\}_n$ is discrete. Therefore $\overline{\{x_n\}_n} \cap \overline{\{x'_n\}_n} = \emptyset$ since $\{x_n\}_n \cap \{x'_n\}_n = \emptyset$ and countable discrete subsets of X are C^* -embedded. We have $f(\overline{\{x_n\}_n}) = f(\overline{\{x'_n\}_n})$ by continuity of f and the fact that $f(\{x_n\}_n) = f(\{x'_n\}_n)$. We also have $\overline{\{x_n\}_n} \setminus \{x_n\}_n \neq \emptyset$ and $\overline{\{x'_n\}_n} \setminus \{x'_n\}_n \neq \emptyset$ since every infinite subset of a compact set has a limit point. By the construction of the x_n 's we see that $\overline{\{x_n\}_n} \setminus \{x_n\}_n \subseteq G_\alpha$.

If $\{z \in Z : z' \in Z\} \cap f^{-1}(W)$ is dense in $f^{-1}(W) \cap Z$, we can choose the x'_n 's in $f^{-1}(W) \cap Z$ so that we also have $\overline{\{x'_n\}_n} \setminus \{x'_n\}_n \subseteq G_\alpha$. In this case if we choose $x \in \overline{\{x_n\}_n} \setminus \{x_n\}_n$, then $x' \in \overline{\{x'_n\}_n} \setminus \{x'_n\}_n$ and $x \neq x'$ since $\overline{\{x_n\}_n} \cap \overline{\{x'_n\}_n} = \emptyset$. Moreover $x \notin J_{A_\beta}$ since $x \in J_{A_\beta}$ implies $x' \notin A_\beta$, which contradicts $x' \in G_\alpha \subseteq A_\beta$. We also have $x \notin I_{B_\beta}$, that is, $x' \notin B_\beta$ since $x'_n \notin B_\beta$ for all n and B_β is a cozero set. Thus we have found an $x \in G_\alpha$ such that $x \notin J_{A_\beta} \cup I_{B_\beta}$.

If $\{z \in Z : z' \in Z\} \cap f^{-1}(W)$ is nowhere dense in $f^{-1}(W) \cap Z$, find a cozero set $A \subseteq A_\beta \subseteq f^{-1}(W) \cap Z$ such that $f|_Z$ is one-to-one on all points of \bar{A} , that is, $f^{-1}(f[\bar{A}])$ meets Z in \bar{A} . In this case we can choose the x_n 's so that $\overline{\{x_n\}_n} \subseteq A$ and hence $\overline{\{x'_n\}_n} \subseteq X \setminus Z$. Then $\overline{\{x_n\}_n} \setminus \{x_n\}_n \subseteq \bar{A}$. Choose an $x \in \overline{\{x_n\}_n} \setminus \{x_n\}_n$. Then $x \in \bar{A}$ and so $x' \notin Z$; in particular $x' \notin Z \setminus A_\beta$ and so $x \notin J_{A_\beta}$. It is also true that $x \notin I_{B_\beta}$, that is, $x' \notin B_\beta$ since $x'_n \notin B_\beta$ for all n and B_β is a cozero set. Therefore $x \in G_\alpha$ and $x \notin J_{A_\beta} \cup I_{B_\beta}$.

We choose a cozero set A_α containing x with $\bar{A}_\alpha \subseteq A_\beta \setminus [J_{A_\beta} \cup I_{B_\beta}]$ and either $\bar{A}_\alpha \subseteq C_\alpha^0$ or $\bar{A}_\alpha \subseteq C_\alpha^1$. Then A_α satisfies all the induction assumptions and this completes the inductive construction.

But now $\bigcap_{\alpha < \omega_1} A_\alpha \neq \emptyset$ since $\bigcap_{\beta < \omega_1} A_\beta \supseteq \bigcap_{\beta < \omega_1} \bar{A}_{\beta+1}$ and X is compact. Moreover the fact that $\bigcap_{\alpha < \omega_1} A_\alpha$ is a singleton is easily seen by the induction hypothesis (2). Let $\bigcap_{\alpha < \omega_1} A_\alpha = \{x\}$.

CLAIM 1. $f^{-1}(f(x)) = \{x\}$.

Proof of Claim. Suppose that for some $x' \neq x$, we have $f(x) = f(x')$. If $x' \in Z$, then $x' \notin A_\alpha$ for some $\alpha < \omega_1$. This implies $x \in J_{A_\alpha}$ and so $x \notin A_{\alpha+1}$. This is a contradiction. If $x' \notin Z$, then $x' \in B_\alpha$ for some $\alpha < \omega_1$. This implies $x \in I_{B_\alpha}$ and so $x \notin A_{\alpha+1}$. This is also a contradiction. ■

This contradicts f being an exactly two-to-one function. Moreover the zero-dimensional case follows immediately from the general case. If X is zero-dimensional, and B is an open set such that $\bar{B} \cap Z = \emptyset$, then there is a clopen set disjoint from Z containing B . ■

COROLLARY 4 (CH). *Let X^* be the Stone-Ćech remainder of a locally compact separable metric space X . If $f : X^* \rightarrow K$ is a two-to-one continuous function, then f is not irreducible. In particular, if $f : \mathbb{N}^* \rightarrow K$ or $f : \mathbb{R}^* \rightarrow K$ is two-to-one and continuous, then f is not irreducible.*

Proof. Suppose that f is irreducible. Taking $X = Z = X^*$ and $W = K$ in Theorem 3 we get a nonempty subset B of the empty set $X^* \setminus Z$. ■

COROLLARY 5 (CH). *If $f : \mathbb{N}^* \rightarrow K$ is a two-to-one continuous function, then K is not ccc.*

