

On analytic flows on the torus which are disjoint from systems of probabilistic origin

by

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Abstract. We describe two methods of obtaining analytic flows on the torus which are disjoint from dynamical systems induced by some classical stationary processes.

Introduction. In [9] the notion of ELF-flows has been introduced. These flows are defined as follows: We look at the set of time t automorphisms, $t \in \mathbb{R}$, of an ergodic flow as Markov operators of the underlying L^2 -space, and the flow has the ELF property if the closure of this set in the weak operator topology consists of indecomposable Markov operators. The ELF-property is automatically satisfied for all mixing systems, however in the weak mixing and non-mixing case it seems to be a common property of systems that might be called of probabilistic origin. Indeed, already in [9] it was remarked that Gaussian systems enjoy the ELF-property. Moreover, in the recent paper [3] it is proved that systems induced by (symmetric) α -stable processes as well as Poisson suspension flows are also examples of ELF-systems. This is still generalized in [25] because one of the main consequences of the developed theory of infinitely divisible joinings for systems generated by infinitely divisible stationary processes is that such systems also have the ELF-property.

It is rather clear that the flow induced by a stationary process given by a function of a classical stationary process should also be regarded as one of probabilistic origin. In other words, factors of systems of probabilistic origin are also of probabilistic origin (note that the ELF-property is closed under taking factors). Therefore if we want to exhibit systems different from ELF-flows, it is natural to require that such systems are disjoint (in the sense of Furstenberg [12]) from the ELF-class. We recall that disjoint systems have no nontrivial common factors.

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In the series of papers [9]–[8] some classes of smooth flows on surfaces are shown to be disjoint from ELF-flows. However, no smooth flow appearing in those papers was analytic. In the present paper we will show how to obtain analytic flows on the torus which are disjoint from the ELF-class. Our first method is “generic”—we show that given a positive analytic function (which is not a trigonometric polynomial) on the circle, for a generic set of irrational rotations the resulting special flows are disjoint from all ELF-flows. Our analysis is similar to the one of A. Katok (see Theorem 5.7 in [15]). The second approach uses the a.a.c.c.p. method from [17]. This method has a prescribed set of parameters and, over each irrational α belonging to a certain residual subset of $[0, 1)$, it leads to a construction of a real-valued step cocycle φ which is cohomologous to an analytic zero mean cocycle. Assuming additionally that $\varphi \in L^2$ and $\varphi + C > 0$ for a constant C , as shown in [9], in order to obtain a special flow (under $\varphi + C$) disjoint from the ELF-class we need two conditions to be satisfied: along a certain rigidity time (w_n) for the irrational rotation by α ,

- (A) the sequence of distributions $((\varphi^{(w_n)})_*)_{n \geq 1}$ weakly converges to a non-Dirac measure on \mathbb{R} ,
- (B) the sequence $(\|\varphi^{(w_n)}\|_{L^2})_{n \geq 1}$ is bounded.

This is the condition (B) which introduces a new restriction on the set of α 's over which a given a.a.c.c.p. can be realized. We argue however that such a set of α 's is still residual.

It should be noticed that the approach when one deals with limit distributions along a rigidity sequence is not an original one. For example the condition (A) is satisfied for analytic constructions of cocycles over irrational rotations in [28] (see also [22]). Also, convergence towards particular distributions including Gaussian distributions appears in [4]. It might be expected that certain modifications in [28] should also give condition (B), and therefore should also lead to constructions of analytic special flows disjoint from the ELF-class.

Our second approach, which uses the a.a.c.c.p. method, will allow us to construct an uncountable family $\{\mathcal{T}^{F_\varepsilon}\}_{\varepsilon \in E}$ of flows obtained by analytic changes of times for the linear flow on $\mathbb{R}^2/\mathbb{Z}^2$, i.e. $\mathcal{T}^{F_\varepsilon}$, $\varepsilon \in E$, is determined by

$$(1) \quad \begin{cases} \frac{dx}{dt} = \frac{\alpha}{F_\varepsilon(x, y)}, \\ \frac{dy}{dt} = \frac{1}{F_\varepsilon(x, y)}, \end{cases}$$

such that each flow of the family is disjoint from the ELF-class. In fact, $E = \{0, 1\}^{\mathbb{N}}$ and on E we consider the equivalence relation \sim :

$$\varepsilon \sim \varepsilon' \Leftrightarrow \varepsilon_i = \varepsilon'_i \text{ eventually.}$$

The family $\{\mathcal{T}^{F_\varepsilon}\}_{\varepsilon \in E}$ enjoys additionally the property that $\mathcal{T}^{F_\varepsilon}$ is disjoint from $\mathcal{T}^{F_{\varepsilon'}}$ whenever $\varepsilon \not\sim \varepsilon'$. Note that, by the Liouville theorem, each flow $\mathcal{T}^{F_\varepsilon}$ preserves a “physical” measure, but this measure depends on ε . By considering the whole family of such flows as non-singular flows for the Lebesgue measure on $\mathbb{R}^2/\mathbb{Z}^2$ we show that the map $\varepsilon \mapsto \mathcal{T}^{F_\varepsilon}$ is Borel, whence we are in the situation which first appeared in [5] (see also e.g. [6]). Consequently, by [5], it is impossible to solve the problem of measure-theoretic classification of flows on $\mathbb{R}^2/\mathbb{Z}^2$ preserving a “physical” measure by a countable set of Borel invariants (in fact, we prove the same result also for the spectral classification).

By further modifications of the a.a.c.c.p. method we will obtain weakly mixing analytic flows $(T_t^f)_{t \in \mathbb{R}}$ such that for every probability Borel measure P on \mathbb{R} the integral Markov operator $\int_{\mathbb{R}} T_t^f dP(t)$ belongs to the weak closure of the time t automorphisms T_t^f , $t \in \mathbb{R}$. In particular, using a result of Ageev ([1]), if we denote by σ the reduced maximal spectral type of $T = T_1^f$, then the Gaussian automorphism determined by σ will have simple spectrum and in particular, convolutions of the reduced maximal spectral type are pairwise singular (see [15], [24], [27] for similar results on convolutions). Such Gaussian automorphisms are interesting because they have the GAG property while Foiaş–Strătilă’s theorem fails for them (see [20]).

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1. Preliminaries. Throughout the paper we will identify \mathbb{R}/\mathbb{Z} with $[0, 1)$ (with addition mod 1). Each function defined on $[0, 1)$ will be treated as a 1-periodic function on \mathbb{R} . The Lebesgue measure on $[0, 1)$ will be denoted by μ . For the complex circle $\{z \in \mathbb{C} : |z| = 1\}$ we will use the notation \mathbb{T} .

1.1. Continued fraction expansion. We will now recall some basic facts about the continued fraction expansion of an irrational number. Each $\alpha \in (0, 1)$ has a representation as a continued fraction

$$\alpha = [0; a_1, a_2, \dots],$$

where the positive integers a_n are called the *partial quotients* of α . This expansion is infinite whenever α is irrational. Put

$$\begin{aligned} q_0 &= 1, & q_1 &= a_1, & q_{n+1} &= a_{n+1}q_n + q_{n-1}, \\ p_0 &= 0, & p_1 &= 1, & p_{n+1} &= a_{n+1}p_n + p_{n-1}. \end{aligned}$$

The rationals p_n/q_n are called the *convergents* of α and the following inequality holds:

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}$$

for $n \geq 0$. Moreover, we have

$$q_{n+1}\|q_n\alpha\| + q_n\|q_{n+1}\alpha\| = 1,$$

where $\|v\| = \text{dist}(v, \mathbb{Z})$ for $v \in \mathbb{R}$. The fractional part of a real number v will be denoted by $\{v\}$. Take now an irrational rotation by α on $X = [0, 1)$, $Tx = x + \alpha$. Recall some facts (contained e.g. in [17]) about the dynamics of T . Fix $n \in \mathbb{N}$ even. Then the union of the disjoint Rokhlin towers

$$\begin{aligned} \zeta_n &= \{[0, \{q_n\alpha\}), T[0, \{q_n\alpha\}), \dots, T^{q_{n+1}-1}[0, \{q_n\alpha\})\}, \\ \bar{\zeta}_n &= \{[\{q_{n+1}\alpha\}, 1), T[\{q_{n+1}\alpha\}, 1), \dots, T^{q_n-1}[\{q_{n+1}\alpha\}, 1)\} \end{aligned}$$

coincides with $[0, 1)$. For a subsequence $(n_k) \subset \mathbb{N}$ define

$$I_k = [0, \{a_{2n_k+1}q_{2n_k}\alpha\}), \quad J_t^k = T^{(t-1)q_{2n_k}}[0, \{q_{2n_k}\alpha\})$$

for $t = 1, \dots, a_{2n_k+1}$. We get

$$I_k = \bigcup_{t=1}^{a_{2n_k+1}} J_t^k,$$

and for all $t = 1, \dots, a_{2n_k+1}$,

$$(2) \quad l_k := |J_t^k| < \frac{1}{a_{2n_k+1}q_{2n_k}}.$$

Then $\{I_k, TI_k, \dots, T^{q_{2n_k}-1}I_k\}$ is a Rokhlin tower and if we put

$$\Sigma_k = [0, 1) \setminus \bigcup_{s=0}^{q_{2n_k}-1} T^s I_k$$

then $\Sigma_k = \bigcup \bar{\zeta}_{2n_k} \cup \bigcup_{r=q_{2n_k+1}-q_{2n_k}-1}^{q_{2n_k+1}-1} T^r J_1^k$, hence

$$(3) \quad \mu(\Sigma_k) = q_{2n_k}\|q_{2n_k+1}\alpha\| + q_{2n_k-1}\|q_{2n_k}\alpha\| < 2/a_{2n_k+1}.$$

We will also need the following lemma.

LEMMA 1 (see e.g. [17]). *Given an infinite increasing sequence (q_n) of natural numbers and a positive real-valued function $r = r(q_n)$, the set*

$$\begin{aligned} \{\alpha \in [0, 1) : \text{for infinitely many } n \text{ we have } |\alpha - p_n/q_n| < r(q_n), \\ \text{and } p_n/q_n \text{ are convergents of } \alpha\} \end{aligned}$$

is residual.

1.2. *The class of flows with the ELF-property.* Consider a measurable flow $\mathcal{S} = (S_t)_{t \in \mathbb{R}}$, that is, for each $t \in \mathbb{R}$, S_t is an automorphism on a standard probability Borel space (X, \mathcal{B}, μ) and the corresponding unitary representation of \mathbb{R} on $L^2(X, \mathcal{B}, \mu)$ given by $U_{S_t}(f) = f \circ S_t$ is (weakly) continuous. Depending on the context, S_t may denote U_{S_t} . Assume \mathcal{S} to be ergodic and take another ergodic flow $\mathcal{R} = (R_t)_{t \in \mathbb{R}}$ on (Y, \mathcal{C}, ν) . Now we define a *joining* of \mathcal{S} and \mathcal{R} as an arbitrary probability measure ϱ on

$(X \times Y, \mathcal{B} \otimes \mathcal{C})$ which is $(S_t \times R_t)_{t \in \mathbb{R}}$ -invariant and whose marginals are μ and ν respectively. The set of all joinings between \mathcal{S} and \mathcal{R} will be denoted by $J(\mathcal{S}, \mathcal{R})$. In case $\mathcal{S} = \mathcal{R}$ we write $J(\mathcal{S})$ and speak about *self-joinings*. The subset of ergodic joinings will be denoted by $J^e(\mathcal{S}, \mathcal{R})$ ($J^e(\mathcal{S})$ for ergodic self-joinings).

