

On expansions of weakly o-minimal non-valuational structures by convex predicates

by

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Abstract. We prove that if $\mathcal{M} = (M, \leq, +, \dots)$ is a weakly o-minimal non-valuational structure expanding an ordered group $(M, \leq, +)$, then its expansion by a family of “non-valuational” unary predicates remains non-valuational. The paper is based on the author’s earlier work on strong cell decomposition for weakly o-minimal non-valuational expansions of ordered groups.

0. Introduction. Several examples of weakly o-minimal structures are obtained as expansions of o-minimal structures by predicates interpreted as certain convex sets [MMS]. Among these we have an expansion of a real closed field by a valuation ring and an expansion of the ordered field of real algebraic numbers by a predicate interpreted as $(-\alpha, \alpha)$, where α is a transcendental number. Structures of the first sort were investigated by L. van den Dries and A. H. Lewenberg (see [DL] and [D2]) who showed for instance that if \mathcal{R} is an o-minimal expansion of a real closed field and V is a proper non-empty convex subring closed under 0-definable continuous functions, then the expansion (\mathcal{R}, V) is weakly o-minimal. The phenomenon occurring in all the mentioned examples was addressed in general by Y. Baisalov and B. Poizat (see [BP]) who proved that an expansion of any o-minimal structure by a family of convex predicates has weakly o-minimal theory (so in particular it is a weakly o-minimal structure). The result of [BP] was generalized by B. Baizhanov (see [Bz]) who proved that an expansion of a model of a weakly o-minimal theory by a family of convex predicates has weakly o-minimal theory. It is worth mentioning that Baizhanov’s theorem also has

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an easy proof when one uses the fact that weakly o-minimal theories do not have the independence property and the result of S. Shelah (see [Sh783]) concerning quantifier elimination for the theory of an expansion of a sufficiently saturated model of a theory without the independence property by all externally definable sets. The question of G. Cherlin whether every expansion of a weakly o-minimal structure (not necessarily with weakly o-minimal theory) by a family of convex predicates is also weakly o-minimal still remains an open problem.

This paper is a sequel to the study of expansions of weakly o-minimal structures by convex predicates. In [We07] I introduced weakly o-minimal non-valuational expansions of ordered groups as a natural generalization of weakly o-minimal non-valuational expansions of real closed fields considered in [MMS]. A weakly o-minimal expansion of a real closed field is said to be non-valuational iff it does not define a non-trivial valuation. Similarly, a weakly o-minimal expansion \mathcal{M} of an ordered group $(M, \leq, +)$ is called non-valuational (or of non-valuational type) iff there is no proper and non-trivial subgroup of $(M, +)$ definable in \mathcal{M} . Being of non-valuational type is equivalent to several conditions which are discussed in [We07]. One of them says that the distance between the two parts of a definable cut is zero (the precise definition appears in §1).

Assume that $\mathcal{M} = (M, \leq, +, \dots)$ is a weakly o-minimal non-valuational expansion of an ordered group. If $P \subseteq M$ is a finite union of convex sets, then P in a natural way determines a finite family of cuts in (M, \leq) . If the parts of all these cuts are “close” to each other, then P is said to be non-valuational. Moreover, the expansion (\mathcal{M}, P) is interdefinable with some expansion of \mathcal{M} by a family of convex predicates. By [We07], $\text{Th}(\mathcal{M})$ is weakly o-minimal. Therefore by [Bz] also any expansion of \mathcal{M} by a family of non-valuational predicates has weakly o-minimal theory. The main result of this paper is Theorem 2.11, which says that every expansion of \mathcal{M} by a family of non-valuational predicates is of non-valuational type. We also show that the theory of such an expansion is weakly o-minimal without using Baizhanov’s theorem (cf. Corollary 2.10).

The paper is organized as follows. In §1 we fix our notation and terminology, and recall some particularly useful results, mainly from [MMS] and [We07]. In §2 we outline the proofs of the results mentioned above.

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1. Notation and preliminaries. Let (M, \leq) be a dense linear ordering without endpoints. A set $I \subseteq M$ is called *convex* in (M, \leq) iff for any $a, b \in I$ and $c \in M$ with $a \leq c \leq b$ we have $c \in I$. If additionally $I \neq \emptyset$ and

$\inf I, \sup I \in M \cup \{-\infty, +\infty\}$, then I is called an *interval* in (M, \leq) . A maximal convex subset of a non-empty subset of M is called a *convex component* of it. A pair $\langle C, D \rangle$ of non-empty subsets of M is called a *cut* in (M, \leq) iff $C < D$, $C \cup D = M$ and D has no lowest element. A first order structure $\mathcal{M} = (M, \leq, \dots)$ expanding (M, \leq) is said to be *weakly o-minimal* iff every subset of M , definable in \mathcal{M} , is a finite union of convex sets. A complete first order theory is called weakly o-minimal iff all its models are weakly o-minimal. The following remark characterizes weakly o-minimal structures in terms of sets definable in them.

REMARK 1.1. Assume that (M, \leq) is a dense linear ordering without endpoints and for $m \in \mathbb{N}_+$, \mathcal{D}_m is a family of subsets of M^m for which the following conditions are satisfied.