Proof. Let W be a nonempty open subset of K . By Zorn's lemma [Wa74], there is a closed subset Z of \mathbb{N}^* such that $f|_Z : Z \rightarrow K$ is irreducible. Then, by Theorem 3, there exists a nonempty clopen set $B \subseteq \mathbb{N}^* \setminus Z$ such that $\text{Int}(f[B]) \cap W \neq \emptyset$ since \mathbb{N}^* is zero-dimensional. Then $f|_B$ is a closed one-to-one function and so B is homeomorphic to $f[B]$. Thus, $f[B]$ has no open ccc subset since \mathbb{N}^* has no open ccc subset. Therefore, K is not ccc. ■

3. Examples of nontrivial two-to-one maps. A two-to-one function $f : X \rightarrow Y$ will be called *trivial* if there exist disjoint clopen sets A and B such that $X = A \cup B$ and $f[A] = f[B] = Y$. In [Do04] the first author proved that all functions defined on \mathbb{N}^* that are two-to-one continuous are trivial under PFA. In this section we will give some nontrivial examples of two-to-one continuous functions defined on \mathbb{N}^* when CH is assumed.

A point is called a *P-point* if the family of its neighborhoods is closed under countable intersections. A subset of a space is a *P-set* if the family of its neighborhoods is closed under countable intersections. CH implies that \mathbb{N}^* has *P-points* and contains a nowhere dense closed *P-set* which is homeomorphic to \mathbb{N}^* [vM84].

EXAMPLE 1 (CH). We give an example of a nontrivial two-to-one continuous function $f : \mathbb{N}^* \rightarrow \mathbb{N}^*$ such that f is locally one-to-one at every point of \mathbb{N}^* except for two P -points.

Consider two copies of \mathbb{N}^* : $\mathbb{N}_1^*, \mathbb{N}_2^*$. Let $p_1 \in \mathbb{N}_1^*$ and $p_2 \in \mathbb{N}_2^*$ be P -points. There is a homeomorphism $g : \mathbb{N}_1^* \rightarrow \mathbb{N}_2^*$ such that $g(p_1) = p_2$ under CH [vM84]. Then $g^{-1} : \mathbb{N}_2^* \rightarrow \mathbb{N}_1^*$ is also a homeomorphism and $g^{-1}(p_2) = p_1$. The free union of the two copies of \mathbb{N}^* , $\mathbb{N}_1^* \cup \mathbb{N}_2^*$, is homeomorphic to \mathbb{N}^* . Let $h_1 : \mathbb{N}_1^* \cup \mathbb{N}_2^* \rightarrow \mathbb{N}_1^* \cup \mathbb{N}_2^*$ be defined by $h_1 = g \cup g^{-1}$. Then $h_1^2 = \text{id}$. In a similar manner define $h_2 : \mathbb{N}_3^* \cup \mathbb{N}_4^* \rightarrow \mathbb{N}_3^* \cup \mathbb{N}_4^*$ so that $h_2 = \tilde{g} \cup \tilde{g}^{-1}$ and $h_2^2 = \text{id}$, where \mathbb{N}_3^* and \mathbb{N}_4^* are other copies of \mathbb{N}^* with corresponding P -points p_3 and p_4 and $\tilde{g} : \mathbb{N}_3^* \rightarrow \mathbb{N}_4^*$ is a homeomorphism with $\tilde{g}(p_3) = p_4$. The quotient spaces

$$(\mathbb{N}_1^* \cup \mathbb{N}_2^*)/p_1 \equiv p_2, \quad (\mathbb{N}_3^* \cup \mathbb{N}_4^*)/p_3 \equiv p_4, \quad (\mathbb{N}_1^* \cup \mathbb{N}_4^*)/p_1 \equiv p_4,$$

identifying p_1 and p_2 , p_3 and p_4 , p_1 and p_4 , as single P -points in their respective spaces, are homeomorphic to \mathbb{N}^* [vM84]. The free union of the first two spaces, $(\mathbb{N}_1^* \cup \mathbb{N}_2^*)/p_1 \equiv p_2 \oplus (\mathbb{N}_3^* \cup \mathbb{N}_4^*)/p_3 \equiv p_4$, is also homeomorphic to \mathbb{N}^* . Now define

$$f : [(\mathbb{N}_1^* \cup \mathbb{N}_2^*)/p_1 \equiv p_2 \oplus (\mathbb{N}_3^* \cup \mathbb{N}_4^*)/p_3 \equiv p_4] \rightarrow (\mathbb{N}_1^* \cup \mathbb{N}_4^*)/p_1 \equiv p_4,$$

$$f(x) = \begin{cases} h_1(x) & \text{if } x \in \mathbb{N}_2^* \setminus \{p_2\}, \\ h_2(x) & \text{if } x \in \mathbb{N}_3^* \setminus \{p_3\}, \\ x & \text{if } x \in (\mathbb{N}_1^* \setminus \{p_1\}) \cup (\mathbb{N}_4^* \setminus \{p_4\}), \\ p_1 \equiv p_4 & \text{if } x \in \{p_1 \equiv p_2, p_3 \equiv p_4\}, \end{cases}$$

Then f is a continuous and exactly two-to-one function. Moreover, the image of f is homeomorphic to \mathbb{N}^* .

We now introduce some notation that will be used in our future discussions about this kind of two-to-one continuous maps. Let

$$X_0 = (\mathbb{N}_1^* \cup \mathbb{N}_2^*)/p_1 \equiv p_2 \oplus (\mathbb{N}_3^* \cup \mathbb{N}_4^*)/p_3 \equiv p_4,$$

$$I_0 = \{A \subseteq X_0 : A = A_0 \dot{\cup} A'_0, A_0, A'_0 \text{ clopen}, f[A_0] = f[A'_0]\}.$$

So I_0 is a family of clopen sets A in X_0 such that $A = f^{-1}(f[A])$, i.e., saturated, and f is locally one-to-one on A . Let \mathcal{U}_0 denote the union of all the A 's in I_0 ,

$$\mathcal{U}_0 = \bigcup_{A \in I_0} A,$$

and in this example $\mathcal{U}_0 = X_0 \setminus \{p_1 \equiv p_2, p_3 \equiv p_4\}$. Thus f is locally one-to-one except at the two P -points $p_1 \equiv p_2$ and $p_3 \equiv p_4$. Let $X_1 = X_0 \setminus \mathcal{U}_0$, which is again for this example given by

$$X_1 = \{p_1 \equiv p_2, p_3 \equiv p_4\}.$$

Then I_1 is the analogous set in X_1 but the points in X_1 are not in \mathcal{U}_0 because as can be seen above f is not locally one-to-one at the points $p_1 \equiv p_2$ and $p_3 \equiv p_4$.

