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Strong initial segments of models of $I\Delta_0$

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Abstract. McAloon showed that if $\mathcal{A}$ is a nonstandard model of $I\Delta_0$, then some initial segment of $\mathcal{A}$ is a nonstandard model of PA. Sommer and D'Aquino characterized, in terms of the Wainer functions, the elements that can belong to such an initial segment. The characterization used work of Ketonen and Solovay, and Paris. Here we give conditions on a model $\mathcal{A}$ of $I\Delta_0$ guaranteeing that there is an *n-elementary* initial segment that is a nonstandard model of PA. We also characterize the elements that can be included.

1. Introduction. Let L be the usual language of arithmetic, with symbols $+$, $\cdot$, 0, 1, and $\leq$. Let $I\Delta_0$ be the subsystem of Peano Arithmetic (PA) in which induction applies only to formulas with bounded quantifiers (Δ_0-formulas). A nonstandard model $\mathcal{A}$ of $I\Delta_0$ satisfies *overspill* for Δ_0-formulas; i.e., if $\varphi(\overline{u}, x)$ is Δ_0, then for any tuple $\bar{b}$ in $\mathcal{A}$, if $\varphi(\bar{b}, x)$ is satisfied by all standard n, then it is satisfied by some nonstandard ν.

By a result of Parikh [14], any Δ_0-definable function that is provably total in $I\Delta_0$ is provably bounded by a polynomial. Bennett [1] found a Δ_0-formula defining in $\mathbb{N}$ the graph of exponentiation. Later, Paris [16] found a Δ_0-formula $E_0(x, y, z)$ defining the relation $x^y = z$, for which the recursive properties of exponentiation are provable in $I\Delta_0$. The lack of exponentiation means that many classical results of elementary number theory are not known to be provable in $I\Delta_0$. In particular, it is an open problem whether $I\Delta_0$ proves Matijasevic's theorem (saying that every c.e. set is Diophantine). A positive answer to this question would have important consequences in complexity theory. If we add to $I\Delta_0$ the axiom $exp = (\forall x > 1)\,(\forall y)\,(\exists z)\,E_0(x, y, z)$, saying that the exponential function is total, then the resulting theory is strong enough to prove all of the results of elementary number theory. In particular, Matijasevic's theorem is provable in $I\Delta_0 + exp$ (see [6]).

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Without exponentiation, we use known sequences to show the existence of others. Let $\mathcal{A}$ be a model of $I\Delta_0$. Let $\varphi(u,x,y)$ be a bounded formula such that for each u and x, there is at most one y such that $\mathcal{A} \models \varphi(u,x,y)$. For a sequence v, we say that v *is determined by* $\varphi(u,x,y)$ if for all $z < \text{length}(v)$, $\mathcal{A} \models \varphi(v\restriction z, z, v(z))$. Let C be a sequence coded in $\mathcal{A}$. If I is the set of $s \leq \text{length}(C)$ such that there exists v of length x determined by $\varphi(u,x,y)$ with a code bounded by that for C, then there is a greatest $s \in I$. We have a maximal sequence J determined by φ such that the length of J, and the code for J, are bounded by those for C. We shall often take $\varphi(u,x,y)$ such that this maximal sequence is a subsequence of C.

The notion of "α-largeness" was defined by Ketonen and Solovay [8]. They connected it with the functions in the Wainer hierarchy, and they also did some Ramsey theory. Sommer [20] developed the theory of ordinals in $I\Delta_0$, and proved many facts about α-largeness in $I\Delta_0 + exp$, including those needed for the connections with the Wainer functions. Sommer did not do the Ramsey theory. In a series of papers [9], [10], [2], [3], [4], [11], [22], Kotlarski, Ratajczyk, Bigorajska, Piekart, and Weiermann gave a thorough development of Ramsey theory for α-largeness, in the setting of PA.