- (a) If $I \subseteq M$ is an interval, then $I \in \mathcal{D}_1$.
- (b) If $X \in \mathcal{D}_1$, then X is a union of finitely many convex sets.
- (c) $\{\langle x, y \rangle \in M^2 : x < y\} \in \mathcal{D}_2$.
- (d) \mathcal{D}_m with the usual set-theoretic operations is a Boolean algebra.
- (e) If $X \in \mathcal{D}_m$, then $X \times M, M \times X \in \mathcal{D}_{m+1}$.
- (f) If $X \in \mathcal{D}_{m+1}$ and $\pi : M^{m+1} \rightarrow M^m$ is the projection dropping the last coordinate, then $\pi[X] \in \mathcal{D}_m$.
- (g) If $1 \leq i < j \leq m$, then $\{(x_1, \dots, x_m) \in M^m : x_i = x_j\} \in \mathcal{D}_m$.

Then there is a weakly o-minimal structure \mathcal{M} expanding (M, \leq) such that for every $X \subseteq M^m$, X is definable in \mathcal{M} iff $X \in \mathcal{D}_m$.

Note that if one replaces “convex sets” in (b) with “intervals”, then (a)–(g) in Remark 1.1 imply that there is an o-minimal expansion \mathcal{M} of (M, \leq) such that for every $X \subseteq M^m$, X is definable in \mathcal{M} iff $X \in \mathcal{D}_m$ (cf. [D1, Chapter 1]).

Assume that $\mathcal{M} = (M, \leq, \dots)$ is a weakly o-minimal structure. A cut $\langle C, D \rangle$ in (M, \leq) is called *definable* in \mathcal{M} iff the sets C, D are definable in \mathcal{M} . The set of all such cuts will be denoted by $\overline{M}^{\mathcal{M}}$. The set M can be regarded as a subset of $\overline{M}^{\mathcal{M}}$ by identifying an element $a \in M$ with the cut $\langle (-\infty, a], (a, +\infty) \rangle$. After such an identification, $\overline{M}^{\mathcal{M}}$ is naturally equipped with a dense linear ordering without endpoints extending that of (M, \leq) , and (M, \leq) is dense in $(\overline{M}^{\mathcal{M}}, \leq)$. For a definable set $X \subseteq M^m$, a function $f : X \rightarrow \overline{M}^{\mathcal{M}}$ is said to be *definable* in \mathcal{M} iff the set $\{\langle \bar{x}, y \rangle \in X \times M : f(\bar{x}) > y\}$ is definable in \mathcal{M} .

Now assume that $\mathcal{M} = (M, \leq, +, \dots)$ is a weakly o-minimal structure expanding an ordered group $(M, \leq, +)$. A cut $\langle C, D \rangle$ is called *non-valuational* iff $\inf\{y - x : x \in C, y \in D\} = 0$. We will say that \mathcal{M} is *non-valuational* (or *of non-valuational type*) iff all cuts definable in \mathcal{M} are non-valuational. If \mathcal{M} is non-valuational, then $\overline{M}^{\mathcal{M}}$ can be naturally equipped with an ordered

group structure extending that of $(M, \leq, +)$. The ordered groups $(M, \leq, +)$ and $(\overline{M}^{\mathcal{M}}, \leq, +)$ are divisible, abelian and torsion free. For details we refer the reader to §1 of [We07].

Assume that $\mathcal{M} = (M, \leq, \dots)$ is a weakly o-minimal structure. Below, for every $m \in \mathbb{N}_+$ we inductively define *strong cells* in M^m and their *completions* in $(\overline{M}^{\mathcal{M}})^m$. The completion in $(\overline{M}^{\mathcal{M}})^m$ of a strong cell $C \subseteq M^m$ will be denoted by \overline{C} .

- (1) A one-element subset of M is a strong $\langle 0 \rangle$ -cell in M and is equal to its completion.
- (2) A non-empty convex open definable subset of M is a strong $\langle 1 \rangle$ -cell in M . If $C \subseteq M$ is a strong $\langle 1 \rangle$ -cell in M , then $\overline{C} := \{x \in \overline{M}^{\mathcal{M}} : (\exists a, b \in C)(a < x < b)\}$.

Assume that $m \in \mathbb{N}_+$, $i_1, \dots, i_m \in \{0, 1\}$ and suppose that we have already defined strong $\langle i_1, \dots, i_m \rangle$ -cells in M^m and their completions in $(\overline{M}^{\mathcal{M}})^m$.

- (3) If $C \subseteq M^m$ is a strong $\langle i_1, \dots, i_m \rangle$ -cell in M^m and $f : C \rightarrow M$ is a continuous definable function which has a continuous extension $\overline{f} : \overline{C} \rightarrow \overline{M}^{\mathcal{M}}$, then $\Gamma(f)$, the graph of f , is a strong $\langle i_1, \dots, i_m, 0 \rangle$ -cell in M^{m+1} . The completion of $\Gamma(f)$ in $(\overline{M}^{\mathcal{M}})^{m+1}$ is defined as $\Gamma(\overline{f})$.
- (4) If $C \subseteq M^m$ is a strong $\langle i_1, \dots, i_m \rangle$ -cell in M^m and $f, g : C \rightarrow \overline{M}^{\mathcal{M}} \cup \{-\infty, +\infty\}$ are continuous definable functions which have continuous extensions $\overline{f}, \overline{g} : \overline{C} \rightarrow \overline{M}^{\mathcal{M}}$ such that $\overline{f}(\overline{x}) < \overline{g}(\overline{x})$ for $\overline{x} \in \overline{C}$, then the set

$$(f, g)_C := \{(\overline{a}, b) \in C \times M : f(\overline{a}) < b < g(\overline{a})\}$$

is called a strong $\langle i_1, \dots, i_m, 1 \rangle$ -cell in M^m . The completion of $(f, g)_C$ in $(\overline{M}^{\mathcal{M}})^{m+1}$ is defined as

$$\overline{(f, g)_C} := (\overline{f}, \overline{g})_{\overline{C}} := \{(\overline{a}, b) \in \overline{C} \times \overline{M}^{\mathcal{M}} : \overline{f}(\overline{a}) < b < \overline{g}(\overline{a})\}.$$