By using a similar construction to Veličković’s poset [Ve93], Example 1 can be done consistent with $\text{MA} + \neg\text{CH}$. But $\text{MA} + \neg\text{CH}$ is not by itself enough to do the construction because of the first author’s PFA result [Do04].

EXAMPLE 2 (CH). We give an example of a nontrivial two-to-one continuous function $f : \mathbb{N}^* \rightarrow \mathbb{N}^*$ such that f is locally one-to-one at every point of \mathbb{N}^* except for two P -sets. We extend the first example by considering nowhere dense closed P -sets instead of P -points.

Consider two copies of \mathbb{N}^* : $\mathbb{N}_1^*, \mathbb{N}_2^*$. Let $P_1 \subseteq \mathbb{N}_1^*$ and $P_2 \subseteq \mathbb{N}_2^*$ be two different closed P -sets such that P_1 is homeomorphic to P_2 . There is a homeomorphism $g_{12} : \mathbb{N}_1^* \rightarrow \mathbb{N}_2^*$ such that $g_{12}(P_1) = P_2$ under CH [vM84]. Therefore $g_{12}^{-1} : \mathbb{N}_2^* \rightarrow \mathbb{N}_1^*$ is also a homeomorphism and $g_{12}^{-1}(P_2) = P_1$. The free union of the two copies of \mathbb{N}^* , $\mathbb{N}_1^* \cup \mathbb{N}_2^*$, is homeomorphic to \mathbb{N}^* . Let

$$h_1 : \mathbb{N}_1^* \cup \mathbb{N}_2^* \rightarrow \mathbb{N}_1^* \cup \mathbb{N}_2^* \quad \text{be defined by} \quad h_1 = g_{12} \cup g_{12}^{-1}.$$

Then $h_1(x) \neq x$ for each x and $h_1^2 = \text{id}$. In a similar manner define $h_2 : \mathbb{N}_3^* \cup \mathbb{N}_4^* \rightarrow \mathbb{N}_3^* \cup \mathbb{N}_4^*$ so that $h_2 = g_{34} \cup g_{34}^{-1}$ and $h_2^2 = \text{id}$, where \mathbb{N}_3^* and \mathbb{N}_4^* are other copies of \mathbb{N}^* with corresponding homeomorphic P -sets P_3 and P_4 . The adjunction spaces

$$\mathbb{N}_1^* \cup_{g_1} \mathbb{N}_2^*, \quad \mathbb{N}_3^* \cup_{g_2} \mathbb{N}_4^*, \quad \mathbb{N}_1^* \cup_{g_3} \mathbb{N}_4^*,$$

where we identify the P -sets P_1 with P_2 , P_3 with P_4 , and P_1 with P_4 , are homeomorphic to \mathbb{N}^* [vM84]. The free union

$$(\mathbb{N}_1^* \cup_{g_1} \mathbb{N}_2^*) \oplus (\mathbb{N}_3^* \cup_{g_2} \mathbb{N}_4^*)$$

is also homeomorphic to \mathbb{N}^* .

Let $\varphi : P_2 \rightarrow P_4$ be a homeomorphism. Now let us define

$$f : [(\mathbb{N}_1^* \cup_{g_1} \mathbb{N}_2^*) \oplus (\mathbb{N}_3^* \cup_{g_2} \mathbb{N}_4^*)] \rightarrow \mathbb{N}_1^* \cup_{g_3} \mathbb{N}_4^*$$

by

$$f(x) = \begin{cases} h_1(x) & \text{if } x \in \mathbb{N}_2^* \setminus P_2, \\ h_2(x) & \text{if } x \in \mathbb{N}_3^* \setminus P_3, \\ x & \text{if } x \in (\mathbb{N}_1^* \setminus P_1) \cup \mathbb{N}_4^*, \\ \varphi(x) & \text{if } x \in P_2. \end{cases}$$

Then f is a continuous and exactly two-to-one function and the image is homeomorphic to \mathbb{N}^* . Let us find the sets $I_0, \mathcal{U}_0, X_1, I_1, \mathcal{U}_1$, and X_2 for this

example, which are introduced in Example 1:

$$\begin{aligned} X_0 &= (\mathbb{N}_1^* \cup_{g_1} \mathbb{N}_2^*) \oplus (\mathbb{N}_3^* \cup_{g_2} \mathbb{N}_4^*), \\ I_0 &= \{A \subseteq X_0 : A = A_0 \dot{\cup} A'_0, A_0, A'_0 \text{ clopen}, f[A_0] = f[A'_0]\}, \\ \mathcal{U}_0 &= \bigcup_{A \in I_0} A = X_0 \setminus (P_2 \cup P_4), \\ X_1 &= X_0 \setminus \mathcal{U}_0 = P_2 \cup P_4. \end{aligned}$$

The function f is not locally one-to-one in the nowhere dense closed sets P_2 and P_4 . But $f|_{X_1}$ is a continuous two-to-one function. I_1 is the analogous set in X_1 and

$$\mathcal{U}_1 = \bigcup_{A \in I_1} A = X_1, \quad X_2 = X_1 \setminus \mathcal{U}_1 = \emptyset.$$