There are some differences in the definitions. Sommer's description of the fundamental sequences looks different from Ketonen and Solovay's, but the definitions really are the same. Sommer's definition of the Wainer functions differs slightly from that of Ketonen and Solovay. Kotlarski and his collaborators defined their fundamental sequences in the same way as Sommer, but they chose a different definition of α-largeness. This choice of definitions yields clean, appealing statements for Ramsey's theorem. We use Sommer's definitions [20] so that we can appeal to the development of the ordinals that he carried out in $I\Delta_0$. We also use facts about α-largeness that Sommer proved in $I\Delta_0 + exp$. We give local versions of these facts, always assuming the existence of a large sequence that bounds the other sequences we need. We take Ketonen and Solovay's definition of the Wainer hierarchy. At the point where we apply Ramsey's theorem for α-largeness, we have already used the Wainer functions to obtain a model of PA.

In Section 2, we give background from Ramsey theory, and we define the Wainer functions and α-largeness. In Section 3, we discuss diagonal indiscernibles. In Section 4, we recall McAloon's original result and the results of Sommer and D'Aquino. In Section 5, we say when a model $\mathcal{A}$ of $I\Delta_0$ has a nonstandard n-elementary initial segment satisfying PA. We first consider the case where $\mathbb{N} \leq_n \mathcal{A}$. Our result here is based on the standard version of Ramsey's theorem. We then drop the assumption that $\mathbb{N} \leq_n \mathcal{A}$. We say when a model of $I\Delta_0$ has an n-elementary initial segment that is a model of PA, and we characterize the elements that can be included in such an initial segment. We work with α-large sets that "bound witnesses" for various sets of formulas, and we define some functions, related to the Wainer functions, that produce these large sets. We close, in Section 6, with a couple of open problems.

2. Ramsey theory and largeness

2.1. *Basic Ramsey theory.* We write $I^{[n]}$ for the set of subsets of I of size n. In our setting, I is a subset of some model of arithmetic, which has a natural ordering, and we may identify sets of size n with increasing n-tuples. A *partition* of $I^{[n]}$ is a function F from $I^{[n]}$ to a set c—we suppose that c has the form $\{0, 1, \ldots, c-1\}$. A set $J \subseteq I$ is *homogeneous for F* if F is constant on $J^{[n]}$. Here is the standard version of Ramsey's theorem.

THEOREM 2.1 (Standard version of Ramsey's theorem). *Let I be an infinite set, and let F be a partition of $I^{[n]}$ into finitely many classes. Then there is an infinite set $J \subseteq I$ that is homogeneous for F.*

The proof proceeds by induction on n. The base case, where $n = 1$, is the standard pigeonhole principle, saying that if F is a partition of an infinite set into finitely many classes, then some class is infinite. There is an inductive lemma, which says that for a partition $F : I^{[n+1]} \to c$, there is an infinite set $I' \subseteq I$ such that for $(x_1, \ldots, x_n, x_{n+1})$ increasing in I', the value of $F(x_1, \ldots, x_n, x_{n+1})$ depends only on $(x_1, \ldots, x_n)$.

The next version of Ramsey's theorem is also well-known (see [7, p. 213]).

THEOREM 2.2 (Infinite Ramsey's theorem for PA). *Let $\mathcal{B}$ be a model of PA. Let I be a cofinal definable set, and let $F : I^{[n]} \to c$ be a definable partition of $I^{[n]}$, where n is standard and $c \in \mathcal{B}$. Then there is a cofinal definable set $J \subseteq I$ that is homogeneous for F.*

There is a well-known finite version of Ramsey's theorem, which we do not use. We want a second finite version, which involves α-largeness [8].