- (5) We say that $C \subseteq M^m$ is a strong cell in M^m iff there are $i_1, \dots, i_m \in \{0, 1\}$ such that C is a strong $\langle i_1, \dots, i_m \rangle$ -cell in M^m .

If $C \subseteq M^m$ is a strong cell, then a definable function $f : C \rightarrow \overline{M}^{\mathcal{M}}$ is called *strongly continuous* iff f has a continuous extension $\overline{f} : \overline{C} \rightarrow \overline{M}^{\mathcal{M}}$. In a standard way we define the notion of *decomposition* of a definable set into strong cells partitioning a given definable set (for details we refer the reader to §2 of [We07]). We will say that \mathcal{M} has the *strong cell decomposition property* iff for any $m, k \in \mathbb{N}_+$ and any definable sets $X_1, \dots, X_k \subseteq M^m$, there is a decomposition of M^m into strong cells partitioning each of the sets X_1, \dots, X_k .

FACT 1.2 ([MMS, Ar]). *Let $\mathcal{M} = (M, \leq, \dots)$ be a weakly o-minimal structure and $A \subseteq M$. If $U \subseteq M$ is an infinite A -definable set and $f :$*

$U \rightarrow \overline{M}^M$ is an A -definable function, then there is a partition of U into A -definable sets X, I_0, \dots, I_m such that X is finite, I_0, \dots, I_m are non-empty convex open sets, and for every $i \leq m$, $f|I_i$ is locally constant or locally strictly increasing or locally strictly decreasing.

The following fact easily follows from the definition of strong cells.

FACT 1.3. *Assume that $\mathcal{M} = (M, \leq, \dots)$ is a weakly o-minimal structure with the strong cell decomposition property and $A \subseteq M$. If $U \subseteq M$ is a non-empty A -definable set and $f : U \rightarrow \overline{M}^M$ is an A -definable function, then there is a partition of U into A -definable sets X, I_0, \dots, I_m such that X is finite, I_0, \dots, I_m are non-empty convex open sets, and for every $i \leq m$, $f|I_i$ is strongly continuous and strictly monotone or constant.*

Proof. By assumption, there is a decomposition \mathcal{C} of M^2 into A -definable strong cells partitioning the set $\{ \langle x, y \rangle \in U \times M : y < f(x) \}$. This yields a decomposition of U into finitely many A -definable convex open sets J_0, \dots, J_k and a finite set X such that

$$\{J_0, \dots, J_k\} \cup \{ \{a\} : a \in X \} = \{ \pi[C] : C \in \mathcal{C} \},$$

where $\pi : M^2 \rightarrow M$ is the projection dropping the second coordinate. By our definition of strong cells, $f|J_i$ is strongly continuous for $i \leq k$. In such a situation, weak o-minimality of \mathcal{M} implies that each of the J_i 's could be decomposed into finitely many A -definable convex open sets and some finite set so that on each of the open sets, f is strictly monotone or constant. This finishes the proof. ■

THEOREM 1.4 ([We07]). *Let $\mathcal{M} = (M, \leq, +, \dots)$ be a weakly o-minimal structure expanding an ordered group $(M, \leq, +)$. Then \mathcal{M} is of non-valuational type iff \mathcal{M} has the strong cell decomposition property.*

2. The main result. Throughout this section we shall work in a weakly o-minimal structure $\mathcal{M} = (M, \leq, \dots)$, usually assuming that \mathcal{M} has the strong cell decomposition property or is a non-valuational expansion of an ordered group. Let C denote a non-empty, convex and non-definable (in \mathcal{M}) proper subset of M such that $\inf C = -\infty$. Then $\langle C, M \setminus C \rangle$ is a cut in (M, \leq) (in [Bz] and [BVT] such cuts are called irrational). By Corollary 2.15 from [We07], we know that if \mathcal{M} is a non-valuational expansion of an ordered group, then $\text{Th}(\mathcal{M})$ is weakly o-minimal. Hence, by [Bz], also $\text{Th}(\mathcal{M}, C)$ is weakly o-minimal. In a series of lemmas below we will show that if \mathcal{M} is a weakly o-minimal non-valuational expansion of an ordered group and the cut $\langle C, M \setminus C \rangle$ is non-valuational, then also the expansion (\mathcal{M}, C) is of non-valuational type. Moreover, we will give a direct proof of the weak o-minimality of $\text{Th}(\mathcal{M}, C)$.

For $m \in \mathbb{N}_+$ and $i \in \{1, \dots, m+1\}$, we denote by π_i^{m+1} the projection from M^{m+1} to M^m dropping the i th coordinate, and by $\mathcal{D}_m(\mathcal{M})$ the family of all subsets of M^m definable in \mathcal{M} .