EXAMPLE 3 (CH). We give an example of a nontrivial two-to-one continuous function $f : \mathbb{N}^* \rightarrow \mathbb{N}^*$ which is locally one-to-one at every point of \mathbb{N}^* except for two P -sets and with the property that $X_2 \neq \emptyset$ and $X_3 = \emptyset$. We know that \mathbb{N}^* can be embedded as a nowhere dense P -set in \mathbb{N}^* assuming CH [vM84]. Consider two copies of \mathbb{N}^* : $\mathbb{N}_5^*, \mathbb{N}_6^*$. Embed $\mathbb{N}_1^* \cup_{g_1} \mathbb{N}_2^*$ and $\mathbb{N}_3^* \cup_{g_2} \mathbb{N}_4^*$ of Example 2 as nowhere dense P -sets P_5 and P_6 in \mathbb{N}_5^* and \mathbb{N}_6^* , respectively:

$$\mathbb{N}_1^* \cup_{g_1} \mathbb{N}_2^* \hookrightarrow \mathbb{N}_5^* \quad \text{and} \quad \mathbb{N}_3^* \cup_{g_2} \mathbb{N}_4^* \hookrightarrow \mathbb{N}_6^*$$

In a similar fashion to Example 1, let $g : \mathbb{N}_5^* \rightarrow \mathbb{N}_6^*$ be a homeomorphism such that $g(P_5) = P_6$. Therefore $g^{-1} : \mathbb{N}_6^* \rightarrow \mathbb{N}_5^*$ is also a homeomorphism and $g^{-1}(P_6) = P_5$. Let

$$h_1 : \mathbb{N}_5^* \cup \mathbb{N}_6^* \rightarrow \mathbb{N}_5^* \cup \mathbb{N}_6^* \quad \text{be defined by} \quad h_1 = g \circ g^{-1}$$

Then $h_1^2 = \text{id}$. Let $\mathbb{N}_7^* \cup \mathbb{N}_8^*$ be another copy of $\mathbb{N}_5^* \cup \mathbb{N}_6^*$. Suppose that $h_2 : \mathbb{N}_7^* \cup \mathbb{N}_8^* \rightarrow \mathbb{N}_7^* \cup \mathbb{N}_8^*$ is defined similarly so that $h_2^2 = \text{id}$.

The adjunction spaces

$$\mathbb{N}_5^* \cup_{g_5} \mathbb{N}_6^*, \quad \mathbb{N}_7^* \cup_{g_6} \mathbb{N}_8^*, \quad \mathbb{N}_5^* \cup_{g_7} \mathbb{N}_8^*,$$

where we identify the P -sets P_5 with P_6 , P_7 with P_8 , and P_5 with P_8 , are homeomorphic to \mathbb{N}^* [vM84]. The free union $(\mathbb{N}_5^* \cup_{g_5} \mathbb{N}_6^*) \oplus (\mathbb{N}_7^* \cup_{g_6} \mathbb{N}_8^*)$ is also homeomorphic to \mathbb{N}^* . Let $\varphi : P_6 \rightarrow P_8$ be the two-to-one function defined in Example 2. Define

$$f : [(\mathbb{N}_5^* \cup_{g_5} \mathbb{N}_6^*) \oplus (\mathbb{N}_7^* \cup_{g_6} \mathbb{N}_8^*)] \rightarrow \mathbb{N}_5^* \cup_{g_7} \mathbb{N}_8^*$$

by

$$f(x) = \begin{cases} h_1(x) & \text{if } x \in \mathbb{N}_6^* \setminus P_6, \\ h_2(x) & \text{if } x \in \mathbb{N}_8^* \setminus P_8, \\ x & \text{if } x \in (\mathbb{N}_5^* \setminus P_5) \cup (\mathbb{N}_7^* \setminus P_7), \\ \varphi(x) & \text{if } x \in P_6 \cup P_8. \end{cases}$$

Let us find the sets $I_0, \mathcal{U}_0, X_1, I_1, \mathcal{U}_1, X_2, I_2, \mathcal{U}_2, X_3$ in this example:

$$\begin{aligned} X_0 &= (\mathbb{N}_5^* \cup_{g_5} \mathbb{N}_6^*) \oplus (\mathbb{N}_7^* \cup_{g_6} \mathbb{N}_8^*), \\ I_0 &= \{A \subseteq X_0 : A = A_0 \dot{\cup} A'_0, A_0, A'_0 \text{ clopen}, f[A_0] = f[A'_0]\}, \\ \mathcal{U}_0 &= \bigcup_{A \in I_0} A = X_0 \setminus (P_6 \cup P_8), \\ X_1 &= X_0 \setminus \mathcal{U}_0 = P_6 \cup P_8. \end{aligned}$$

The function f is not locally one-to-one in the nowhere dense closed sets P_6 and P_8 . Now $f|_{X_1}$ is an exactly two-to-one continuous function which is the same as the function in Example 2. Therefore,

$$\begin{aligned} I_1 &= \{A \subseteq X_1 : A = A_1 \dot{\cup} A'_1, A_1, A'_1 \text{ clopen}, f[A_1] = f[A'_1]\}, \\ \mathcal{U}_1 &= \bigcup_{A \in I_1} A = X_1 \setminus (P_2 \cup P_4), \\ X_2 &= X_1 \setminus \mathcal{U}_1 = P_2 \cup P_4, \\ I_2 &= \{A \subseteq X_2 : A = A_2 \dot{\cup} A'_2, A_2, A'_2 \text{ clopen}, f[A_2] = f[A'_2]\}, \\ \mathcal{U}_2 &= \bigcup_{A \in I_2} A = X_2, \\ X_3 &= X_2 \setminus \mathcal{U}_2 = \emptyset. \end{aligned}$$

It is clear that we can continue this process for any finite number of steps in the following sense: If $f : \mathbb{N}^* \rightarrow K$ is a two-to-one continuous function and $X_0 = \mathbb{N}^*$, then for each integer n ,

$$\begin{aligned} I_n &= \{A \subseteq X_n : A = A_n \dot{\cup} A'_n, A_n, A'_n \text{ clopen}, f[A_n] = f[A'_n]\}, \\ \mathcal{U}_n &= \bigcup_{A \in I_n} A, \quad X_{n+1} = X_n \setminus \mathcal{U}_n. \end{aligned}$$

Then for each integer n there is an f so that $X_n \neq \emptyset$ while $X_{n+1} = \emptyset$.