Having a joining $\varrho \in J(\mathcal{S}, \mathcal{R})$ we define a map $\Phi_\varrho : L^2(X, \mathcal{B}, \mu) \rightarrow L^2(Y, \mathcal{C}, \nu)$ such that for each $f \in L^2(X, \mathcal{B}, \mu)$ and $g \in L^2(Y, \mathcal{C}, \nu)$,

$$\int_Y \Phi_\varrho(f)g \, d\nu = \int_{X \times Y} f \otimes g \, d\varrho.$$

Notice that we get a Markov operator $\Phi_\varrho : L^2(X, \mathcal{B}, \mu) \rightarrow L^2(Y, \mathcal{C}, \nu)$, i.e. a bounded linear operator satisfying

$$\Phi_\varrho 1 = \Phi_\varrho^* 1 = 1 \quad \text{and} \quad \Phi_\varrho f \geq 0 \quad \text{whenever} \quad f \geq 0.$$

Conversely, having a Markov operator $\Phi : L^2(X, \mathcal{B}, \mu) \rightarrow L^2(Y, \mathcal{C}, \nu)$ we can obtain a unique measure on $(X \times Y, \mathcal{B} \otimes \mathcal{C})$ whose projections on X and Y are equal to μ and ν respectively: indeed, we put

$$\varrho(A \times B) = \int_B \Phi(\chi_A) \, d\nu$$

for all $A \in \mathcal{B}$ and $B \in \mathcal{C}$. Moreover, the $(S_t \times R_t)_{t \in \mathbb{R}}$ -invariance of ϱ is equivalent to

$$(4) \quad \Phi \circ U_{S_t} = U_{R_t} \circ \Phi$$

for each $t \in \mathbb{R}$. Thus we can identify the set of all Markov operators satisfying (4) (denoted by $\mathcal{J}(\mathcal{S}, \mathcal{R})$) with the set $J(\mathcal{S}, \mathcal{R})$. The set of Markov operators corresponding to ergodic joinings will be denoted by $\mathcal{J}^e(\mathcal{S}, \mathcal{R})$ (such Markov operators are *indecomposable*, that is, they are extremal elements of the simplex $\mathcal{J}(\mathcal{S}, \mathcal{R})$). The notation $\mathcal{J}(\mathcal{S})$ and $\mathcal{J}^e(\mathcal{S})$ is used for self-joinings. Note that $U_{S_t} \in \mathcal{J}^e(\mathcal{S})$, $t \in \mathbb{R}$. Denote by \int the Markov operator corresponding to the product measure, i.e. $\int(f) = \int_X f \, d\mu$.

Following [12], \mathcal{S} and \mathcal{R} are called *disjoint (in the sense of Furstenberg)* if $J(\mathcal{S}, \mathcal{R}) = \{\mu \otimes \nu\}$.

Recall also (see [13]) that whenever the maximal spectral types $\sigma_{\mathcal{S}}$ and $\sigma_{\mathcal{R}}$ of the corresponding Koopman representations on $L^2_0(X, \mathcal{B}, \mu)$ and $L^2_0(Y, \mathcal{C}, \nu)$ are mutually singular then \mathcal{S} and \mathcal{R} are disjoint.

Finally, following [9], we say that an ergodic flow $\mathcal{S} = (S_t)_{t \in \mathbb{R}}$ has the *ELF-property* (briefly, is an *ELF-flow*) if $\overline{\{U_{S_t} : t \in \mathbb{R}\}} \subset \mathcal{J}^e(\mathcal{S})$ (the closure in the weak operator topology).

1.3. Special flows and ELF-property. Denote by $\mathcal{P}(\mathbb{R})$ the space of all Borel probability measures on \mathbb{R} . Recall that a sequence (σ_n) of measures

converges weakly in $\mathcal{P}(\mathbb{R})$ to a measure σ if for each function $h \in CB(\mathbb{R})$,

$$\int_{\mathbb{R}} h d\sigma_n \rightarrow \int_{\mathbb{R}} h d\sigma.$$

Let (X, \mathcal{B}, μ) be a standard probability Borel space. Given a measurable function $f : X \rightarrow \mathbb{R}$, we denote by $f_*\mu$ the image of μ via f , i.e. $f_*\mu(A) = \mu(f^{-1}(A))$ for an arbitrary Borel set A in \mathbb{R} .

Note that whenever $(f_k), (g_k)$ are sequences of real measurable functions on X satisfying $(f_k)_*\mu \rightarrow \sigma$ as $k \rightarrow \infty$ for some $\sigma \in \mathcal{P}(\mathbb{R})$ and $\mu\{x \in X : f_k(x) \neq g_k(x)\} \rightarrow 0$ as $k \rightarrow \infty$, then $(g_k)_*\mu \rightarrow \sigma$ as $k \rightarrow \infty$.

Now we recall basic facts concerning special flows.

Let T be an ergodic automorphism of (X, \mathcal{B}, μ) . We will denote by λ Lebesgue measure on \mathbb{R} . Assume $f : X \rightarrow \mathbb{R}$ to be a measurable positive function with $\int_X f d\mu < \infty$.

We define an \mathbb{R} -action, denoted by T^f , on the space $(X^f, \mathcal{B}^f, \mu^f)$ where

$$X^f = \{(x, t) \in X \times \mathbb{R} : 0 \leq t < f(x)\},$$

\mathcal{B}^f is the restriction of $\mathcal{B} \otimes \mathcal{B}(\mathbb{R})$ and μ^f is the restriction to X^f of the product measure $\mu \otimes \lambda$ of $X \times \mathbb{R}$. This action, denoted by $T^f = ((T^f)_t)_{t \in \mathbb{R}}$, will be called the *special flow* built from T and the roof function f and it is as follows. Each point from X^f moves vertically upwards with unit speed and as soon as it reaches the graph of f , the point $(x, f(x))$ is identified with $(Tx, 0)$ (see e.g. [2, Chapter 11]). Put

$$(5) \quad f^{(k)}(x) = \begin{cases} f(x) + f(Tx) + \cdots + f(T^{k-1}x) & \text{for } k > 0, \\ 0 & \text{for } k = 0, \\ -(f(T^k x) + \cdots + f(T^{-1}x)) & \text{for } k < 0. \end{cases}$$

Only functions f satisfying $f \geq c > 0$ will be considered. It follows that for a.e. $x \in X$,

$$f^{(k)}(x) \geq ck, \quad k \in \mathbb{N}.$$

Recall that a sequence (q_n) is said to be a *rigidity time* for the automorphism T if $U_T^{q_n} \rightarrow \text{Id}$ as $n \rightarrow \infty$ in the weak (or, equivalently, strong) operator topology.

We now recall some results from [9] that will be needed in what follows.

PROPOSITION 1 ([9]). *Suppose that $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ is an ergodic flow on (Y, \mathcal{C}, ν) such that for a sequence $(t_n) \subset \mathbb{R}$ with $t_n \rightarrow \infty$,*

$$T_{t_n} \rightarrow \int_{\mathbb{R}} T_s dP(s),$$

where $P \in \mathcal{P}(\mathbb{R})$. Then

- (i) \mathcal{T} is disjoint from all mixing flows,

- (ii) \mathcal{T} is disjoint from all weakly mixing ELF-flows whenever P is not a Dirac measure.

PROPOSITION 2 ([9]). Let (q_n) be a rigidity sequence for T . Suppose that $f \in L^2(X, \mu)$, $f \geq c > 0$ and $\int_X f(x) d\mu(x) = d < \infty$. Put $f_0 = f - d$. Moreover, suppose that the sequence $(f_0^{(q_n)})$ is bounded in $L^2(X, \mu)$ and $(f_0^{(q_n)})_*\mu \rightarrow P$ weakly in $\mathcal{P}(\mathbb{R})$. Then

$$(T^f)_{dq_n} \rightarrow \int_{\mathbb{R}} (T^f)_{-t} dP(t).$$

From the above propositions we obtain the following.

COROLLARY 1 ([9]). Under the assumptions of Proposition 2 suppose additionally that T^f is weakly mixing and that the limit measure P is not a Dirac measure. Then the special flow T^f is disjoint from all ELF-flows.

Recall also that if (q_n) is a rigidity time for T and $(f_0^{(q_n)})_*\mu \rightarrow P$ with P continuous then the special flow T^f is weakly mixing (see [21]).

We will need some disjointness results concerning flows having integral Markov operators in the weak closure of their time t automorphisms, $t \in \mathbb{R}$.

LEMMA 2. Let $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ and $\mathcal{S} = (S_t)_{t \in \mathbb{R}}$ be \mathbb{R} -actions on (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) respectively. Assume additionally that \mathcal{T} is weakly mixing and \mathcal{S} is ergodic. Moreover, suppose that for a sequence $(t_n) \subset \mathbb{R}$ with $t_n \rightarrow \infty$,

$$T_{t_n} \rightarrow \int_{\mathbb{R}} T_t dP(t) \quad \text{and} \quad S_{t_n} \rightarrow \int_{\mathbb{R}} S_t dQ(t).$$

If $P \neq Q$ then the flows \mathcal{T} and \mathcal{S} are disjoint in the sense of Furstenberg.

Proof. Let $J : L^2(X, \mathcal{B}, \mu) \rightarrow L^2(Y, \mathcal{C}, \nu)$ be the Markov operator corresponding to an ergodic joining between \mathcal{T} and \mathcal{S} . Thus

$$(6) \quad J \circ T_t = S_t \circ J \quad \text{for all } t \in \mathbb{R}.$$

By taking $t = t_n$ and passing to the limits we obtain

$$J \circ \int_{\mathbb{R}} T_t dP(t) = \left(\int_{\mathbb{R}} S_t dQ(t) \right) \circ J.$$

Hence from (6) we have

$$(7) \quad \int_{\mathbb{R}} J \circ T_t dP(t) = \int_{\mathbb{R}} J \circ T_t dQ(t).$$

Suppose that $J \neq \int$. Denote by P' and Q' the images via the (continuous) mapping

$$(8) \quad t \mapsto J \circ T_t$$

of the measures P and Q respectively. We obtain

$$\int_{\mathbb{R}} J \circ T_t dP(t) = \int_{\mathcal{J}^e(\mathcal{S}, \mathcal{T})} \Phi dP'(\Phi),$$

$$\int_{\mathbb{R}} J \circ T_t dQ(t) = \int_{\mathcal{J}^e(\mathcal{S}, \mathcal{T})} \Phi dQ'(\Phi).$$

In view of (7) and from the uniqueness of the ergodic decomposition we have $P' = Q'$. Now using weak mixing of \mathcal{T} , we conclude that the continuous map (8) is an injection, so (by the Suslin theorem) $P = Q$. The result follows. ■

LEMMA 3. *Under the assumptions of Lemma 2, if $P \neq \delta_0$ and $Q = \delta_0$, then the flows \mathcal{T} and \mathcal{S} are spectrally disjoint.*

Proof. Assume that the (reduced) maximal spectral types $\sigma_{U_{\mathcal{T}}}$ and $\sigma_{U_{\mathcal{S}}}$ of \mathcal{T} and \mathcal{S} are not mutually singular. Then there exists a probability measure σ such that $\sigma \ll \sigma_{U_{\mathcal{T}}}$ and $\sigma \ll \sigma_{U_{\mathcal{S}}}$. We have $\sigma = \sigma_{f, U_{\mathcal{T}}} = \sigma_{g, U_{\mathcal{S}}}$ for some functions $f \in L^2_0(X, \mathcal{B}, \mu)$ and $g \in L^2_0(Y, \mathcal{C}, \nu)$ and $\|f\| = \|g\| = 1$. From the assumptions we obtain

$$\hat{\sigma}(t_n) = \langle T_{t_n} f, f \rangle \rightarrow \int_{\mathbb{R}} \langle T_t f, f \rangle dP(t),$$

$$\hat{\sigma}(t_n) = \langle S_{t_n} g, g \rangle \rightarrow \int_{\mathbb{R}} \langle S_t g, g \rangle d\delta_0(t) = \|g\|^2 = 1.$$

A convexity argument shows immediately that $\langle T_t f, f \rangle = 1$ for P -a.e. $t \in \mathbb{R}$. Now the equality in the Schwarz inequality and the weak mixing of \mathcal{T} yield $f = 0$. ■

REMARK 1. Note that the proof of Corollary 5.2 from [9] gives rise to the following: If for flows $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$, $\mathcal{S} = (S_t)_{t \in \mathbb{R}}$, one of which is weakly mixing, we have

$$T_{t_n} \rightarrow \int T_t dP(t), \quad S_{t_n} \rightarrow \int S_t dQ(t)$$

for some $t_n \rightarrow \infty$, $P \neq Q$ and P and Q have compact supports then \mathcal{T} and \mathcal{S} are spectrally disjoint.