LEMMA 2.1. *Assume that $\mathcal{M} = (M, \leq, \dots)$ is a weakly o-minimal structure and $I \subseteq M$ is a convex open definable set such that $I \cap C \neq \emptyset$ and $I \setminus C \neq \emptyset$. If $f : I \rightarrow \overline{M}^{\mathcal{M}}$ is a definable function such that $(\forall x \in I)(f(x) > c)$ $[(\forall x \in I)(f(x) < c)]$ for some $c \in M$, then there are an open interval $J \subseteq I$ and $\alpha > c$ [$\alpha < c$], $\alpha \in M$, such that $J \cap C \neq \emptyset$, $J \setminus C \neq \emptyset$ and $(\forall x \in J)(f(x) > \alpha)$ $[(\forall x \in J)(f(x) < \alpha)]$.*

Proof. Let $f : I \rightarrow \overline{M}^{\mathcal{M}}$ be a definable function such that $(\forall x \in I)(f(x) > c)$, where $c \in M$. By Fact 1.2 and the non-definability of C , there is an open interval $I_1 \subseteq I$ such that $I_1 \cap C \neq \emptyset$, $I_1 \setminus C \neq \emptyset$ and $f|_{I_1}$ is either strictly monotone or constant. Fix $a \in I_1 \cap C$, $b \in I_1 \setminus C$ and $\alpha \in M$ such that $c < \alpha < \min\{f(a), f(b)\}$. It is clear that $f(x) > \alpha$ whenever $x \in (a, b)$. The other case is proved in a similar way. ■

LEMMA 2.2. *Let $\mathcal{M} = (M, \leq, \dots)$ be a weakly o-minimal non-valuational expansion of an ordered group. Assume that $I \subseteq M$ is a non-empty convex open definable set such that $I \cap C \neq \emptyset$ and $I \setminus C \neq \emptyset$, and $f, g : I \rightarrow \overline{M}^{\mathcal{M}}$ are definable functions such that $(\forall x \in I)(f(x) < g(x))$. Then there are an element $a \in M$ and an open interval $J \subseteq I$ such that $J \cap C \neq \emptyset$, $J \setminus C \neq \emptyset$ and $f(x) < a < g(x)$ for $x \in J$.*

Proof. By Theorem 1.4 and Fact 1.3, without loss of generality we can assume that the functions f, g are strongly continuous. By Lemma 2.1, there are an open interval $I_1 \subseteq I$ and an element $\alpha > 0$ in M such that $I_1 \cap C \neq \emptyset$, $I_1 \setminus C \neq \emptyset$ and $(\forall x \in I_1)(g(x) - f(x) > \alpha)$. Fix $\varepsilon > 0$ in M such that $2\varepsilon < \alpha$. For $x_0 \in I_1$ define

$$\begin{aligned} \delta_f(x_0) &= \min(\varepsilon, \sup\{d \in M : d > 0, \\ &\quad (\forall x \in (x_0 - d, x_0 + d) \cap I_1)(|f(x) - f(x_0)| < \varepsilon)\}), \\ \delta_g(x_0) &= \min(\varepsilon, \sup\{d \in M : d > 0, \\ &\quad (\forall x \in (x_0 - d, x_0 + d) \cap I_1)(|g(x) - g(x_0)| < \varepsilon)\}). \end{aligned}$$

Again, by Lemma 2.1, there are an open interval $I_2 \subseteq I_1$ and an element $\beta > 0$ in M such that $I_2 \cap C \neq \emptyset$, $I_2 \setminus C \neq \emptyset$ and $\min(\delta_f(x_0), \delta_g(x_0)) > \beta$ for $x_0 \in I_2$. Fix $c_1 \in I_2 \cap C$ and $c_2 \in I_2 \setminus C$ such that $c_2 - c_1 < \beta$. For $x_1, x_2 \in (c_1, c_2)$ we have

$$g(x_2) - f(x_1) = (g(x_2) - g(x_1)) + (g(x_1) - f(x_1)) > -\varepsilon + 2\varepsilon = \varepsilon.$$

Consequently,

$$\inf\{g(x) : x \in (c_1, c_2)\} - \sup\{f(x) : x \in (c_1, c_2)\} \geq \varepsilon.$$

If $a \in M$ is such that $\sup\{f(x) : x \in (c_1, c_2)\} < a < \inf\{g(x) : x \in (c_1, c_2)\}$, then for $x \in (c_1, c_2)$ we have $f(x) < a < g(x)$. ■

LEMMA 2.3. *Let \mathcal{M} be a weakly o-minimal non-valuational expansion of an ordered group. Assume that $X \subseteq M^2$ is a set definable in \mathcal{M} such that for any $a, b \in M$, if $\langle a, b \rangle \in X$, then there are $a_1, a_2 \in M$ such that $a_1 < a < a_2$ and $(a_1, a_2) \times \{b\} \subseteq X$. The following conditions are equivalent.*

- (a) *There are $a_1 \in C$, $a_2 \in M \setminus C$ and $b \in M$ such that $(a_1, a_2) \times \{b\} \subseteq X$.*
- (b) *There are $a_1 \in C$ and $a_2 \in M \setminus C$ such that $(a_1, a_2) \subseteq \pi_2^2[X]$.*

Proof. The implication (a) \Rightarrow (b) is obvious, so assume that (b) holds. Let \mathcal{C} be a decomposition of M^2 into strong cells in M^2 partitioning the set X . There is a convex open definable set $I \subseteq M$ such that $I \cap C \neq \emptyset$, $I \setminus C \neq \emptyset$ and for every $D \in \mathcal{C}$, either $I = \pi_2^2[D]$ or $\pi_2^2[D] \cap I = \emptyset$. Below we consider two cases.