2.2. *Largeness.* Recall that ϵ_0 is the least ordinal α such that $\omega^\alpha = \alpha$. Each $\alpha < \epsilon_0$ can be expressed in Cantor normal form as $\omega^{\beta_1} \cdot x_1 + \cdots + \omega^{\beta_k} \cdot x_k$, where $\alpha > \beta_1 > \cdots > \beta_k$. Sommer [20] formalized the whole theory of ordinals below ϵ_0 in a Δ_0-way, including the notion of fundamental sequence. In particular, he provided a Cantor normal form for all those elements which are ordinals in a model of $I\Delta_0$.

DEFINITION 1. To each ordinal $0 < \alpha < \epsilon_0$, we assign a *fundamental sequence* $\{\alpha\}(x)$ as follows.

- For $\alpha = \beta + 1$, $\{\alpha\}(x) = \beta$ for all x.
- For $\alpha = \omega^{\beta+1}$, $\{\alpha\}(x) = \omega^\beta \cdot x$.
- For $\alpha = \omega^\beta$, where β is a limit ordinal, $\{\alpha\}(x) = \omega^{\{\beta\}(x)}$.
- For $\alpha = \omega^\beta \cdot (a + 1)$, where $a \neq 1$, $\{\alpha\}(x) = \omega^\beta \cdot a + \{\omega^\beta\}(x)$.

- For α with Cantor normal form ending in $\omega^\beta \cdot a$, say $\alpha = \gamma + \omega^\beta \cdot a$, $\{\alpha\}(x) = \gamma + \{\omega^\beta \cdot a\}(x)$.

Below we define a special sequence $(\omega_n)_{n\in\omega}$ of ordinals, cofinal in the interval below ϵ_0.

Definition 2.

$$\omega_0 = 1, \quad \omega_{n+1} = \omega^{\omega_n}.$$

It is also convenient to have a name for a tower of n ω's, with α on top.

Definition 3.

$$\omega_0(\alpha) = \alpha, \quad \omega_{n+1}(\alpha) = \omega^{\omega_n(\alpha)}.$$

We are ready to define α-largeness. We identify a set X, finite or infinite, with the sequence of elements of X, given in increasing order.

Definition 4. The set X is *α-large*, for $\alpha < \epsilon_0$, if there is a sequence $C = (\alpha_0, x_0, \alpha_1, x_1, \ldots, \alpha_{r-1}, x_{r-1}, \alpha_r)$ such that

- $\alpha_0 = \alpha$,
- $\alpha_r = 0$,
- x_0 is the first element of X,
- for $0 < i < r$, x_i is the first element of X that is $> x_{i-1}$,
- for $i < r$, $\alpha_{i+1} = \{\alpha_i\}(x_i)$.

We say that C *witnesses that X is α-large.*

Example. The set $\{3, 4, 5, 6\}$ is ω-large, witnessed by the sequence

$$C = (\omega, 3, 3, 4, 2, 5, 1, 6, 0).$$

We can easily see the following.

Proposition 2.3. *A set X is ω-large if the cardinality of X is greater than the least element.*

In the standard setting, an infinite subset of ω is α-large for all $\alpha < \epsilon_0$. The following is not difficult to prove.

Proposition 2.4. *Let $\mathcal{A}$ be a model of PA, and let X be a cofinal definable set. Then X is α-large for all $\alpha < \epsilon_0$.*

Sommer [20] developed the notion of α-largeness in $I\Delta_0 + exp$. Through most of the present paper, we work in $I\Delta_0$, not assuming that exp is total. We work locally, making sure that the sequences we actually need are bounded by some known element, usually a sequence C witnessing that some set is α-large. Suppose J is α-large, witnessed by the sequence C. Suppose the ordinal β occurs in C. Let C' be the tail of C that begins with β, and let J' be the corresponding tail of J, consisting of the elements of J that do not occur before β in C. Then C' witnesses that J' is β-large. The sequence C' is defined by recursion using a bounded formula. Each initial segment of C'

is bounded by the corresponding initial segment of C (with the same last term).