Indeed, otherwise, as in the proof of the above lemma, we obtain

$$\hat{\sigma}(t_n) \rightarrow \int_{\mathbb{R}} \langle T_t f, f \rangle dP(t) = \int_{\mathbb{R}} \langle S_t g, g \rangle dQ(t).$$

But

$$\int_{\mathbb{R}} \langle T_t f, f \rangle dP(t) = \int_{\mathbb{R}} \hat{P}(s) d\sigma_{f, U_{\mathcal{T}}}(s) = \int_{\mathbb{R}} \hat{P}(s) d\sigma(s)$$

and hence $\int_{\mathbb{R}} \hat{P}(s) d\sigma(s) = \int_{\mathbb{R}} \hat{Q}(s) d\sigma(s)$. Moreover, we can repeat the same reasoning for each probability measure $\eta \ll \sigma$ and therefore $\hat{P}(s) = \hat{Q}(s)$ for σ -a.e. $s \in \mathbb{R}$. Since P and Q have compact supports, \hat{P} and \hat{Q} have

analytic extensions to the whole complex plane. Since σ is continuous, $\widehat{P} = \widehat{Q}$ everywhere, and hence $P = Q$.

Finally, notice that the set of integral Markov operators is closed under the weak convergence of measures: if P_n converges weakly to P in $\mathcal{P}(\mathbb{R})$ then

$$(9) \quad \int_{\mathbb{R}} T_t dP_n(t) \rightarrow \int_{\mathbb{R}} T_t dP(t).$$

Indeed, we only need to check that for each $f, g \in L^2(X, \mathcal{B}, \mu)$,

$$\begin{aligned} \left\langle \left(\int_{\mathbb{R}} T_t dP_n(t) \right) f, g \right\rangle &= \int_{\mathbb{R}} \langle T_t f, g \rangle dP_n(t) \\ &\rightarrow \int_{\mathbb{R}} \langle T_t f, g \rangle dP(t) = \left\langle \left(\int_{\mathbb{R}} T_t dP(t) \right) f, g \right\rangle, \end{aligned}$$

which is immediate as the function $\mathbb{R} \ni t \mapsto \langle T_t f, g \rangle$ is continuous and bounded.

2. Analytic special flows disjoint from the ELF-class

2.1. Generic method. Let $f : [0, 1) \rightarrow \mathbb{R}$ be a real-analytic function. Consider its Fourier expansion

$$f(x) = \sum_{n=-\infty}^{\infty} b_n e^{2\pi i n x}$$

where $b_n = o(A^{|n|})$ for some $0 < A < 1$, $n \in \mathbb{Z}$. We additionally require that $\int f = 0$ ($b_0 = 0$) and that $\#\{n \in \mathbb{Z} : b_n \neq 0\} = \infty$ (so that f is not a trigonometric polynomial).

LEMMA 4. *There exists an infinite sequence (q_n) such that $b_{q_n} \neq 0$ and*

$$\lim_{n \rightarrow \infty} \frac{|b_{q_n}|}{|b_{q_n}| + |b_{2q_n}| + \dots} = 1.$$

Proof. Choose $q_1 \geq 1$ and $\delta_1 > 0$ so that

$$(10) \quad \frac{|b_{q_1}|}{A^{q_1}} \geq \delta_1 \quad \text{and} \quad \frac{|b_s|}{A^s} < \delta_1 \quad \text{for all } s > q_1.$$

Then

$$\sum_{i=2}^{\infty} |b_{iq_1}| < \sum_{i=2}^{\infty} A^{iq_1} \delta_1 = \delta_1 A^{q_1} \frac{A^{q_1}}{1 - A^{q_1}} \leq |b_{q_1}| \frac{A^{q_1}}{1 - A^{q_1}}.$$

Now choose $q_2 > q_1$ and $\delta_2 > 0$ so that (10) holds with q_2 instead of q_1 and δ_2 instead of δ_1 . It follows that we can choose an increasing sequence (q_n) for which

$$\sum_{i=2}^{\infty} |b_{iq_n}| \leq |b_{q_n}| \frac{A^{q_n}}{1 - A^{q_n}} = o(|b_{q_n}|),$$

and the proof is complete. ■

For a fixed integer $q \geq 1$ consider now a $1/q$ -periodic function $f_q : [0, 1) \rightarrow \mathbb{R}$ given by

$$f_q(x) = f(x) + f\left(x + \frac{1}{q}\right) + \dots + f\left(x + \frac{q-1}{q}\right).$$

By the Koksma inequality, $f_q \rightarrow 0$ uniformly as $q \rightarrow \infty$, in particular

$$(f_q)_*\mu \rightarrow \delta_0 \quad \text{as } q \rightarrow \infty.$$

Moreover,

$$f_q(x) = q \sum_{l=-\infty}^{\infty} b_{lq} e^{2\pi i l q x}.$$

We will consider functions of the form $s f_q$ for some positive integer s . We have

$$s f_q(x) = s q 2 \operatorname{Re}(b_q e^{2\pi i q x}) + s q 2 \operatorname{Re}(b_{2q} e^{2\pi i 2q x} + b_{3q} e^{2\pi i 3q x} + \dots).$$

Since $b_q \rightarrow 0$ exponentially fast, $q b_q \rightarrow 0$ as $q \rightarrow \infty$. Hence $q_n b_{q_n} \rightarrow 0$, where (q_n) is the sequence from Lemma 4. We now select a sequence (s_n) of natural numbers so that

$$(11) \quad s_n q_n |b_{q_n}| \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

It follows from (11) and Lemma 4 that

$$(12) \quad s_n q_n (|b_{2q_n}| + |b_{3q_n}| + \dots) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

PROPOSITION 3. *The sequence of distributions $(s_n f_{q_n})_* \mu$ converges weakly to σ , where σ is an absolutely continuous measure on $[-2, 2]$. Furthermore*

$$\|s_n f_{q_n}\|_{C([0,1])} \leq 3 \quad \text{for } n \geq n_0.$$

Proof. Since $|s_n q_n b_{q_n}| \rightarrow 1$ as $n \rightarrow \infty$, the sequence of distributions $(s_n q_n b_{q_n} e^{2\pi i q_n x})_* \mu$ tends to Lebesgue measure on \mathbb{T} . It follows that $(2 \operatorname{Re}(s_n q_n b_{q_n} e^{2\pi i q_n x}))_* \mu \rightarrow \tilde{\sigma}$, where $\tilde{\sigma}$ is the image of Lebesgue measure from \mathbb{T} via the map $z \mapsto 2 \operatorname{Re} z$. In view of (12),

$$(13) \quad 2 \operatorname{Re}(s_n q_n b_{2q_n} e^{2\pi i 2q_n x} + s_n q_n b_{3q_n} e^{2\pi i 3q_n x} + \dots) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

uniformly, so the corresponding sequence of distributions converges to the Dirac measure δ_0 . In particular $s_n f_{q_n}$ and $2 \operatorname{Re}(s_n q_n b_{q_n} e^{2\pi i q_n x})$ have the same limit distributions as their difference tends to zero in measure. Therefore $(s_n f_{q_n})_* \mu \rightarrow \tilde{\sigma}$. The uniform bound of $s_n f_{q_n}$ is obvious from (13) and (11). ■

In view of Lemma 1 the set of α 's for which $|\alpha - p_n/q_n| < r(q_n)$ for infinitely many n 's, where $r(q_n) = o(|b_{q_n}|^2)$ and p_n/q_n are the convergents of α , is residual. For an irrational α in this set, $(s_n q_n)$ is a rigidity time for $Tx = x + \alpha$. Moreover, for some infinite subsequence (n_k) we get

$$(14) \quad |s_{n_k} f_{q_{n_k}}(x) - f^{(s_{n_k} q_{n_k})}(x)| \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

uniformly in $x \in [0, 1)$. Indeed (to simplify notation we write q and s instead of q_{n_k} and s_{n_k} respectively),

$$\begin{aligned}
 sf_q(x) - f^{(sq)}(x) &= \sum_{j=1}^s \left(\left[f(x) + f\left(x + \frac{1}{q}\right) + \cdots + f\left(x + \frac{q-1}{q}\right) \right] \right. \\
 &\quad - [f(x + (j-1)q\alpha) + f(x + (j-1)q\alpha + \alpha) \\
 &\quad \left. + \cdots + f(x + (j-1)q\alpha + (q-1)\alpha)] \right).
 \end{aligned}$$

Given $1 \leq j \leq s$ and $0 \leq k \leq q-1$, for some $i \in \{(j-1)q, \dots, jq-1\}$ we have $k/q = ip/q \pmod{1}$ and then

$$\left| f\left(x + \frac{k}{q}\right) - f(x + i\alpha) \right| \leq \|f'\|_{C([0,1])} i \left| \frac{p}{q} - \alpha \right| \leq \|f'\|_{C([0,1])} sqr(q).$$

Now in view of (11),

$$|sf_q(x) - f^{(sq)}(x)| \leq sq \|f'\|_{C([0,1])} sqr(q) = \|f'\|_{C([0,1])} (sq|b_q|)^2 \frac{r(q)}{|b_q|^2} \rightarrow 0$$

as $q \rightarrow \infty$, which completes the proof of (14).

Thus we have proved the following.

PROPOSITION 4. *There exists a residual subset of $[0, 1)$ such that whenever an irrational α belongs to this set there exists a sequence (q_n) (a subsequence of denominators of α) such that for some sequence (s_n) , $(s_n q_n)$ is a rigidity time for T , the sequence of distributions $(f^{(s_n q_n)})_* \mu$ tends to an absolutely continuous measure on $[-2, 2]$ and the sequence of norms $(\|f^{(s_n q_n)}\|_{C([0,1])})$ is bounded. ■*

Take now the function $\tilde{f} = f + d$ for some d so that $\tilde{f} > 0$. Then $\tilde{f}_0 = f$ and \tilde{f} (with the sequence $(s_n q_n)_n$) satisfies the assumptions of Corollary 1.