4. Zero-dimensional spaces. A space X is called *zero-dimensional* if it has a base consisting of clopen sets, that is, if for every point $x \in X$ and for every neighborhood U of x there exists a clopen subset $C \subseteq X$ such that $x \in C \subseteq U$. \mathbb{N}^* is a zero-dimensional space and in this section we show that every two-to-one continuous image of \mathbb{N}^* is zero-dimensional under CH.

Suppose that $f : \mathbb{N}^* \rightarrow K$ is a two-to-one continuous function. As in the examples given in Section 3, let

$$\begin{aligned} X_0 &= \mathbb{N}^*, \quad K_0 = K, \\ I_0 &= \{A \subseteq X_0 : A = A_0 \dot{\cup} A'_0, A_0, A'_0 \text{ clopen}, f[A_0] = f[A'_0]\}, \\ \mathcal{U}_0 &= \bigcup_{A \in I_0} A. \end{aligned}$$

CLAIM 2. $I_0 \neq \emptyset$.

Proof of Claim. By Theorem 3, there is a clopen set $B \subseteq \mathbb{N}^*$ such that $f|_B$ is one-to-one and $\text{Int}f[B] \neq \emptyset$. Therefore, $f[B]$ is homeomorphic to B and hence there is a clopen set $B' \subseteq \text{Int}f[B]$ and $f^{-1}[B']$ is clopen since $f|_B$ is one-to-one and it can be written as a union of two disjoint clopen sets $f^{-1}[B'] = A_0 \dot{\cup} A'_0$ such that $f[A_0] = f[A'_0]$. Therefore, $f^{-1}[B'] \in I_0$. This shows I_0 is nonempty and $f[\mathcal{U}_0]$ is dense in K_0 . ■

Now let $X_1 = X_0 \setminus \mathcal{U}_0$ and $K_1 = K_0 \setminus f(\mathcal{U}_0)$. Then X_1 is a closed subset of X_0 . If $X_1 \neq \emptyset$, then $f|_{X_1} : X_1 \rightarrow K_1$ is an exactly two-to-one continuous function. In a similar way as before let

$$I_1 = \{A \subseteq X_1 : A = A_1 \dot{\cup} A'_1, A_1, A'_1 \text{ clopen}, f[A_1] = f[A'_1]\},$$

$$\mathcal{U}_1 = \bigcup_{A \in I_1} A, \quad X_2 = X_1 \setminus \mathcal{U}_1, \quad K_2 = K_1 \setminus f(\mathcal{U}_1).$$

If $X_1 \neq \emptyset$, then $I_1 \neq \emptyset$ by Theorem 3. If $X_2 \neq \emptyset$, then $f|_{X_2} : X_2 \rightarrow K_2$ is an exactly two-to-one continuous function. Continuing in a similar fashion, for each n we define

$$I_n = \{A \subseteq X_n : A = A_n \dot{\cup} A'_n, A_n, A'_n \text{ clopen}, f[A_n] = f[A'_n]\},$$

$$\mathcal{U}_n = \bigcup_{A \in I_n} A, \quad X_{n+1} = X_n \setminus \mathcal{U}_n, \quad K_{n+1} = K_n \setminus f(\mathcal{U}_n).$$

Then $X_\omega = \bigcap_n X_n$ and $K_\omega = \bigcap_n K_n$. Recall that we showed in Section 3 that X_n may be nonempty for any given natural number n . Therefore, the next result is quite a surprise.

THEOREM 6 (CH). $X_\omega = \emptyset$ and $K_\omega = \emptyset$.

Proof. Suppose $X_\omega \neq \emptyset$. Then $I_\omega \neq \emptyset$ where

$$I_\omega = \{A \subseteq X_\omega : A = A_\omega \dot{\cup} A'_\omega, A_\omega, A'_\omega \text{ clopen}, f[A_\omega] = f[A'_\omega]\}.$$

Therefore, there exist two nonempty disjoint clopen sets $A_\omega, A'_\omega \subseteq X_\omega$ such that $f[A_\omega] = f[A'_\omega]$. Since X_ω is compact and X_0 is zero-dimensional there are disjoint clopen sets $B_0, B'_0 \subseteq X_0$ such that $B_0 \cap X_\omega = A_\omega$ and $B'_0 \cap X_\omega = A'_\omega$. Thus $X_0 - (B_0 \cup B'_0)$ is clopen in X_0 and

$$(A_\omega \cup A'_\omega) \cap f^{-1}(f(X_0 - (B_0 \cup B'_0))) = \emptyset$$

by the definition of A_ω and A'_ω . But

$$A_\omega \cup A'_\omega = (B_0 \cup B'_0) \cap X_\omega = (B_0 \cup B'_0) \cap \bigcap_n X_n = \bigcap_n (B_0 \cup B'_0) \cap X_n.$$

Therefore $f^{-1}(f[X_0 \setminus (B_0 \cup B'_0)]) \cap ((B_0 \cup B'_0) \cap X_m) = \emptyset$ for some m .

CLAIM 3. *There exists $n_0 > m$ such that $f^{-1}(f[B_0]) \supseteq X_n \cap B'_0$ for all $n > n_0$.*

Proof of Claim. Otherwise for every $n > n_0$ there are $x_n, x'_n \in (X_n \cap B'_0) \setminus f^{-1}(f[B_0])$ such that $f(x_n) = f(x'_n)$. Then $\{x_n\} \cup \{x'_n\}$ is a discrete subset of \mathbb{N}^* and therefore $\overline{\{x_n\}} \cap \overline{\{x'_n\}} = \emptyset$. Moreover $\overline{\{x_n\}} \setminus \{x_n\} \cup \overline{\{x'_n\}} \setminus \{x'_n\} \subseteq X_\omega \cap B'_0 = A'_\omega$ and $\overline{\{x_n\}} \setminus \{x_n\} \cup \overline{\{x'_n\}} \setminus \{x'_n\}$ is nonempty since every infinite discrete set in a compact space has a limit point.