LEMMA 2.5. *Let J be α-large, witnessed by C. Suppose α has Cantor normal form*

$$\omega^{\beta_1}x_1 + \cdots + \omega^{\beta_n}x_n.$$

Then $J = J_n{}^\frown \cdots {}^\frown J_1$, where J_i is an $\omega^{\beta_i}x_i$-large segment of J. The elements of J_n come first, those in J_1 come last, and, in general, the elements of J_{i+1} come before those of J_i.

Proof. We indicate what happens with the initial segment J_n. The witnessing sequence C for J starts with ordinals of the form

$$\alpha_k = \omega^{\beta_1}x_1 + \cdots + \omega^{\beta_{n-1}}x_{n-1} + \gamma_k,$$

with Cantor normal form matching that of α through the first $n-1$ terms. The last part, which we call γ_k, starts with the value $\omega^{\beta_n}x_n$ and decreases to 0. The witnessing sequence C_n for J_n is obtained from this initial segment of C by replacing each ordinal α_k by γ_k. The sequence C_n can be defined by recursion, using a bounded formula. The initial segments of C_n are bounded by the corresponding initial segments of C. ∎

It is tempting to think that if X is α-large and $\beta < \alpha$, then X should be β-large. However, this need not be true. For example, suppose X is an ω-large set consisting of standard numbers, and let c be nonstandard. Then, thinking of c as a finite ordinal, we have $c < \omega$, but X is not c-large. The following result of Sommer (see [20, p. 149]) says that if X is α-large, then for each $x \le \min(X)$, there is a subsequence X' that is $\{\alpha\}(x)$-large.

PROPOSITION 2.6. *Suppose C witnesses that J is α-large. If $x \le \min(J)$, then $\{\alpha\}(x)$ occurs in C.*

Proof. We do not need exp here. We show by induction on the ordinals β that appear in C that if β is followed in C by j (where $j \in J$), then for all numbers $x \le j$ (not necessarily in J), $\{\beta\}(x)$ appears in C. Everything is bounded by C. ∎

The next two lemmas are proved simultaneously.

LEMMA 2.7. *Suppose C witnesses that J is ω^α-large. Then there exists $J' \subseteq J$ with C' bounded by C witnessing that J' is α-large.*

LEMMA 2.8. *If J is $\omega^\alpha \cdot x$-large, witnessed by C, then for all $y < x$, the ordinal $\omega^\alpha \cdot y$ appears in C.*

Proof of Lemmas 2.7 and 2.8. We proceed by induction on ordinals appearing in the given sequence C.

CASE 1. For $\alpha = 0$, the statements are trivially true.

CASE 2. Consider $\alpha = \beta+1$, where both statements hold for β. First, we prove Lemma 2.7 for $\beta + 1$. Let x be the first term of J. The next ordinal is $\omega^\beta \cdot x$. By the induction hypothesis for Lemma 2.8, ω^β appears later in C. The part of J after this is ω^β-large. By the induction hypothesis for Lemma 2.8, there is a β-large subset J'. Then $x\hat{\ }J'$ is $(\beta+1)$-large. Next, we prove Lemma 2.8 for α. We show by induction on x that for all $y \leq x$, $\omega^\alpha \cdot y$ appears in C. The statement is clear for $x = 0$. Supposing the statement for x, we show it for $x + 1$. We have $\omega^\alpha \cdot (x + 1) = \omega^\alpha \cdot x + \omega^\alpha$. If this appears in C, followed by the element z, then the next term is $\omega^\alpha \cdot x + \omega^\beta \cdot z$. The next few terms have the form $\omega^\alpha \cdot x + \gamma$, where $\gamma < \omega^\beta \cdot z$. We see the γ parts reduce. By Lemma 2.8 for β, we arrive at $\gamma = 0$. So, we have $\omega^\alpha \cdot x$ in C, and by induction, we get all $\omega^\alpha \cdot y$ for all $y < x$.