2.2. *The a.a.c.c.p. (“almost analytic cocycle construction procedure”).* First, we briefly recall the a.a.c.c.p. from [17], with a small modification for the purpose of this paper (see Remark 2 below).

The aim of an a.a.c.c.p. is to construct a (non-trivial) cocycle of the form $\varphi = \sum_{k=1}^\infty \varphi_k$, where φ_k 's are step-cocycles which are coboundaries and φ is cohomologous to some analytic cocycle f .

We are given a collection of the following parameters: a sequence $(M_k) \subset \mathbb{N}$ and an array $((d_{k1}, \dots, d_{kM_k}))_{k \geq 1}$ with $d_{ki} \in \mathbb{R}$ such that for each k ,

$$(15) \quad \sum_{i=1}^{M_k} d_{ki} = 0$$

and

$$(16) \quad M_k \geq 2 \left(\sum_{i=1}^{k-1} D_i M_i \right)^2$$

where $D_k = \max_{1 \leq i \leq M_k} |d_{ki}|$. Then we choose a sequence $(\varepsilon_k) \subset \mathbb{R}_+$ such that $\sum_{k=1}^{\infty} \sqrt{\varepsilon_k} M_k < \infty$, $\sum_{k=1}^{\infty} \varepsilon_k < 1$ and $\varepsilon_k < 1/D_k^2$ for $k = 1, 2, \dots$. The last needed parameter is a real number $A > 1$.

Having collected the above parameters, we say that *the a.a.c.c.p. is realized over an irrational number $\alpha = [0; a_1, a_2, \dots]$* if there exists a strictly increasing sequence $(n_k) \subset \mathbb{N}$ satisfying

$$(17) \quad A^{N_k} \frac{D_k M_k}{a_{2n_k+1} q_{2n_k}} < \frac{1}{2^k},$$

where N_k is the degree of a real trigonometric polynomial

$$P_k(t) = \sum_{s=-N_k}^{N_k} b_s^{(k)} e^{2\pi i s t}$$

such that $\int_0^1 P_k(t) dt = 1$, $P_k \geq 0$, and $P_k(t) < \varepsilon_k$ for $t \in (\eta_k/2, 1)$ (notice that $|b_s^{(k)}| \leq \|P_k\|_{L^1} = 1$), where the η_k 's satisfy $4M_k \eta_k < \varepsilon_k / q_{2n_k}$. Furthermore, we require that $a_{2n_k+1} > 2$ and $1/a_{2n_k+1} q_{2n_k} < \frac{1}{2} \eta_k$.

Now we can define a cocycle $\varphi = \sum_{k=1}^{\infty} \varphi_k$ in the following way. In the interval $I_k = [0, \{a_{2n_k+1} q_{2n_k} \alpha\})$ we will choose pairwise disjoint intervals $\omega_{k1}, \dots, \omega_{kM_k}$ of the same length $\lambda_k \in (\eta_k, 2\eta_k)$ such that each ω_{ki} consists of $e_k \geq 3$ (odd) consecutive subintervals J_t^k . For $i = 1, \dots, M_k - 1$ the interval $\omega_{k,i+1}$ follows ω_{ki} in the natural ordering of $[0, 1)$ and, in general, they are separated by a certain number of consecutive subintervals J_j^k . Denoting by $J_{s_k,i}^k$ the central subinterval in ω_{ki} define

$$\varphi_k(x) = \begin{cases} d_{ki} & \text{for } x \in J_{s_k,i}^k, \quad i = 1, \dots, M_k, \\ 0 & \text{otherwise.} \end{cases}$$

Since $I_{k+1} \subset J_1^k$, the φ_k 's have disjoint supports and $\varphi = \sum_{k=1}^{\infty} \varphi_k$ is well defined.

REMARK 2. In comparison with [17] we slightly reduced the number of conditions imposed on a.a.c.c.p. (for example (8) in [17] follows from (7) and (10) there, $\|P_k\|_{\mathcal{F}} = 1$), however we added a new condition (16). In what follows we will still add some extra conditions on M_k and d_{ki} . It is however clear that the argument contained in the proof of Proposition 1 in [17] persists, and the set of α 's over which an a.a.c.c.p. (in the sense of the present paper) is realized is a G_δ and dense subset of $[0, 1)$. Recall that the argument used in [17] was a simple observation that the a.a.c.c.p. can be

carried out over an $\alpha = [0; a_1, a_2, \dots]$ whenever for a subsequence (n_k) the a_{2n_k+1} 's are sufficiently large. This also says that we can find an α for which we can require the a_{2n_k+1} 's to have a special form (for example be multiples of M_k , etc.)

REMARK 3. Assume that an a.a.c.c.p. with an array (d_{ki}) is realized over an irrational α . We now take another a.a.c.c.p. which has the same sequences of parameters except for an array (d_{ki}) which is now (d'_{ki}) . Then the second a.a.c.c.p. is realized over the same α provided that $D'_k \leq D_k$, $k \geq 1$. In other words, to define α we need the sequence (D_k) rather than the sequence $((d_{k1}, \dots, d_{kM_k}))$.

Moreover, the following theorem holds true.

THEOREM 1 ([17]). *Suppose that for an irrational α an a.a.c.c.p. is realized. Then there exists an analytic cocycle $f : [0, 1) \rightarrow \mathbb{R}$ which is α -cohomologous to φ .*

REMARK 4. In the proof of Theorem 1 in [17] an explicit form of the analytic cocycle f (cohomologous to φ) has been given. Namely

$$(18) \quad f(t) = \sum_{k=1}^{\infty} l_k \sum_{r=1}^{M_k} d_{kr} P_k(t - m_{kr}\alpha),$$

where m_{kr} is determined by $J_{s_{k,r}}^k = T^{m_{kr}} J_1^k$. Recall that $l_k = |J_1^k|$. Thus the Fourier expansion of f is given by

$$(19) \quad f(t) = \sum_{s=-\infty}^{\infty} \left(\sum_{\{k: |s| \leq N_k\}} b_s^{(k)} l_k \sum_{r=1}^{M_k} d_{kr} e^{-2\pi i s m_{kr} \alpha} \right) e^{2\pi i s t}.$$

This allows us to show (see the lemma below) that the analytic functions f obtained from the a.a.c.c.p. enjoy a universal bound which depends only on $A > 1$.

LEMMA 5. *For each f given by (18), $|f(t)| \leq 2/(A - 1)$ for each $t \in [0, 1)$.*

Proof. From (19), (2) and (17), for $s \neq 0$ we have

$$\begin{aligned} |\widehat{f}_s| &= \left| \sum_{\{k: |s| \leq N_k\}} l_k b_s^{(k)} \sum_{r=1}^{M_k} d_{kr} e^{-2\pi i s m_{kr} \alpha} \right| \\ &\leq \sum_{\{k: |s| \leq N_k\}} l_k M_k D_k \frac{A^{N_k}}{A^{N_k}} \leq \frac{1}{A^{|s|}} \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{A^{|s|}}. \end{aligned}$$

Thus

$$|f(t)| \leq \sum_{s=-\infty}^{\infty} |\widehat{f}_s| \leq 2 \sum_{s=1}^{\infty} \frac{1}{A^s} = \frac{2}{A-1}. \quad \blacksquare$$

We will also need the following property of the a.a.c.c.p.

LEMMA 6 ([17]). *For an arbitrary a.a.c.c.p., an arbitrary α over which it is realized and for any k , the cocycle φ is constant on each interval $T^i(I_k)$, $i = 1, \dots, q_{2n_k} - 1$. Moreover, if we put $b_{k,i} = \varphi|_{T^i(I_k)}$ then*

$$\sum_{i=1}^{q_{2n_k}-1} b_{k,i} = 0.$$

Now, we will carry out an a.a.c.c.p., selecting the parameters so that the sequence of distributions $(\varphi^{(b_k q_{2n_k})})_* \mu$ tends to a continuous measure on $[-1, 1]$, where $(b_k q_{2n_k})$ is a certain rigidity time for T .

Fix an arbitrary probability continuous measure ν on $[-1, 1]$. We will put some extra conditions on M_k, d_{ki} :

$$(20) \quad |d_{ki}| \leq 1, \quad i = 1, \dots, M_k, k \geq 1;$$

$$(21) \quad M_k^{-1} \sum_{i=1}^{M_k} \delta_{d_{ki}} \rightarrow \nu \quad \text{weakly in } \mathcal{P}([-1, 1]) \text{ as } k \rightarrow \infty,$$

in particular (from (21))

$$(22) \quad M_k \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

Let α be an irrational over which the a.a.c.c.p. is carried out. We require additionally that a_{2n_k+1} is a multiple of M_k , more precisely

$$(23) \quad a_{2n_k+1} = M_k b_k,$$

where b_k is odd, $k \geq 1$.

We now define φ_k 's by putting

$$s_{k,i} = (i - 1)b_k + \frac{b_k + 1}{2}, \quad i = 1, \dots, M_k,$$

and then defining

$$\varphi_k(x) = d_{ki} \quad \text{if } x \in J_{s_{k,i}}^k, \quad i = 1, \dots, M_k,$$

and 0 elsewhere (notice that $b_k l_k$ is of order

$$\frac{a_{2n_k+1}}{M_k} \cdot \frac{1}{a_{2n_k+1} q_{2n_k}} = \frac{1}{M_k q_{2n_k}}$$

while η_k is at most of order $\varepsilon_k / M_k q_{2n_k}$, so the intervals $\omega_{k1}, \dots, \omega_{kM_k}$ are disjoint).

Set $\xi_j = J_{(j-1)b_k+1}^k \cup \dots \cup J_{j b_k}^k$, $j = 1, \dots, M_k$ (that is, ξ_1, \dots, ξ_{M_k} are consecutive intervals of equal length $b_k l_k$ partitioning I_k).

Consider now the union

$$\bigcup_{s=1}^{q_{2n_k}} T^{s - q_{2n_k} (b_k + 1)/2} \xi_j =: U_{k,j}, \quad j = 1, \dots, M_k.$$

We obtain

$$(24) \quad \varphi_k^{(b_k q_{2n_k})}(x) = d_{kj}, \quad x \in U_{k,j}, \quad j = 1, \dots, M_k.$$

Recall that $\Sigma_k = [0, 1) \setminus \bigcup_{s=0}^{q_{2n_k}-1} T^s I_k$ and from (3) and (22),

$$(25) \quad \mu(\Sigma_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Observe that $\varphi_k^{(b_k q_{2n_k})} = 0$ on the complement of the set $\bigcup_{j=1}^{M_k} U_{k,j}$ which has the same measure as Σ_k .

Consider now the sequence of distributions $(\varphi_k^{(b_k q_{2n_k})})_* \mu$. We will show that $(\varphi_k^{(b_k q_{2n_k})})_* \mu \rightarrow \nu$ as $k \rightarrow \infty$. Indeed, observe that

$$(\varphi_k^{(b_k q_{2n_k})})_* \mu = \sum_{j=1}^{M_k} \mu(U_{k,j}) \delta_{d_{kj}} + \mu(\Sigma_k) \delta_0,$$

where $\mu(U_{k,j}) = (1 - \mu(\Sigma_k))/M_k$. Now using (25) and (21) we get

$$(\varphi_k^{(b_k q_{2n_k})})_* \mu = \frac{1 - \mu(\Sigma_k)}{M_k} \sum_{j=1}^{M_k} \delta_{d_{kj}} + \mu(\Sigma_k) \delta_0 \rightarrow \nu \quad \text{as } k \rightarrow \infty.$$

So we have constructed a cocycle $\varphi = \sum_{k=1}^{\infty} \varphi_k$ for which the sequence of distributions $(\varphi_k^{(b_k q_{2n_k})})_* \mu$ converges weakly to ν .