CASE 1: The set $(I \times M) \cap X$ has empty interior. The following claim is a consequence of Lemma 2.1 from [We06] but for the sake of completeness we give the proof in our particular situation.

CLAIM. *For every $x \in I$, the set $\{y \in M : \langle x, y \rangle \in X\}$ is finite.*

Proof of the Claim. Suppose for a contradiction that for some $a \in I$, the set $\{y \in M : \langle a, y \rangle \in X\}$ is infinite, so it contains an open interval J . For $b \in J$ define

$$f(b) = \sup\{c \in M : c > a \text{ and } \{a\} \times (b, c) \subseteq (I \times M) \cap X\}.$$

The function f assumes values greater than a in $\overline{M}^M \cup \{+\infty\}$. By Fact 1.3, there is an open interval $J' \subseteq J$ such that $f|_{J'}$ is strongly continuous and strictly monotone or constant. It is clear that the set $\{\langle x, y \rangle : a < x < f(y), y \in J'\}$ is contained in $(I \times M) \cap X$ and contains an open box itself. This means that $(I \times M) \cap X$ has non-empty interior, a contradiction.

Using the Claim, for $x \in I$, we can define $f(x) = \min\{y \in M : \langle x, y \rangle \in X\}$. Our assumptions guarantee that f is constant (say $f(x) = b$ whenever $x \in I$), so for any $a_1 \in I \cap C$ and $a_2 \in I \setminus C$, we have $(a_1, a_2) \times \{b\} \subseteq X$.

CASE 2: There are definable strongly continuous functions $f : I \rightarrow \overline{M}^M \cup \{-\infty\}$ and $g : I \rightarrow \overline{M}^M \cup \{+\infty\}$ such that $\bar{f}(a) < \bar{g}(a)$ for $a \in \bar{I}$ and $D := (f, g)_I \subseteq X \cap (I \times M)$. By Lemma 2.2, there are $a_1 \in I \cap C$, $a_2 \in I \setminus C$ and $b \in M$ such that $(a_1, a_2) \times \{b\} \subseteq D \subseteq X$. ■

For a weakly o-minimal structure \mathcal{M} and a set $X \in \mathcal{D}_{m+1}(\mathcal{M})$ let

$$I(X, C) = \{\bar{a} \in M^m : (\exists b \in C)(\exists c \in M \setminus C)(\{\bar{a}\} \times (b, c) \subseteq X)\}.$$

Define also $\mathcal{E}_m(\mathcal{M}, C) = \{I(X, C) : X \in \mathcal{D}_{m+1}(\mathcal{M})\}$. Below we will show that if \mathcal{M} is a non-valuational expansion of an ordered group, then $\mathcal{E}_m(\mathcal{M}, C)$ is exactly the family of subsets of M^m definable in (\mathcal{M}, C) .

LEMMA 2.4. *Let \mathcal{M} be a weakly o-minimal structure with the strong cell decomposition property.*

- (a) *If $X \in \mathcal{D}_2(\mathcal{M})$ is a strong cell, then $I(\{\langle x, y \rangle \in M^2 : \langle y, x \rangle \in X\}, C)$ is a convex set.*
- (b) *If $Y \in \mathcal{E}_1(\mathcal{M}, C)$, then Y is a finite union of convex sets.*
- (c) *$C \in \mathcal{E}_1(\mathcal{M}, C)$.*

Proof. (a) is obvious from the definition of strong cells. (b) follows from (a) and the strong cell decomposition property of \mathcal{M} . For the proof of (c), note that

$$C = I(\{\langle x, y \rangle \in M^2 : y > x\}, C).$$

Here the strong cell decomposition property is not needed. ■

LEMMA 2.5. *Assume that \mathcal{M} is a weakly o-minimal structure and $m \in \mathbb{N}_+$.*

- (a) $\mathcal{D}_m(\mathcal{M}) \subseteq \mathcal{E}_m(\mathcal{M}, C)$.
- (b) $\mathcal{E}_m(\mathcal{M}, C)$ is closed under Boolean operations.
- (c) *If $X \in \mathcal{E}_m(\mathcal{M}, C)$, then $X \times M, M \times X \in \mathcal{E}_{m+1}(\mathcal{M}, C)$.*

Proof. (a) If $m \in \mathbb{N}_+$ and $X \in \mathcal{D}_m(\mathcal{M})$, then $X = I(X \times M, C)$.

(b) Fix $m \in \mathbb{N}_+$ and $X, Y \in \mathcal{E}_m(\mathcal{M}, C)$. There are $X_1, Y_1 \in \mathcal{D}_{m+1}(\mathcal{M})$ such that $X = I(X_1, C)$ and $Y = I(Y_1, C)$. Clearly, $X \cup Y = I(X_1 \cup Y_1, C)$, $X \cap Y = I(X_1 \cap Y_1, C)$ and $M^m \setminus X = I(M^{m+1} \setminus X_1, C)$.