But then there are elements $x \in \overline{\{x_n\}} \setminus \{x_n\}$ and $x' \in \overline{\{x'_n\}} \setminus \{x'_n\}$ such that $f(x) = f(x')$. This is a contradiction since $f \upharpoonright_{A'_\omega}$ is one-to-one. ■

By symmetry there exists $k_0 > m$ such that $f^{-1}(f[B_0]) \supseteq X_n \cap B'_0$ for all $n > k_0$. Let $k = \max\{k_0, n_0\}$. Then $f(B_0 \cap X_{k+1}) = f(B'_0 \cap X_{k+1})$. This implies $A_\omega \subseteq U_{k+1}$ and $A'_\omega \subseteq U_{k+1}$. This is a contradiction since $A_\omega, A'_\omega \subseteq X_\omega \subseteq X_{k+1} \setminus U_{k+1}$. Hence $X_\omega = \emptyset$ and $K_\omega = \emptyset$. ■

LEMMA 7. *If $A \subseteq X_1$ is clopen with $f^{-1}(f[A]) = A$ and $U \subseteq X_0$ is open with $A \subseteq U$, then there is a clopen set $A' \subseteq U$ in X_0 such that $A' \cap X_1 = A$ and $f^{-1}(f[A']) = A'$.*

Proof. Since A is clopen in X_1 and X_1 is a subspace of X_0 , there is a clopen set $B \subseteq X_0$ such that $B \cap X_1 = A$ and $B \subseteq U$. Then $f[X_0 \setminus B] \cap f[A] = \emptyset$.

Let $A' = B \setminus f^{-1}(f[X_0 \setminus B])$. We now show that A' is the clopen subset of X_0 we are looking for. Clearly A' is open in X_0 , $A' \subseteq U$, $A' \cap X_1 = A$, and $f^{-1}(f[A']) = A'$. It remains to show that A' is closed in X_0 . This is equivalent to showing that $B \cap f^{-1}(f[X_0 \setminus B])$ is open.

Let $x \in B \cap f^{-1}(f[X_0 \setminus B])$ and let $x' \in X_0$ be such that $f(x) = f(x')$. This implies $x, x' \in \mathcal{U}_0 = X_0 \setminus X_1$. Therefore, by the definition of \mathcal{U}_0 , there are disjoint clopen sets $A_0, A'_0 \subseteq \mathcal{U}_0$ in X_0 such that $x \in A_0$, $x' \in A'_0$, and $f[A_0] = f[A'_0]$. Now shrink A_0 and A'_0 to clopen sets B_0 and B'_0 , respectively, so that $x \in B_0 \subseteq B$, $x' \in B'_0 \subseteq f^{-1}(f[X_0 \setminus B])$, and $f[B_0] = f[B'_0]$. Then $x \in B_0 \subseteq B \cap f^{-1}(f[X_0 \setminus B])$.

Therefore, $B \cap f^{-1}(f[X_0 \setminus B])$ is open and A' is closed in X_0 . ■

LEMMA 8. *If $A \subseteq X_{n+1}$ is clopen with $f^{-1}(f[A]) = A$ and $U \subseteq X_n$ is open with $A \subseteq U$, then there is a clopen set $A' \subseteq U$ in X_n such that $A' \cap X_{n+1} = A$ and $f^{-1}(f[A']) = A'$.*

Proof. The proof is similar to the proof of Lemma 7 with X_{n+1} and X_n playing the roles of X_1 and X_0 , respectively. ■

LEMMA 9. *If $A \subseteq X_n$ is clopen with $f^{-1}(f[A]) = A$ and $U \subseteq X_0$ is open with $A \subseteq U$, then there is a clopen set $A' \subseteq U$ in X_0 such that $A' \cap X_n = A$ and $f^{-1}(f[A']) = A'$.*

Proof. This follows from Lemmas 7 and 8 by induction. ■

THEOREM 10 (CH). *If $f : \mathbb{N}^* \rightarrow K$ is a two-to-one continuous function, then K is zero-dimensional.*

Proof. Let $y \in V$ where V is an open subset of K . Then $y \in K_n \setminus K_{n+1}$ since $K_\omega = \emptyset$ by Theorem 6. This implies $y \in f(\mathcal{U}_n) = K_n \setminus K_{n+1}$. Therefore, $y \in f[A_n] \subseteq f(\mathcal{U}_n) = K_n \setminus K_{n+1}$ for some clopen set $A_n \subseteq X_n$ such that $f[A_n]$ is clopen and $f|_{A_n}$ is one-to-one. This is by the definition of \mathcal{U}_n . Then $f[A_n]$ is homeomorphic to A_n and so there is a clopen set $B \subseteq \text{Int}f[A_n]$ containing y . Shrink B so that $y \in B \subseteq V \cap K_n$.

Let $A = f^{-1}(B)$ and $U = f^{-1}(V)$. Then $A \subseteq X_n$ is clopen with $f^{-1}(f[A]) = A$ and $U \subseteq X_0$ is open with $A \subseteq U$. By Lemma 7, there exists $A' \subseteq U$ clopen in X_0 such that $A' \cap X_n = A$ and $f^{-1}(f[A']) = A'$. Then $y \in f[A'] \subseteq V$ and $f[A']$ is clopen in K since $f^{-1}(f[A']) = A'$. Hence K is zero-dimensional. ■

5. F -spaces. A space is called an F -space if every pair of disjoint cozero subsets are completely separated. It is well known that \mathbb{N}^* is an F -space [Wa74] and in this section we show that every two-to-one continuous image of \mathbb{N}^* is also an F -space under CH.