We will now show that

$$(26) \quad \mu(\{x \in [0, 1) : \varphi_k^{(b_k q_{2n_k})}(x) \neq \varphi^{(b_k q_{2n_k})}(x)\}) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Indeed, putting $w_k = b_k q_{2n_k}$ we get

$$\varphi^{(w_k)}(x) = \sum_{i=1}^{\infty} \varphi_i^{(w_k)}(x).$$

Notice that the support of $\sum_{i \geq k+1} \varphi_i$ is contained in $I_{k+1} \subset J_1^k$, therefore it follows from (2) and (22) that

$$\mu\left(\left\{x \in [0, 1) : \sum_{i \geq k+1} \varphi_i^{(w_k)}(x) \neq 0\right\}\right) \leq l_k b_k q_{2n_k} < \frac{b_k}{a_{2n_k+1}} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

On the other hand, $\sum_{i=1}^{k-1} \varphi_i$ is zero on I_k , which combined with Lemma 6 implies

$$(27) \quad \mu\left(\left\{x \in [0, 1) : \sum_{i=1}^{k-1} \varphi_i^{(w_k)}(x) \neq 0\right\}\right) \leq \mu(\Sigma_k) + \mu\left(\bigcup_{s=0}^{w_k-1} T^s J_{a_{2n_k+1}-b_k+1}^k\right),$$

hence

$$\mu\left(\left\{x \in [0, 1) : \sum_{i=1}^{k-1} \varphi_i^{(w_k)}(x) \neq 0\right\}\right) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Thus (26) follows. We hence proved that

$$(\varphi^{(b_k q_{2n_k})})_* \mu \rightarrow \nu \quad \text{as } k \rightarrow \infty.$$

Clearly

$$\|b_k q_{2n_k} \alpha\| \leq \frac{b_k}{q_{2n_k+1}} \leq \frac{a_{2n_k+1}}{M_k} \cdot \frac{1}{a_{2n_k+1} q_{2n_k}} \rightarrow 0$$

so $(b_k q_{2n_k})$ is a rigidity sequence for $Tx = x + \alpha$. For all assumptions of Corollary 1 to be satisfied we need to check that $(\|\varphi^{(w_k)}\|_{L^2})_k$ is bounded.

Notice that since $|\varphi_i| \leq 1$ for each $i \in \mathbb{N}$ and the φ_i 's have disjoint supports, we have

$$(28) \quad \left\| \sum_{i \geq k+1} \varphi_i \right\|_{\infty} \leq 1.$$

We have

$$\varphi^{(w_k)}(x) = \left(\sum_{i=1}^{k-1} \varphi_i \right)^{(w_k)}(x) + \varphi_k^{(w_k)}(x) + \left(\sum_{i \geq k+1} \varphi_i \right)^{(w_k)}(x).$$

From the construction it follows that

$$|\varphi_k^{(w_k)}(x)| \leq 1 \quad \text{for all } x \in [0, 1).$$

Taking into account (28) and the facts that $\text{supp}(\sum_{i \geq k+1} \varphi_i) \subset I_{k+1}$ and that $\text{card}(\{x, x + \alpha, \dots, x + (w_k - 1)\alpha\} \cap I_{k+1}) \leq 1$ for all $x \in [0, 1)$, we get

$$\left\| \left(\sum_{i \geq k+1} \varphi_i \right)^{(w_k)} \right\|_{\infty} \leq 1.$$

Consider $(\sum_{i=1}^{k-1} \varphi_i)^{(w_k)}$. Notice that because of (15), given $i \geq 1$ and $x \in [0, 1)$, we have $|\varphi_i^{(n)}(x)| \leq \sum_{j=1}^{M_i} |d_{ij}| \leq D_i M_i$ for all $n \geq 0$. Hence in view of (27), (3) and (16), we obtain

$$\begin{aligned} \left\| \left(\sum_{i=1}^{k-1} \varphi_i \right)^{(w_k)} \right\|_{L^2} &\leq \left(\sum_{i=1}^{k-1} D_i M_i \right) \cdot \left(\mu \left(\left\{ x \in [0, 1) : \sum_{i=1}^{k-1} \varphi_i^{(w_k)}(x) \neq 0 \right\} \right) \right)^{1/2} \\ &\leq \left(\sum_{i=1}^{k-1} D_i M_i \right) \cdot \left(\mu(\Sigma_k) + \mu \left(\bigcup_{s=0}^{w_k-1} T^s J_{a_{2n_k+1}-b_k+1}^k \right) \right)^{1/2} \\ &\leq \left(\sum_{i=1}^{k-1} D_i M_i \right) \cdot \left(\frac{2}{a_{2n_k+1}} + \frac{b_k}{a_{2n_k+1}} \right)^{1/2} \\ &\leq \left(\sum_{i=1}^{k-1} D_i M_i \right) \sqrt{\frac{2}{M_k}} \leq 1. \end{aligned}$$

We have obtained a zero mean cocycle $\varphi : [0, 1) \rightarrow \mathbb{R}$ with $|\varphi| \leq 1$ satisfying $(\varphi^{(b_k q_{2n_k})})_* \mu \rightarrow \nu$, where $(b_k q_{2n_k})$ is a rigidity time for the rotation T and moreover $(\|\varphi^{(b_k q_{2n_k})}\|_{L^2})_k$ is bounded. Moreover, from Theorem 1 such a

cocycle is cohomologous to some real-analytic function f . Since $\int \varphi d\mu = 0$, we have $\int f d\mu = 0$. Let $d > 0$ be such that $\tilde{\varphi} = \varphi + d > 0$ and $\tilde{f} = f + d > 0$. Then $\int \tilde{\varphi} d\mu = d$ and $\int \tilde{f} d\mu = d$. Note that $\tilde{\varphi}_0 = \tilde{\varphi} - \int \tilde{\varphi} = \varphi$ and similarly $\tilde{f}_0 = f$.

Thus we have proved the following.

COROLLARY 2. *The special flow $T^{\tilde{\varphi}}$ built from the rotation T and the roof function $\tilde{\varphi}$ is disjoint from all ELF-flows.*

Since \tilde{f} and $\tilde{\varphi}$ are cohomologous, $T^{\tilde{f}}$ and $T^{\tilde{\varphi}}$ are isomorphic. In particular $T^{\tilde{f}}$ is disjoint from an arbitrary ELF-flow.

3. Analytic flows on the torus which are disjoint in the sense of Furstenberg from all ELF-flows. In Section 2 we have presented two different methods to obtain a special flow $T^{\tilde{f}}$ with an analytic roof function \tilde{f} and over an irrational rotation T , which is disjoint in the sense of Furstenberg from an arbitrary ELF-flow. We will now show that whenever $d > \pi/(A - 1)$ then $T^{\tilde{f}}$ ($\tilde{f} = f + d$) is a natural special representation of some flow $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ on the torus $[0, 1]^2$ given by

$$T_t(x, y) = (x(t), y(t))$$

where $(x(t), y(t))$ is the only solution of the system of differential equations

$$(29) \quad \begin{cases} \frac{dx}{dt} = \frac{\alpha}{F(x, y)}, \\ \frac{dy}{dt} = \frac{1}{F(x, y)}. \end{cases}$$

Here $F : [0, 1]^2 \rightarrow \mathbb{R}_+$ is analytic and $(x(0), y(0)) = (x, y)$ and the function F will be precisely defined by (30) below. Recall first some facts concerning flows arising from (29) (see [2, Chapter 16] for details).

Let us assume that the function F in (29) is smooth. Notice that from the Liouville theorem, the flow \mathcal{T} preserves the measure $F(x, y)dx dy$. Furthermore \mathcal{T} is ergodic since it arises from a change of time of the linear flow which is ergodic. Moreover, the following holds.

PROPOSITION 5 ([2]). *The flow $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ corresponding to the system (29) is isomorphic to the special flow T^h , where*

$$h(x) = \int_0^1 F(x + s\alpha, s) ds.$$

Conversely, having a special flow T^h , where h is smooth, we can find a smooth function $F : [0, 1]^2 \rightarrow \mathbb{R}_+$ such that T^h is isomorphic to $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ coming from (29).

Assume now that $h(x) = \sum_{k=-\infty}^{\infty} \widehat{h}_k e^{2\pi i k x}$ is analytic and $\int_0^1 h d\mu = \widehat{h}_0 > (\pi/2) \sum_{k \neq 0} |\widehat{h}_k|$, then as shown in [2, proof of Lemma 1, p. 435] the function F given by

$$(30) \quad F(x, y) = \widehat{h}_0 + \sum_{s \neq 0} \frac{2\pi i (s\alpha + l_s)}{e^{2\pi i (s\alpha + l_s)} - 1} \widehat{h}_s e^{2\pi i (sx + l_s y)},$$

where l_s is the integer nearest to $-\alpha s$, is analytic. Observe that

$$(31) \quad \left| \frac{2\pi i (s\alpha + l_s)}{e^{2\pi i (s\alpha + l_s)} - 1} \right| \leq \frac{\pi}{2}$$

and hence

$$(32) \quad F(x, y) \geq \widehat{F}_{0,0} - \sum_{(k,l) \neq (0,0)} |\widehat{F}_{k,l}| \geq \widehat{h}_0 - \frac{\pi}{2} \sum_{s \neq 0} |\widehat{h}_s| > 0.$$

We end up this section by the observation that there exists a universal d (depending only on $A > 1$) such that all functions f obtained from the a.a.c.c.p. (and given by (18)) satisfy

$$(33) \quad d > \frac{\pi}{2} \sum_{s \neq 0} |\widehat{f}_s|.$$

Indeed, from the proof of Lemma 5 it follows that whenever $d > \pi/(A - 1)$ then for each f given by (18) the inequality (33) holds.

4. An uncountable family of analytic flows. Put $E = \{0, 1\}^{\mathbb{N}}$. In this section we will construct an uncountable family of cocycles $\{\varphi_\varepsilon\}_{\varepsilon \in E}$ so that for some equivalence relation $\sim \subset E \times E$ the corresponding special flows over an irrational rotation $Tx = x + \alpha$ are disjoint whenever $\varepsilon \not\sim \varepsilon'$ and isomorphic whenever $\varepsilon \sim \varepsilon'$.

Define now a cocycle φ_ε in the following way. Given $\varepsilon = (\varepsilon_i)_{i \geq 1} \in E$ put

$$\varphi_\varepsilon = \sum_{i=1}^{\infty} \widetilde{\varphi}_i,$$

where $\widetilde{\varphi}_i = \varepsilon_i \varphi_i$ and for $\varepsilon_i = 1$ we carry out the i th step of the a.a.c.c.p. from Subsection 2.2. The uncountable family of a.a.c.c.p.'s is now realized over a common α (indeed, recall that in view of Remark 3 we can replace some of the d_{k_i} 's by 0 and the a.a.c.c.p. will be realized over the same α). Using similar arguments to those proving (26) we get $\mu(\{x \in [0, 1) : \widetilde{\varphi}_k^{(b_k q_{2n_k})}(x) \neq \varphi_\varepsilon^{(b_k q_{2n_k})}(x)\}) \rightarrow 0$ as $k \rightarrow \infty$. Therefore

$$(\varphi_\varepsilon^{(b_k q_{2n_k})})_* \mu \rightarrow \nu \quad \text{as } k \rightarrow \infty, k \in \{i : \varepsilon_i = 1\}$$

if $\#\{i : \varepsilon_i = 1\} = \infty$, and

$$(\varphi_\varepsilon^{(b_k q_{2n_k})})_* \mu \rightarrow \delta_0 \quad \text{as } k \rightarrow \infty, k \in \{i : \varepsilon_i = 0\}$$

if $\#\{i : \varepsilon_i = 0\} = \infty$.