(c) Let $m \in \mathbb{N}_+$ and $X \in \mathcal{E}_m(\mathcal{M}, C)$. Then $X = I(X_1, C)$ for some $X_1 \in \mathcal{D}_{m+1}(\mathcal{M})$. Hence $M \times X = I(M \times X_1, C)$ and $X \times M = I(\{\langle \bar{x}, y, z \rangle \in M^{m+1} : \langle \bar{x}, z \rangle \in X\}, C)$. ■

LEMMA 2.6. *Let \mathcal{M} be a weakly o-minimal expansion of an ordered group. If $X \in \mathcal{E}_{m+1}(\mathcal{M}, C)$, then $\pi_{m+1}^{m+1}[X] \in \mathcal{E}_m(\mathcal{M}, C)$.*

Proof. Fix $m \in \mathbb{N}_+$ and $X \in \mathcal{E}_{m+1}(\mathcal{M}, C)$. There is $X_1 \in \mathcal{D}_{m+2}(\mathcal{M})$ such that $X = I(X_1, C)$. Let

$$X_2 = \bigcup \{ \{\bar{a}\} \times (b, c) : \bar{a} \in M^{m+1}, b \in C, c \in M \setminus C \text{ and } \{\bar{a}\} \times (b, c) \subseteq X_1 \}.$$

Clearly, $X = I(X_2, C)$ and $X_2 \in \mathcal{D}_{m+2}(\mathcal{M})$. We claim that $\pi_{m+1}^{m+1}[X] = I(\pi_{m+1}^{m+2}[X_2], C)$.

In order to prove that $\pi_{m+1}^{m+1}[X] \subseteq I(\pi_{m+1}^{m+2}[X_2], C)$, fix $\bar{a} \in \pi_{m+1}^{m+1}[X]$. There is $b \in M$ such that $\langle \bar{a}, b \rangle \in X = I(X_2, C)$. So there are $c \in C$ and $d \in M \setminus C$ such that $\{\langle \bar{a}, b \rangle\} \times (c, d) \subseteq X_2$. Hence $\{\bar{a}\} \times (c, d) \subseteq \pi_{m+1}^{m+2}[X_2]$ and $\bar{a} \in I(\pi_{m+1}^{m+2}[X_2], C)$.

For the reverse inclusion, let $\bar{a} \in I(\pi_{m+1}^{m+2}[X_2], C)$. There are $c \in C$ and $d \in M \setminus C$ such that $\{\bar{a}\} \times (c, d) \subseteq \pi_{m+1}^{m+2}[X_2]$. Let $Z = \{\langle x, y \rangle \in M^2 : \langle \bar{a}, y, x \rangle \in X_2\}$. Clearly, $(c, d) \subseteq \pi_2^2[Z]$. By Lemma 2.3, there are $c' \in C$,

$d' \in M \setminus C$ and $e \in M$ such that $(c, d) \times \{e\} \subseteq Z$. Consequently, $\{(\bar{a}, e)\} \times (c, d) \subseteq X_2$. The latter implies that $(\bar{a}, e) \in X$ and $\bar{a} \in \pi_{m+1}^{m+1}[X]$. ■

Lemmas 2.4–2.6 imply that if \mathcal{M} is a weakly o-minimal non-valuational expansion of an ordered group, then:

- (a) all intervals in (M, \leq) belong to $\mathcal{E}_1(\mathcal{M}, C)$;
- (b) every set belonging to $\mathcal{E}_1(\mathcal{M}, C)$ is a union of finitely many convex sets;
- (c) $\{(x, y) \in M^2 : x < y\} \in \mathcal{E}_2(\mathcal{M}, C)$;
- (d) for every $m \in \mathbb{N}_+$, $\emptyset, M^m \in \mathcal{E}_m(\mathcal{M}, C)$ and $(\mathcal{E}(\mathcal{M}, C), \cap, \cup, ^c)$ is a Boolean algebra;
- (e) if $X \in \mathcal{E}_m(\mathcal{M}, C)$, then $X \times M, M \times X \in \mathcal{E}_{m+1}(\mathcal{M}, C)$;
- (f) if $1 \leq i < j \leq m$, then $\{(x_1, \dots, x_m) \in M^m : x_i = x_j\} \in \mathcal{E}_m(\mathcal{M}, C)$;
- (g) if $X \in \mathcal{E}_{m+1}(\mathcal{M}, C)$ and $i \in \{1, \dots, m\}$, then $\pi_i^{m+1}[X] \in \mathcal{E}_m(\mathcal{M}, C)$;
- (h) $C \in \mathcal{E}_1(\mathcal{M}, C)$.

Therefore, by Remark 1.1, there is a weakly o-minimal structure \mathcal{M}' expanding \mathcal{M} such that a set $X \subseteq M^m$ is definable in \mathcal{M}' iff $X \in \mathcal{E}_m(\mathcal{M}, C)$. In the following lemma, $\mathcal{D}_m(\mathcal{M}, C)$ denotes the family of all subsets of M^m which are definable in the structure (\mathcal{M}, C) .