THEOREM 11 (CH). *If $f : \mathbb{N}^* \rightarrow K$ is a two-to-one continuous function, then K is an F -space.*

Proof. Let C_1 and C_2 be two disjoint cozero sets in K . Then $f^{-1}(C_1)$ and $f^{-1}(C_2)$ are disjoint cozero sets in \mathbb{N}^* . Since \mathbb{N}^* is an F -space we have $\overline{f^{-1}(C_1)} \cap \overline{f^{-1}(C_2)} = \emptyset$. We must show that $\overline{C_1} \cap \overline{C_2} = \emptyset$. It is sufficient to show that for any $y \in \overline{C_1}$ there are two elements $x, x' \in \overline{f^{-1}(C_1)}$ such that $f(x) = y = f(x')$. This shows that $y \notin \overline{C_2}$. Otherwise, if $y \in \overline{C_2} \subseteq \overline{f[f^{-1}(C_2)]}$, there exists an $x'' \in \overline{f^{-1}(C_2)}$ such that $f(x'') = y$ and $x'' \neq x, x'$ since $\overline{f^{-1}(C_1)} \cap \overline{f^{-1}(C_2)} = \emptyset$ and $x, x' \in \overline{f^{-1}(C_1)}$. So three different points x, x', x'' map to y . This is a contradiction to the fact that the function f is exactly two-to-one.

Let $y \in \overline{C_1}$. Then $y \in \overline{C_1} \subseteq \overline{f[f^{-1}(C_1)]}$ since $C_1 = f[f^{-1}(C_1)] \subseteq \overline{f[f^{-1}(C_1)]}$. This implies there exists an $x \in \overline{f^{-1}(C_1)}$ such that $f(x) = y$. By Theorem 6, $K_\omega = \emptyset$ and so $y \in K_n \setminus K_{n+1}$ for some integer n . Let $m \leq n$ be maximal such that $y \in \overline{C_1} \cap \overline{K_m}$. Then $y \notin \overline{C_1} \cap \overline{K_{m+1}}$ and so there is a cozero set $C_y \subseteq K_m$ such that $y \in C_y$ and $C_y \cap \overline{C_1} \cap \overline{K_{m+1}} = \emptyset$. Thus $C_y \cap C_1 \cap K_{m+1} = \emptyset$.

Therefore, without loss of generality, we can assume that $C_1 \cap K_{m+1} = \emptyset$ and C_1 is a cozero set in K_m since we can take C_1 to be the cozero set $C_y \cap C_1$. Then $f^{-1}(C_1)$ is a cozero set in X_m and $f^{-1}(C_1) \cap X_{m+1} = \emptyset$, that is, $f^{-1}(C_1) \subseteq \mathcal{U}_m$ where \mathcal{U}_m is defined as in Section 4 by

$$\mathcal{U}_m = \bigcup_{A \in I_m} A,$$

$$I_m = \{A \subseteq X_m : A = A_m \dot{\cup} A'_m, A_m, A'_m \text{ clopen, } f[A_m] = f[A'_m]\}.$$

Since $f^{-1}(C_1)$ is a cozero set in X_m and X_m is a compact zero-dimensional F -space, it can be written as a countable union of disjoint clopen sets in such a way that $f^{-1}(C_1) = \bigcup_{n=0}^{\infty} [A_n \cup A'_n]$ where A_n and A'_n are disjoint clopen sets in X_m and $f[A_n] = f[A'_n]$. Therefore, $f^{-1}(C_1)$ can be written as a union of two disjoint sets

$$f^{-1}(C_1) = \left(\bigcup_{n=0}^{\infty} A_n \right) \cup \left(\bigcup_{n=0}^{\infty} A'_n \right)$$

and by the definition of A_n and A'_n we get

$$f \left[\bigcup_{n=0}^{\infty} A_n \right] = f \left[\bigcup_{n=0}^{\infty} A'_n \right].$$

Thus

$$\overline{f^{-1}(C_1)} = \overline{\bigcup_{n=0}^{\infty} A_n} \cup \overline{\bigcup_{n=0}^{\infty} A'_n}.$$

The sets $\bigcup_{n=0}^{\infty} A_n$ and $\bigcup_{n=0}^{\infty} A'_n$ are cozero sets since a countable union of clopen sets is a cozero set, and they are disjoint by construction. Therefore, since X_m is an F -space we get

$$\overline{\bigcup_{n=0}^{\infty} A_n} \cap \overline{\bigcup_{n=0}^{\infty} A'_n} = \emptyset.$$

Now since

$$x \in \overline{f^{-1}(C_1)} = \overline{\bigcup_{n=0}^{\infty} A_n} \cup \overline{\bigcup_{n=0}^{\infty} A'_n}$$

we assume, without loss of generality, that

$$x \in \overline{\bigcup_{n=0}^{\infty} A_n} \quad \text{and} \quad x \notin \overline{\bigcup_{n=0}^{\infty} A'_n}.$$

By continuity of f and the fact that $f[\bigcup_{n=0}^{\infty} A_n] = f[\bigcup_{n=0}^{\infty} A'_n]$ we get $f[\overline{\bigcup_{n=0}^{\infty} A_n}] = f[\overline{\bigcup_{n=0}^{\infty} A'_n}]$ and

$$y \in \overline{C_1} \subseteq f[\overline{f^{-1}(C_1)}] = f \left[\overline{\bigcup_{n=0}^{\infty} A_n} \right] = f \left[\overline{\bigcup_{n=0}^{\infty} A'_n} \right].$$

Therefore, there exists an $x' \in \overline{\bigcup_{n=0}^{\infty} A'_n}$ such that $f(x') = y$.