Define $\sim \subset E \times E$ by putting

$$\varepsilon \sim \varepsilon' \quad \text{whenever } \varepsilon_i = \varepsilon'_i \text{ for } i \text{ large enough.}$$

Assume now that $\varepsilon \not\sim \varepsilon'$. So we can assume that the set $\{i \geq 1 : \varepsilon_i = 1, \varepsilon'_i = 0\}$ is infinite. Take a positive d so that $\tilde{\varphi}_\varepsilon = \varphi_\varepsilon + d > 0$ and $\tilde{\varphi}_{\varepsilon'} = \varphi_{\varepsilon'} + d > 0$. In view of Proposition 2,

$$(T^{\tilde{\varphi}_\varepsilon})_{db_k q_{2n_k}} \rightarrow \int_{\mathbb{R}} (T^{\tilde{\varphi}_\varepsilon})_{-t} d\nu(t), \quad (T^{\tilde{\varphi}_{\varepsilon'}})_{db_k q_{2n_k}} \rightarrow \int_{\mathbb{R}} (T^{\tilde{\varphi}_{\varepsilon'}})_{-t} d\delta_0(t),$$

when $\varepsilon_k = 1, \varepsilon'_k = 0$ and $k \rightarrow \infty$.

By Lemma 3 the special flows $(T_t^{\tilde{\varphi}_\varepsilon})_{t \in \mathbb{R}}$ and $(T_t^{\tilde{\varphi}_{\varepsilon'}})_{t \in \mathbb{R}}$ are spectrally disjoint, hence they are disjoint in the sense of Furstenberg. Note that if $\varepsilon \sim \varepsilon'$ then φ_ε and $\varphi_{\varepsilon'}$ are cohomologous and hence the corresponding special flows are isomorphic.

REMARK 5. Now, the argument used to prove (33) and the fact that $|\varphi| \leq 1$ imply that whenever $d > \max(1, \pi/(A - 1))$,

$$\varphi_\varepsilon + d > 0 \quad \text{and} \quad d > \frac{\pi}{2} \sum_{s \neq 0} |(f_\varepsilon)_s^\wedge|$$

for each $\varepsilon \in E$. Notice that the second inequality immediately implies that $f_\varepsilon + d > 0$ for each $\varepsilon \in E$.

Put now $\tilde{f}_\varepsilon = f_\varepsilon + d$ and $\tilde{f}_{\varepsilon'} = f_{\varepsilon'} + d$ and observe that the corresponding special flows also have the property that they are isomorphic if $\varepsilon \sim \varepsilon'$ and disjoint (even spectrally disjoint) if $\varepsilon \not\sim \varepsilon'$.

REMARK 6. In view of Section 3 and Remark 5 for every (analytic) function $\tilde{f}_\varepsilon = f_\varepsilon + d$ we obtain an analytic function $F_\varepsilon : [0, 1]^2 \rightarrow \mathbb{R}_+$ such that the special flow $T^{\tilde{f}_\varepsilon}$ and the flow $\mathcal{T}^{F_\varepsilon}$ coming from (29) are isomorphic. Remembering that the function F_ε is given by (30) we deduce that for all $\varepsilon \in E$,

$$(34) \quad F_\varepsilon(x, y) > d - \frac{\pi}{A - 1} > 0.$$

On the other hand, from (30) and (31) we get, for all $(x, y) \in [0, 1]^2$ and for all $\varepsilon \in E$,

$$(35) \quad |F_\varepsilon(x, y)| \leq 2d.$$

From now on we fix $d > \max(1, \pi/(A - 1))$.

REMARK 7. Observe that the family $\{\mathcal{T}^{F_\varepsilon} : \varepsilon \in E\}$ of analytic flows on $[0, 1]^2$ obtained as above has the property that $\mathcal{T}^{F_\varepsilon}$ and $\mathcal{T}^{F_{\varepsilon'}}$ are isomorphic whenever $\varepsilon \sim \varepsilon'$, and disjoint (in fact, spectrally disjoint) for $\varepsilon \not\sim \varepsilon'$.

5. Non-existence of a countable set of Borel invariants in the class of flows on $\mathbb{R}^2/\mathbb{Z}^2$. The aim of this section is to show that a consequence of the results from the previous section is that it is not possible to find a countable complete set of Borel invariants for the measure-theoretic classification in the class of flows coming from (29). Such flows in general do not preserve Lebesgue measure, but they are non-singular with respect to it. Therefore we first recall some basic facts about topology of non-singular flows.

For a standard probability Borel space $(X, \mathcal{B}, \lambda)$ (we assume that X is a compact space with a metric ϱ) consider the space $\widetilde{\text{Aut}}(X, \mathcal{B}, \lambda)$ of all invertible non-singular automorphisms on X , i.e. $S \in \widetilde{\text{Aut}}(X, \mathcal{B}, \lambda)$ if S is invertible, bimeasurable and the measures λ and λ_S are equivalent, where $\lambda_S(A) = \lambda(SA)$ for every Borel set $A \subset X$.

We now define a Polish topology in $\widetilde{\text{Aut}}(X, \mathcal{B}, \lambda)$ (see [23] for details).

For $S \in \widetilde{\text{Aut}}(X, \mathcal{B}, \lambda)$ define

$$(36) \quad \widetilde{U}_S f = \frac{d\lambda_S}{d\lambda} f \circ S, \quad f \in L^1(X, \mathcal{B}, \lambda).$$

We then observe that \widetilde{U}_S belongs to the set $\text{Iso}(L^1(X, \lambda))$ of invertible isometries of $L^1(X, \mathcal{B}, \lambda)$. Then by the Lamperti theorem the map

$$(37) \quad \widetilde{\text{Aut}}(X, \mathcal{B}, \lambda) \ni S \mapsto \widetilde{U}_S \in \text{Iso}(L^1(X, \lambda))$$

is injective. On $\text{Iso}(L^1(X, \lambda))$ we consider the strong operator topology.

We endow $\widetilde{\text{Aut}}(X, \mathcal{B}, \lambda)$ with a topology which makes the map (37) a homeomorphism. This topology is metrisable with a (complete) metric given by

$$d(S, T) = \sum_{n=1}^{\infty} 2^{-n} (\|\widetilde{U}_S f_n - \widetilde{U}_T f_n\|_1 + \|\widetilde{U}_{S^{-1}} f_n - \widetilde{U}_{T^{-1}} f_n\|_1),$$

where $\{f'_n\}_{n \geq 1}$ is a dense family in $L^1(X, \mathcal{B}, \lambda)$ and $f_n = f'_n / \|f'_n\|$.

REMARK 8. Assume that S preserves the measure $F d\lambda$, where $F : X \rightarrow \mathbb{R}_+$ is measurable and in $L^1(\lambda)$. Observe that this is equivalent to

$$(38) \quad F \circ S d\lambda_S = F d\lambda.$$

Indeed, putting $\mu = F d\lambda$ we have

$$\mu(A) = \mu(SA) = \int_{SA} F d\lambda = \int_X \chi_A \circ S^{-1} \cdot F d\lambda = \int_A F \circ S d\lambda_S.$$

It follows that in the continuous case to show the convergence in the metric d it is enough to show the uniform convergence of both: automorphisms and densities. Indeed, assume that $F_n \rightarrow F > 0$, F_n, F continuous and $S_n \rightarrow S$, $S_n^{-1} \rightarrow S^{-1}$ (uniformly) and that for each $n \in \mathbb{N}$, S_n preserves the measure $F_n d\lambda$ (which by (38) is equivalent to the condition $d\lambda_{S_n}/d\lambda = F_n/F_n \circ S_n$). Then $F_n/F_n \circ S_n \rightarrow F/F \circ S$ uniformly and hence $d(S_n, S) \rightarrow 0$, as $n \rightarrow \infty$.

Put now $X = [0, 1]^2$, $\mathcal{B} = \mathcal{B}([0, 1]^2)$ and let λ stand for Lebesgue measure on $[0, 1]^2$. Denote by $\mathcal{NF}([0, 1]^2, \lambda)$ the space of all flows non-singular with respect to λ , that is, for each flow $\mathcal{T} = (T_t)_{t \in \mathbb{R}} \in \mathcal{NF}([0, 1]^2, \lambda)$ and all $t \in \mathbb{R}$, we have $T_t \in \widetilde{\text{Aut}}([0, 1]^2, \mathcal{B}([0, 1]^2), \lambda)$. Moreover, we require that the unitary representation $t \mapsto \widetilde{U}_{T_t}$ given by $\widetilde{U}_{T_t} f = \sqrt{d\lambda_{T_t}/d\lambda} \cdot f \circ T_t$, $f \in L^2([0, 1]^2, \lambda)$, is strongly continuous. The space $\mathcal{NF}([0, 1]^2, \lambda)$ with the metric

$$D(\mathcal{T}, \mathcal{S}) = \sup_{t \in [0, 1]} d(T_t, S_t)$$

becomes a Polish space (see [26] for the measure preserving case).

Consider now the subspace $\mathcal{NF}'([0, 1]^2, \lambda) \subset \mathcal{NF}([0, 1]^2, \lambda)$ of all flows arising from the systems of differential equations of the form (29) where $F \in C^1([0, 1]^2)$. Denote by $\mathcal{NF}^\omega([0, 1]^2, \lambda)$ the space (contained in $\mathcal{NF}'([0, 1]^2, \lambda)$) of flows arising from (29) with F analytic. On $\mathcal{NF}'([0, 1]^2, \lambda)$ and $\mathcal{NF}^\omega([0, 1]^2, \lambda)$ we consider the restriction of the metric D .

REMARK 9. Observe that each element $\mathcal{T}^F = (T_t^F)_t \in \mathcal{NF}'([0, 1]^2, \lambda)$ preserves a measure equivalent to Lebesgue measure λ , namely $F d\lambda$. Hence using similar arguments to those in Remark 8 we find that the convergence in this space follows from the uniform convergence of densities and time t automorphisms. In other words, to obtain $D(\mathcal{T}^{F_n}, \mathcal{T}^F) \rightarrow 0$ as $n \rightarrow \infty$ it suffices to show that $T_t^{F_n}(x, y) \rightarrow T_t^F(x, y)$ uniformly with respect to (t, x, y) , $t \in [-1, 1]$ and that $F_n(x, y) \rightarrow F(x, y)$ uniformly with respect to (x, y) .

Recall that α defines the common rotation for the whole family $\{F_\varepsilon\}_{\varepsilon \in E}$ obtained in Section 4. Without changing the notation let us go back to the construction in which we have obtained the family $\{\mathcal{T}^{F_\varepsilon} : \varepsilon \in E\}$ of (analytic) flows on $[0, 1]^2$. Consider the map

$$E = \{0, 1\}^{\mathbb{N}} \ni \varepsilon \xrightarrow{\Theta} \mathcal{T}^{F_\varepsilon} \in \mathcal{NF}^\omega([0, 1]^2, \lambda).$$

We equip $E = \{0, 1\}^{\mathbb{N}}$ with the product topology (which is metrisable with, for instance, the product metric $d_E((x_n), (y_n)) = \sum_{k \geq 1} 2^{-k} |x_k - y_k|$) so that E becomes a compact topological space.