LEMMA 2.7. *Let \mathcal{M} be a weakly o-minimal non-valuational expansion of an ordered group. For every $m \in \mathbb{N}_+$, $\mathcal{E}_m(\mathcal{M}, C) = \mathcal{D}_m(\mathcal{M}, C)$.*

Proof. The inclusion \subseteq is obvious. That $\mathcal{D}_m(\mathcal{M}, C) \subseteq \mathcal{E}_m(\mathcal{M}, C)$ follows easily by induction from the above remark. ■

COROLLARY 2.8. *If \mathcal{M} is a weakly o-minimal non-valuational expansion of an ordered group, then (\mathcal{M}, C) is weakly o-minimal.*

LEMMA 2.9. *Let $\mathcal{M} = (M, \leq, +, \dots)$ be a weakly o-minimal non-valuational expansion of an ordered group $(M, \leq, +)$. Assume that I is a non-empty convex open definable (in \mathcal{M}) set with $I \cap C \neq \emptyset$ and $I \setminus C \neq \emptyset$. Let $f : I \rightarrow \overline{M}^{\mathcal{M}}$ be a definable strongly continuous and strictly monotone function.*

- (a) *If f is strictly increasing, then*
 $\{a \in M : (\exists c \in C)(f(c) > a)\}, \{a \in M : (\exists d \in M \setminus C)(f(d) < a)\}$
is a non-definable and non-valuational cut in $(M, \leq, +)$.
- (b) *If f is strictly decreasing, then*
 $\{a \in M : (\exists d \in M \setminus C)(f(d) > a)\}, \{a \in M : (\exists c \in C)(f(c) < a)\}$
is a non-definable and non-valuational cut in $(M, \leq, +)$.

Proof. As both cases are similar, we will only prove (a). Assume that f is strictly increasing and let

$$C' = \{a \in M : (\exists c \in C)(f(c) > a)\}$$

and

$$D' = \{a \in M : (\exists d \in M \setminus C)(f(d) < a)\}.$$

It is clear that C' and D' are both convex definable sets with $\inf C' = -\infty$ and $\sup D' = +\infty$. The sets C', D' are disjoint since otherwise we would have $f(d) < a < f(c)$ for some $a \in M, c \in C$ and $d \in M \setminus C$.

To show that $C' \cup D' = M$, suppose for a contradiction that there exists an element $a \in M \setminus (C' \cup D')$. This means that $f(c) \leq a \leq f(d)$ whenever $c \in C$ and $d \in M \setminus C$. Note that if there was a $c \in C$ with $f(c) = a$, then there would also be a $c' \in C$ with $c' > c$ and $f(c') > f(c) = a$, a contradiction. So $(\forall c \in C)(f(c) < a)$ and similarly $(\forall d \in M \setminus C)(f(d) > a)$. Now,

$$C = \{x \in M : x \leq \inf I\} \cup \{x \in I : f(x) < a\},$$

which means that C is definable in \mathcal{M} , a contradiction. In this way we have shown that $\langle C', D' \rangle$ is a cut in (M, \leq) . Its non-definability is a consequence of the non-definability of C .

In order to complete the proof, suppose for a contradiction that

$$\inf\{z - y : y \in C', z \in D'\} = \inf\{f(d) - f(c) : c \in C, d \in M \setminus C\} > 0$$

and fix $\varepsilon > 0$ in M such that $\varepsilon < \inf\{f(d) - f(c) : c \in C, d \in M \setminus C\}$. So clearly $f(x_2) - f(x_1) > \varepsilon$ whenever $x_1 \in I \cap C$ and $x_2 \in I \setminus C$. For $x_0 \in I$ define

$$\delta(x_0) = \min\{\varepsilon, \sup\{d \in M : d > 0 \text{ and } |f(x) - f(x_0)| < \varepsilon \\ \text{for } x \in (x_0 - d, x_0 + d) \cap I\}\}.$$

By Lemma 2.1, there are an open interval $J \subseteq I$ and an element $\alpha > 0$ in M such that $J \cap C \neq \emptyset, J \setminus C \neq \emptyset$ and $(\forall x_0 \in J)(\delta(x_0) > \alpha)$. Since the cut $\langle C, M \setminus C \rangle$ is non-valuational, we can choose $x_1 \in J \cap C$ and $x_2 \in J \setminus C$ so that $x_2 - x_1 < \alpha$. We then have $\delta(x_1) > \alpha$ and $|f(x_2) - f(x_1)| < \varepsilon$, which contradicts our choice of ε . ■

COROLLARY 2.10. *Let \mathcal{M} be a weakly o-minimal non-valuational expansion of an ordered group.*

- (a) *The structure (\mathcal{M}, C) is of non-valuational type.*
- (b) *$\text{Th}(\mathcal{M}, C)$ is weakly o-minimal.*

Proof. We already know by Corollary 2.8 that (\mathcal{M}, C) is weakly o-minimal. To demonstrate that (\mathcal{M}, C) is of non-valuational type, fix a cut $\langle D, D' \rangle$ in (M, \leq) which is definable in (\mathcal{M}, C) . We have to show that $\langle D, D' \rangle$ is non-valuational. As there is nothing to do in case $D \in \mathcal{D}_1(\mathcal{M})$, suppose that D is not definable in \mathcal{M} . By Lemma 2.7, there exists a set $X \in \mathcal{D}_2(\mathcal{M})$ such that $D = I(X, C)$. Denote by X' the union of all sets of the form $\{a\} \times (b, c)$, where $a, b, c \in M, b < c$ and $\{a\} \times (b, c) \subseteq X$. Obviously, X' is definable in \mathcal{M} and $I(X', C) = D$. Fix a decomposition \mathcal{C} of M^2 into strong cells

partitioning X' and let I_0, \dots, I_n be an enumeration of all convex sets of the form $\pi_2^2[Y]$, where $Y \in \mathcal{C}$ and $Y \subseteq X'$, such that I_i precedes I_j whenever $i < j \leq n$. Without loss of generality we can assume that the functions appearing in definitions of cells from \mathcal{C} are strictly monotone or constant. Non-definability of D guarantees that there is $k \leq n$ such that I_k is a convex open set intersecting both D and $M \setminus D$. There are strongly continuous functions $f_0, g_0, \dots, f_m, g_m : I_k \rightarrow \overline{M}^{\mathcal{M}} \cup \{-\infty, +\infty\}$ such that