Now $x' \neq x$ because $x' \in \overline{\bigcup_{n=0}^{\infty} A'_n}$, $x \in \overline{\bigcup_{n=0}^{\infty} A_n}$, and

$$\overline{\bigcup_{n=0}^{\infty} A_n} \cap \overline{\bigcup_{n=0}^{\infty} A'_n} = \emptyset.$$

Thus, there are two different points $x, x' \in \overline{f^{-1}(C_1)}$ such that $f(x) = y = f(x')$. Hence K is an F -space. ■

6. Nonempty G_δ -sets. The intersection of countably many open sets is called a G_δ -set. Nonempty G_δ -sets on \mathbb{N}^* have nonempty interiors. In this section we prove that two-to-one continuous images of \mathbb{N}^* have the same property.

THEOREM 12 (CH). *If $f : \mathbb{N}^* \rightarrow K$ is a two-to-one continuous function, then nonempty G_δ -sets in K have nonempty interior.*

Proof. Suppose that $\{b_n : n \in \omega\}$ is a descending sequence of clopen subsets of K with $b_0 = K$. It suffices to deal with clopen sets since we have shown that K is zero-dimensional. Assume to the contrary that $\bigcap_n b_n$ is nowhere dense. Let $Z \subseteq \mathbb{N}^*$ be such that $f \upharpoonright_Z$ is irreducible. This is possible by Zorn's lemma.

For each n let $a_n = f^{-1}(b_n) \cap Z$. Then $\{a_n : n \in \omega\}$ is a descending sequence of clopen subsets of Z and $a_0 = Z$. Therefore, by Theorem 3, for each n pick clopen sets $e_n \subseteq a_n \setminus a_{n+1}$ and $e'_n \subseteq \mathbb{N}^* \setminus Z$ such that $f[e_n] = f[e'_n]$. Then, in Z , $\overline{\bigcup_n (a_n \setminus (e_n \cup a_{n+1}))} \cap \bigcup_n e_n = \emptyset$ since Z is an F -space.

Since we assumed $\bigcap_n b_n$ is nowhere dense in K , $f^{-1}(\bigcap_n b_n) \cap Z$ is nowhere dense and so

$$Z = \overline{\bigcup_n (a_n \setminus (e_n \cup a_{n+1}))} \cup \overline{\bigcup_n e_n}.$$

Thus $\overline{\bigcup_n e_n}$ is clopen in Z . Clearly $\overline{\bigcup_n (a_n \setminus a_{n+1})} \cap \bigcup_n e'_n = \emptyset$ since Z is closed and $e'_n \subseteq \mathbb{N}^* \setminus Z$.

Let us show that $\bigcup_n (a_n \setminus a_{n+1}) \cap \overline{\bigcup_n e'_n} = \emptyset$. For each n , $f^{-1}(K \setminus b_n)$ is clopen in \mathbb{N}^* since b_n is clopen and f is continuous. By construction $\bigcup_{m \geq n} e'_m \cap f^{-1}(K \setminus b_n) = \emptyset$ and $\bigcup_{m \in \omega} e'_m \cap f^{-1}(K \setminus b_n)$ is clopen in \mathbb{N}^* since it is a finite intersection of clopen sets. Because

$$a_n \setminus a_{n+1} \subseteq f^{-1}(K \setminus b_{n+1}) \quad \text{and} \quad [a_n \setminus (a_{n+1} \cup e_n)] \cap \bigcup_{n \in \omega} e'_n = \emptyset$$

we have $\bigcup_n (a_n \setminus a_{n+1}) \cap \overline{\bigcup_n e'_n} = \emptyset$. Thus $\overline{\bigcup_n (a_n \setminus a_{n+1})} \cap \overline{\bigcup_n e'_n} = \emptyset$ since \mathbb{N}^* is an F -space. Therefore $\overline{\bigcup_n e_n} \cap \overline{\bigcup_n e'_n} = \emptyset$ and hence f is a two-to-one function on $\overline{\bigcup_n e_n} \cup \overline{\bigcup_n e'_n}$.

Now since $f[\overline{\bigcup_n e_n}] = f[\overline{\bigcup_n e'_n}]$ we have $K = f[\overline{\bigcup_n e_n}] \cup f[Z \setminus \overline{\bigcup_n e_n}]$ and $f[\overline{\bigcup_n e_n}] \cap f[Z \setminus \overline{\bigcup_n e_n}] = \emptyset$. Therefore $f[\overline{\bigcup_n e_n}] \subseteq K$ is clopen in K . But then

$$f^{-1}\left(f\left[\overline{\bigcup_n e'_n}\right]\right) = \overline{\bigcup_n e_n} \cup \overline{\bigcup_n e'_n}$$

is clopen. This is a contradiction since $\overline{\bigcup_n e_n} \cup \overline{\bigcup_n e'_n}$ is not clopen by the fact that in \mathbb{N}^* nonempty G_δ -sets have nonempty interior. ■

If $f : \mathbb{N}^* \rightarrow K$ is a two-to-one continuous function, Levy [Le04] proved that countable discrete subsets of K are C^* -embedded and the weight of K is \mathfrak{c} . This completes everything needed to prove that a two-to-one continuous image of \mathbb{N}^* is \mathbb{N}^* up to a homeomorphism.

COROLLARY 13 (CH). *If $f : \mathbb{N}^* \rightarrow K$ is a two-to-one continuous function, then K is homeomorphic to \mathbb{N}^* .*

Proof. Follows from Theorems 1, 10, 11, and 12. ■

OPEN PROBLEMS. Our results are all under the set-theoretic assumption CH. Is it possible to eliminate CH? In particular, if $f : \mathbb{N}^* \rightarrow K$ is a two-to-one continuous function:

- (1) Is it true that f is not irreducible?
- (2) Is K homeomorphic to \mathbb{N}^* ?
- (3) Is every countable subset of K C^* -embedded?
- (4) Can K be separable or ccc? (Levy question [vD93]).
- (5) If f is n -to-one continuous with $n > 2$, is K homeomorphic to \mathbb{N}^* under CH or ZFC?

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