Put

$$\mathcal{A} = \Theta(E) = \{\mathcal{T}^{F_\varepsilon} : \varepsilon \in E\}.$$

The main result of this section is the following.

THEOREM 2. *The set $\mathcal{A} \subset \mathcal{NF}^\omega([0, 1]^2, \lambda)$ is a Borel subset of $\mathcal{NF}([0, 1]^2, \lambda)$. Moreover, if $\tilde{\varphi} : \mathcal{A} \rightarrow \mathbb{R}^\infty$ is an arbitrary Borel function such that $\tilde{\varphi}(\mathcal{T}) = \tilde{\varphi}(\mathcal{S})$ whenever $\mathcal{T} \in \mathcal{A}$ and $\mathcal{S} \in \mathcal{A}$ are isomorphic, then there exist spectrally disjoint flows $\mathcal{T}' \in \mathcal{A}$ and $\mathcal{S}' \in \mathcal{A}$ such that $\tilde{\varphi}(\mathcal{T}') = \tilde{\varphi}(\mathcal{S}')$.*

From Theorem 2 we immediately get the following.

COROLLARY 3. *There does not exist a Borel map $\tilde{\varphi} : \mathcal{NF}([0, 1]^2, \lambda) \rightarrow \mathbb{R}^\infty$ which is constant on the (spectral) isomorphism classes and simultaneously takes different values for (spectrally) non-isomorphic arguments.*

In other words, it is not possible to find a countable complete set of Borel invariants in $\mathcal{NF}([0, 1]^2, \lambda)$ for the problem of (spectral) isomorphism.

Proof of Theorem 2. It is sufficient to show that

- (i) Θ is continuous (hence \mathcal{A} is compact),
- (ii) if $\tilde{\varphi} : \mathcal{A} \rightarrow \mathbb{R}^\infty$ is a Borel function constant on isomorphism classes, then there exist non-isomorphic flows $\mathcal{T}^{F_\varepsilon}$ and $\mathcal{T}^{F_{\varepsilon'}}$ such that $\tilde{\varphi}(\mathcal{T}^{F_\varepsilon}) = \tilde{\varphi}(\mathcal{T}^{F_{\varepsilon'}})$.

First observe that (ii) follows from the 0-1 Kolmogorov law (by considering properties of $\tilde{\varphi} \circ \Theta$).

It remains to show (i). Assume that $\varepsilon_n \rightarrow \varepsilon$, as $n \rightarrow \infty$. In view of Remark 9, to show that $D(\mathcal{T}^{F_{\varepsilon_n}}, \mathcal{T}^{F_\varepsilon}) \rightarrow 0$ it is enough to show that

$$(39) \quad T_t^{F_{\varepsilon_n}}(x, y) \rightarrow T_t^{F_\varepsilon}(x, y)$$

uniformly with respect to (t, x, y) , $t \in [-1, 1]$ and that

$$(40) \quad F_{\varepsilon_n}(x, y) \rightarrow F_\varepsilon(x, y)$$

uniformly with respect to (x, y) .

To see (40) we will prove the continuity of the map

$$E \ni \varepsilon \mapsto F_\varepsilon \in C([0, 1]^2).$$

This can be replaced by a stronger condition that the mapping $E \ni \varepsilon \mapsto \widehat{F}_\varepsilon \in l^1(\mathbb{Z}^2)$ is continuous. Furthermore in view of (30) and (31) it is enough to show the continuity of the map $E \ni \varepsilon \mapsto \widehat{f}_\varepsilon \in l^1(\mathbb{Z})$, and this latter condition is proved in much the same way as Lemma 5.

To prove (39) recall that a differential equation on $[0, 1]^2$ may be lifted to an equation on \mathbb{R}^2 simply by periodic extension. For $\varepsilon \in E$ consider then a differential equation of the form (29), that is,

$$(41) \quad \frac{d\mathcal{X}}{dt} = \Gamma_\varepsilon(\mathcal{X}),$$

where $\mathcal{X} = (x, y)$, $\Gamma_\varepsilon : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $\Gamma_\varepsilon(\mathcal{X}) = (\alpha/F_\varepsilon(\mathcal{X}), 1/F_\varepsilon(\mathcal{X}))$.

Since F_ε is analytic, it is Lipschitz (with a constant L_ε), which implies that Γ_ε is also Lipschitz with the constant

$$\tilde{L}_\varepsilon = \frac{\sqrt{\alpha^2 + 1}}{\left(d - \frac{\pi}{A-1}\right)^2} L_\varepsilon,$$

i.e. for all $(x, y) \in \mathbb{R}^2$ and $(x', y') \in \mathbb{R}^2$,

$$\|\Gamma_\varepsilon(x, y) - \Gamma_\varepsilon(x', y')\|_{\mathbb{R}^2} \leq \tilde{L}_\varepsilon \|(x, y) - (x', y')\|_{\mathbb{R}^2}.$$

Then the equation (41) has a solution defined for all $t \in \mathbb{R}$, of the form

$$\overline{T}_t^{F_\varepsilon}(x, y) = (x, y) + \int_0^t \Gamma_\varepsilon(x(\tau), y(\tau)) d\tau$$

for each $(x, y) \in \mathbb{R}^2$. Observe that for the flows given by solutions of (41) and (29) we have

$$(42) \quad \|T_t^{F_\varepsilon}(x, y) - T_t^{F_{\varepsilon'}}(x, y)\|_{[0,1]^2} \leq \|\overline{T}_t^{F_\varepsilon}(x, y) - \overline{T}_t^{F_{\varepsilon'}}(x, y)\|_{\mathbb{R}^2},$$

because $T_t^{F_\varepsilon} = \overline{\pi} \circ \overline{T}_t^{F_\varepsilon}$, where $\overline{\pi} : \mathbb{R}^2 \rightarrow [0, 1]^2$ is the natural quotient map.

Now from the Gronwall inequality it follows that

$$(43) \quad \|\overline{T}_t^{F_{\varepsilon'}}(x, y) - \overline{T}_t^{F_\varepsilon}(x, y)\|_{\mathbb{R}^2} \leq \sqrt{\alpha^2 + 1} \left(d - \frac{\pi}{A-1}\right)^{-2} \|F_\varepsilon - F_{\varepsilon'}\|_\infty e^{\tilde{L}_{\varepsilon'} t}.$$

Indeed, observe that

$$\begin{aligned} \|\overline{T}_t^{F_{\varepsilon'}}(x, y) - \overline{T}_t^{F_\varepsilon}(x, y)\|_{\mathbb{R}^2} &= \left\| \int_0^t (\Gamma_{\varepsilon'}(\overline{T}_\tau^{F_{\varepsilon'}}(x, y)) - \Gamma_\varepsilon(\overline{T}_\tau^{F_\varepsilon}(x, y))) d\tau \right\|_{\mathbb{R}^2} \\ &\leq \int_0^t \|\Gamma_{\varepsilon'}(\overline{T}_\tau^{F_{\varepsilon'}}(x, y)) - \Gamma_{\varepsilon'}(\overline{T}_\tau^{F_\varepsilon}(x, y))\|_{\mathbb{R}^2} d\tau \\ &\quad + \int_0^t \|\Gamma_{\varepsilon'}(\overline{T}_\tau^{F_\varepsilon}(x, y)) - \Gamma_\varepsilon(\overline{T}_\tau^{F_\varepsilon}(x, y))\|_{\mathbb{R}^2} d\tau \\ &\leq \int_0^t \tilde{L}_{\varepsilon'} \|\overline{T}_\tau^{F_{\varepsilon'}}(x, y) - \overline{T}_\tau^{F_\varepsilon}(x, y)\|_{\mathbb{R}^2} d\tau + t \sqrt{\alpha^2 + 1} \left\| \frac{1}{F_{\varepsilon'}} - \frac{1}{F_\varepsilon} \right\|_\infty \\ &\leq \int_0^t \tilde{L}_{\varepsilon'} \|\overline{T}_\tau^{F_{\varepsilon'}}(x, y) - \overline{T}_\tau^{F_\varepsilon}(x, y)\|_{\mathbb{R}^2} d\tau + \sqrt{\alpha^2 + 1} \left(d - \frac{\pi}{A-1}\right)^{-2} \|F_\varepsilon - F_{\varepsilon'}\|_\infty. \end{aligned}$$

Hence (43) follows. And from (43) we immediately get (39). ■

6. Weak closure of time t automorphisms of an analytic flow.

In this section we will construct a weakly mixing flow $T^{\overline{\varphi}}$ with the following property:

(44) for each $P \in \mathcal{P}(\mathbb{R})$ the Markov operator $\int_{\mathbb{R}} T_t^{\tilde{\varphi}} dP(t)$ belongs to the weak closure of $\{T_t^{\tilde{\varphi}} : t \in \mathbb{R}\}$.

This flow will be isomorphic to an analytic flow on $[0, 1)^2$. By (9) to prove (44) we only need to consider P running over a dense subset of $\mathcal{P}(\mathbb{R})$, for example measures of the form $\sum_{j=1}^Q B_j \delta_{r_j}$, where $r_j \in \mathbb{Q}$, $B_j \in \mathbb{Q}_+$ and $\sum_{j=1}^Q B_j = 1$. Our goal is to construct a flow $T^{\tilde{\varphi}}$ satisfying:

(45) For each $\tilde{M} \in \mathbb{N}_+$, $s \in \mathbb{N}$, $r_j \in \mathbb{Q}$, $A_j \in \mathbb{Q}_+$, $j = 1, \dots, \tilde{M}$, $\tilde{A} := \sum_{j=1}^{\tilde{M}} A_j < 1$, the Markov operator $\int_{\mathbb{R}} T_t^{\tilde{\varphi}} dP(t)$ with

$$P = \sum_{j=1}^{\tilde{M}} A_j \delta_{r_j} + \frac{1 - \tilde{A}}{2} (\delta_{s - \frac{2}{1 - \tilde{A}} \sum_{j=1}^{\tilde{M}} r_j A_j} + \delta_{-s})$$

belongs to the weak closure of the set $\{T_t^{\tilde{\varphi}} : t \in \mathbb{R}\}$.