- (a) each of $f_0, g_0, \dots, f_m, g_m$ is either strictly monotone or constant;
- (b) $\bar{f}_i(x) < \bar{g}_i(x)$ for $i \leq m$ and $x \in \bar{I}_k$;
- (c) $\bar{g}_i(x) < \bar{f}_{i+1}(x)$ for $i < m$ and $x \in \bar{I}_k$;
- (d) $X' \cap (I_k \times M) = (f_0, g_0)_{I_k} \cup \dots \cup (f_m, g_m)_{I_k}$.

There is a unique $i \leq m$ such that $\sup I((f_i, g_i)_{I_k}, C) = \sup D \cap I_k$ and exactly one of the following two conditions holds.

- (1) f_i is strictly increasing and $D \cap I_k = I((f_i, +\infty)_{I_k}, C)$.
- (2) g_i is strictly decreasing and $D \cap I_k = I((-\infty, g_i)_{I_k}, C)$.

Since the reasoning is similar in both cases, we will only consider (1). To simplify notation let $f := f_i$. Fix $a \in C$ such that $a > \inf\{f(x) : x \in I_k\}$ and $b \in M \setminus C$ such that $b < \sup\{f(x) : x \in I_k\}$, and let $J = (a, b)$. For $y \in J$ define

$$h(y) = \sup\{x \in M : \langle x, y \rangle \in (f, +\infty)_{I_k}\}.$$

The function h is strictly increasing and strongly continuous. As

$$D = \{x \in M : (\exists x' \in C)(h(x') > x)\},$$

by Lemma 2.9 the cut $\langle D, D' \rangle$ is non-valuational.

The weak o-minimality of $\text{Th}(\mathcal{M}, C)$ now follows from the fact that (\mathcal{M}, C) is a weakly o-minimal non-valuational structure and from Corollary 2.15 from [We07]. ■

Before formulating the main result, we will introduce so called unary non-valuational predicates. Assume that $\mathcal{M} = (M, \leq, \dots)$ is a weakly o-minimal non-valuational expansion of an ordered group and let $X \subseteq M$ be a finite union of convex sets. For $a \in X$ denote by $R(a, X)$ the convex component of X containing a . Similarly, for $a \in M \setminus X$, let $R(a, X)$ be the convex component of $M \setminus X$ containing a . For $a \in M$ define

$$D(a, X) = \bigcup_{\alpha \in R(a, X)} (\alpha, +\infty).$$

Clearly, if $D(a, X) \neq M$, then $\langle M \setminus D(a, X), D(a, X) \rangle$ is a cut in (M, \leq) . A cut $\langle C, D \rangle$ in (M, \leq) is said to be *determined by* X if it is of the form $\langle M \setminus D(a, X), D(a, X) \rangle$ for some $a \in M$.

We say that a set $X \subseteq M$ is a *unary non-valuational predicate* iff X is a union of finitely many convex sets and all cuts determined by X are non-valuational.

THEOREM 2.11. *Assume that \mathcal{M} is a weakly o-minimal non-valuational expansion of an ordered group and \mathcal{N} is an expansion of \mathcal{M} by a family of non-valuational unary predicates. Then \mathcal{N} is of non-valuational type.*

Proof. There is a family of convex open sets $C_i \subsetneq M$, $i \in I$, such that

- for every $i \in I$, C_i is not definable in \mathcal{M} and $\inf C_i = -\infty$;
- the structures \mathcal{N} and $(\mathcal{M}, C_i : i \in I)$ have the same definable sets.

Without loss of generality we can assume that I is finite, in which case the theorem follows easily by induction on $|I|$ from Corollary 2.10(b). ■

Theorem 2.11 actually shows that for weakly o-minimal expansions of ordered groups, the property of a structure having the strong cell decomposition is preserved under expansions by families of unary non-valuational predicates. Having in mind that non-valuational predicates are those which determine non-valuational cuts, one can speak of valuatinal/non-valuational cuts in an arbitrary weakly o-minimal structure with the strong cell decomposition property, not necessarily expanding an ordered group. More precisely, if $\mathcal{M} = (M, \leq, \dots)$ is a weakly o-minimal structure with the strong cell decomposition property, then a cut $\langle C, D \rangle$ in (M, \leq) could be called non-valuational if the structure (\mathcal{M}, C) has the strong cell decomposition property. This gives us notions of being “close” and being “far” for the parts of a cut $\langle C, D \rangle$ in (M, \leq) and it would be interesting to further investigate this topic, probably relating it to the canonical o-minimal extension of a weakly o-minimal structure with the strong cell decomposition property constructed in [We07].

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