The construction goes as follows. Consider the set

$$\left\{ ((r_1, \dots, r_{\tilde{M}}), (A_1, \dots, A_{\tilde{M}}), s) : r_j \in \mathbb{Q}, A_j \in \mathbb{Q}_+, \sum_{j=1}^{\tilde{M}} A_j < 1, s \in \mathbb{N}, \tilde{M} \in \mathbb{N}_+ \right\}.$$

Note that this set is countable and denote by $(U_k)_{k \geq 1}$ a sequence

$$U_k = ((r_{1k}, \dots, r_{\tilde{M}_k k}), (A_{1k}, \dots, A_{\tilde{M}_k k}), s_k)$$

in which for each element $((r_1, \dots, r_{\tilde{M}}), (A_1, \dots, A_{\tilde{M}}), s)$ of the above set we have $\#\{k \in \mathbb{N}_+ : U_k = ((r_1, \dots, r_{\tilde{M}}), (A_1, \dots, A_{\tilde{M}}), s)\} = \infty$. We need more notation: $A_{jk} = g_{jk}/f_{jk}$ ($g_{jk}, f_{jk} \in \mathbb{N}_+$), $j = 1, \dots, \tilde{M}_k$, $\tilde{A}_k = \sum_{j=1}^{\tilde{M}_k} A_{jk}$, $(1 - \tilde{A}_k)/2 = g_{\tilde{A}_k}/f_{\tilde{A}_k}$, $g_{\tilde{A}_k}, f_{\tilde{A}_k} \in \mathbb{N}_+$, $\tilde{m}_k = \text{LCM}(f_{1k}, \dots, f_{\tilde{M}_k k}, f_{\tilde{A}_k})$, $N_{jk} = \min\{m \in \mathbb{N}_+ : |r_{jk}/m| < 1\}$,

$$\tilde{N}_k = \min \left\{ m \in \mathbb{N}_+ : \left| \frac{s_k - \frac{2}{1 - \tilde{A}_k} \sum_{j=1}^{\tilde{M}_k} r_{jk} A_{jk}}{m} \right| < 1 \right\}.$$

We will now carry out an a.a.c.c.p. with extra conditions on the parameters; namely, we require that $D_k \leq 1$ (see Remark 3) and that

$$(46) \quad M_k > m_k \left(N_{1k} A_{1k} + \dots + N_{\tilde{M}_k k} A_{\tilde{M}_k k} + \frac{1 - \tilde{A}_k}{2} (\tilde{N}_k + s_k + 1) \right),$$

where $m_k = c_k \tilde{m}_k$, $c_k \in \mathbb{N}$, $c_k \rightarrow \infty$ and

$$(47) \quad \left(\sum_{i=1}^{k-1} M_i \right) \sqrt{\frac{2}{m_k}} \leq \text{const.}$$

Observe that

$$N_{1k}A_{1k} + \dots + N_{\widetilde{M}_k k}A_{\widetilde{M}_k k} + \frac{1 - \widetilde{A}_k}{2}(\widetilde{N}_k + s_k + 1) \geq 1,$$

hence $m_k \varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$ (recall that $\sum_{k=1}^{\infty} \sqrt{\varepsilon_k} \cdot M_k < \infty$). As $m_k \rightarrow \infty$, we have $M_k \rightarrow \infty$. Let α be an irrational over which the a.a.c.c.p. is carried out. We require additionally that a_{2n_k+1} is a multiple of m_k , more precisely

$$a_{2n_k+1} = m_k b_k,$$

where $b_k \geq 1$ for all $k \geq 1$. Recall that

$$(48) \quad \lambda_k M_k < \frac{\varepsilon_k}{2q_{2n_k}},$$

so $\lambda_k M_k = o(\mu(I_k))$. Denote by ξ_1, \dots, ξ_{m_k} the consecutive subintervals of I_k of length $\mu(I_k)/m_k$. This means that any such subinterval consists of b_k intervals J_t^k . We now define φ_k as follows. If $\varepsilon_k m_k > 2/3$ we put arbitrary d_{ki} according to general rules for a.a.c.c.p. Assume now that $\varepsilon_k m_k \leq 2/3$. Notice that

$$(49) \quad \lambda_k M_k \leq \frac{3}{2} \cdot \frac{\varepsilon_k}{3q_{2n_k}} \leq \frac{3}{2} \varepsilon_k \mu(I_k) \leq \frac{1}{m_k} \mu(I_k) = |\xi_i|,$$

$i = 1, \dots, m_k$. Given $j = 1, \dots, \widetilde{M}_k$, in each interval $\xi_{(A_{1k} + \dots + A_{j-1,k})m_k + v}$, $v = 1, \dots, A_{jk}m_k$, we choose N_{jk} consecutive intervals ω_{ki} so that the interval $\bigcup \omega_{ki}$ is placed “centrally”, i.e. in the middle of $\xi_{(A_{1k} + \dots + A_{j-1,k})m_k + v}$. In the central interval $J_{s_{k,i}^k} \subset \omega_{ki}$ we put the value r_{jk}/N_{jk} . Analogously in each interval $\xi_{\widetilde{A}_k m_k + v}$, $v = 1, \dots, (1 - \widetilde{A}_k)m_k/2$, we select “centrally” \widetilde{N}_k consecutive intervals ω_{ki} and in the central subintervals we put the value

$$\frac{s_k - \frac{2}{1 - \widetilde{A}_k} \sum_{j=1}^{\widetilde{M}_k} r_{jk} A_{jk}}{\widetilde{N}_k}.$$

Finally, in each of $\xi_{(\widetilde{A}_k + 1)m_k/2 + v}$, $v = 1, \dots, (1 - \widetilde{A}_k)m_k/2$, we select “centrally” $s_k + 1$ consecutive ω_{ki} on which in the central subintervals we put $-s_k/(s_k + 1)$. In order to complete the a.a.c.c.p. we choose ω_{ki} arbitrarily and put $d_{ki} = 0$ for the remaining

$$i \in \left\{ m_k \left(\sum_{i=1}^{\widetilde{M}_k} N_{ik} A_{ik} + \frac{1 - \widetilde{A}_k}{2} (\widetilde{N}_k + s_k + 1) \right) + 1, \dots, M_k \right\},$$

which is possible by (49). Notice that $\sum_{i=1}^{M_k} d_{ki} = 0$.

Now, we fix $U = ((r_1, \dots, r_{\widetilde{M}}), (A_1, \dots, A_{\widetilde{M}}), s)$ and consider only k such that $U_k = U$. First notice that $m_k \varepsilon_k < 2/3$ for sufficiently large such k . Moreover,

$$\frac{\max(N_1, \dots, N_{\widetilde{M}}, \widetilde{N}_k, s + 1) \lambda_k m_k}{\mu(I_k)} \rightarrow 0$$

as $k \rightarrow \infty$ with $U_k = U$ ($\tilde{N}_k = \tilde{N}$) by (48). Thus

$$(\varphi_k^{(b_k q_{2n_k})}(x))_* \mu \rightarrow \sum_{j=1}^{\tilde{M}} A_j \delta_{r_j} + \frac{1 - \tilde{A}}{2} (\delta_{s - \frac{2}{1 - \tilde{A}} \sum_{j=1}^{\tilde{M}} r_j A_j} + \delta_{-s})$$

as $k \rightarrow \infty$ with $U_k = U$.

Now using similar arguments to those showing (26) and the fact that $m_k \rightarrow \infty$ as $k \rightarrow \infty$, we obtain

$$\mu(\{x \in [0, 1) : \varphi_k^{(b_k q_{2n_k})}(x) \neq \varphi^{(b_k q_{2n_k})}(x)\}) \rightarrow 0$$

as $k \rightarrow \infty$. Thus

$$(\varphi^{(b_k q_{2n_k})}(x))_* \mu \rightarrow \sum_{j=1}^{\tilde{M}} A_j \delta_{r_j} + \frac{1 - \tilde{A}}{2} (\delta_{s - \frac{2}{1 - \tilde{A}} \sum_{j=1}^{\tilde{M}} r_j A_j} + \delta_{-s})$$

as $k \rightarrow \infty$ with $U_k = U$.

Obviously $|\varphi(x)| \leq 1$ for $x \in [0, 1)$. Take now a constant d such that $\tilde{\varphi} = \varphi + d > 0$. Using similar arguments to those in Subsection 2.2 (with (47) instead of (16)) we will show that along the subsequence $(b_k q_{2n_k})$ where $k \in \{i : U_i = U = ((r_1, \dots, r_{\tilde{M}}), (A_1, \dots, A_{\tilde{M}}), s)\}$, $\|\varphi^{(b_k q_{2n_k})}\|_{L^2} < \text{const.}$ Indeed, the only thing to notice is that for all $x \in [0, 1)$ and $k \in \{i : U_i = U = ((r_1, \dots, r_{\tilde{M}}), (A_1, \dots, A_{\tilde{M}}), s)\}$,

$$\begin{aligned} & |\varphi_k^{(b_k q_{2n_k})}(x)| \\ & \leq \max \left\{ e + f : e, f \in \left\{ |r_1|, \dots, |r_{\tilde{M}}|, \left| s - \frac{2}{1 - \tilde{A}} \sum_{j=1}^{\tilde{M}} r_j A_j \right|, |s| \right\} \right\} \\ & = \text{const.} \end{aligned}$$

The construction of $T^{\tilde{\varphi}}$ is complete, and by Proposition 2, (45) has been proved.

In this way we see that for each $k \geq 1$ there exists an increasing sequence $(q_n^{(k)})$ such that

$$T_{dq_n^{(k)}}^{\tilde{\varphi}} \rightarrow \int_{\mathbb{R}} T_{-t}^{\tilde{\varphi}} dP_k(t),$$

where $\{P_k\}$ is a dense subset in $\mathcal{P}(\mathbb{R})$.

Now in view of (9) we get (44):

COROLLARY 4. *For each measure $P \in \mathcal{P}(\mathbb{R})$ there exists a sequence $(q_n^{(P)}) \subseteq \mathbb{N}$ such that*

$$T_{dq_n^{(P)}}^{\tilde{\varphi}} \rightarrow \int_{\mathbb{R}} T_{-t}^{\tilde{\varphi}} dP(t).$$

Assume now that $d \in \mathbb{N}$. Then we obtain the following.

COROLLARY 5. For each measure $P \in \mathcal{P}(\mathbb{Z})$ there exists a sequence $(q_n^{(P)}) \subseteq \mathbb{N}$ such that

$$(T_1^{\tilde{\varphi}})^{q_n^{(P)}} \rightarrow \sum_{j=-\infty}^{\infty} P(j)(T_1^{\tilde{\varphi}})^{-j}.$$

REMARK 10. Recall (see e.g. [15]) that a flow $(S_t)_{t \in \mathbb{R}}$ is called κ -weakly mixing ($0 \leq \kappa \leq 1$) if there exists a sequence $(t_n) \rightarrow \infty$ such that

$$S_{t_n} \rightarrow \kappa \text{Id} + (1 - \kappa) \int.$$

By taking $P_n = \frac{1}{4}\delta_{-n} + \frac{1}{2}\delta_0 + \frac{1}{4}\delta_n$ and considering $n = n_k$ so that $T_{n_k}^{\tilde{\varphi}} \rightarrow \int$ (which is justified by weak mixing of $T^{\tilde{\varphi}}$, by Corollary 5, $\frac{1}{2} \text{Id} + \frac{1}{2} \int$ belongs to the weak closure of the set $\{(T^{\tilde{\varphi}})_t : t \in \mathbb{R}\}$, i.e. $T^{\tilde{\varphi}}$ is $\frac{1}{2}$ -weakly mixing. This implies mutual singularity of convolutions of maximal spectral type (see [15], [24], [27]).

However, using a result of O. N. Ageev (see [1]) we can obtain much more. If $T^{\tilde{\varphi}}$ satisfies the assumptions of Corollary 5 then the maximal spectral type of $T_1^{\tilde{\varphi}}$, denoted by $\sigma_{T_1^{\tilde{\varphi}}}$, is a continuous measure such that the Gaussian system S determined by it has a simple spectrum. In particular, convolutions of the reduced maximal spectral type of $T_1^{\tilde{\varphi}}$ are mutually singular. Since $T_1^{\tilde{\varphi}}$ is disjoint from all ELF-automorphisms (see [3]), it is disjoint from all Gaussian automorphisms. Hence the Gaussian system S has the GAG property, but it cannot satisfy the Foias–Strătilă theorem (indeed, if a system satisfies the Foias–Strătilă theorem then it is Gaussian; see [20] for details).

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