CASE 3. Let α be a limit ordinal, where both statements hold for $\beta < \alpha$ appearing in C. First, we prove Lemma 2.7 for α. In C, suppose that after ω^α, we have x. The next ordinal is ω^β, where $\beta = \{\alpha\}(x)$. Let J' be the result of removing x from the front of J. Then J' is ω^β-large. By the induction hypothesis, there is a subsesquence J'' of J' that is β-large. Then $x\hat{\ }J''$ is α-large. Next, we prove Lemma 2.8 for α. We show that if $\omega^\alpha \cdot x$ appears in C, then $\omega^\alpha \cdot y$ appears for all $y < x$. The statement is clear for $x = 0$. Supposing that it holds for x, we show it for $x + 1$. Let z be the first term in J. The next ordinal is $\omega^\alpha \cdot x + \omega^\beta$, where $\beta = \{\alpha\}(z)$. Let J' be the result of removing x from the front of J. Then J' is $\omega^\alpha \cdot x + \omega^\beta$-large. Watching the next few terms in C, we see ordinals $\omega^\alpha \cdot x + \gamma$, for $\gamma < \omega^\beta$, with γ reducing to 0. Since we have $\omega^\alpha \cdot x$, we also have $\omega^\alpha \cdot y$ for all $y < x$. ∎

Looking at the proof above, we obtain the following further result.

LEMMA 2.9. *Suppose J is ω^α-large, witnessed by C. Then there is an α-large subsequence $J' = (x_0, x_1, \ldots, x_r)$. Moreover, there is a subsequence of C of the form*

$$(\omega^\alpha, x_0, \omega^{\beta_1}, x_1, \ldots, \omega^{\beta_{r-1}} x_r, 1),$$

where the corresponding sequence

$$C' = (\alpha, x_0, \beta_1, x_1, \ldots, \beta_{r-1}, x_r, 0)$$

witnesses that J' is α-large.

By iterating Lemma 2.9, we obtain the following.

LEMMA 2.10. *Suppose J is $\omega_n(\alpha)$-large, witnessed by C. Then there is a subsequence J' that is α-large.*

2.3. *Connecting largeness with Ramsey theory.* Ketonen and Solovay [8] developed Ramsey theory for α-largeness. Their results can be formalized in PA. We do not need anything more. Given a standard n and $\alpha < \epsilon_0$, we want a standard $\beta < \epsilon_0$ such that if J is β-large and $F : [J]^n \to c$, where

$c \le \min(J)$, then there is an α-large $I \subseteq J$ such that I is homogeneous for F. We do not need a sharp result.

THEOREM 2.11 (Ramsey theorem for α-largeness). *Suppose $n \ge 1$. For each k there exists m such that if $F : J^{[n]} \to c$, where J is ω_m-large and $c \le \min(J)$, then there is an ω_k-large, or even $(\omega_k + 1)$-large, homogeneous set $I \subseteq J$.*

Ketonen and Solovay [8] did not state Theorem 2.11. They were primarily interested in the case where the homogeneous set I is ω-large, and they gave a pigeonhole principle for that case. However, their inductive lemma is perfectly general. To state it, we need one more definition.

DEFINITION 5. For $\alpha < \epsilon_0$, the *norm* of α, denoted by $\|\alpha\|$, is defined inductively as follows:

- $\|0\| = 0$.
- If $\alpha = \omega^{\alpha_1} m_1 + \cdots + \omega^{\alpha_k} m_k$, then $\|\alpha\| = \sum_{j=1}^{k} m_j \cdot (\|\alpha_j\| + 1)$.

Here is Ketonen and Solovay's inductive lemma (see also [15]).

THEOREM 2.12 (Inductive lemma). *Let $n \ge 1$ and let $\omega \le \alpha < \epsilon_0$. Suppose $F : J^{[n+1]} \to c$. If J is θ-large, where $\theta = \omega^\alpha + \omega^3 + \max\{c, \|\alpha\|\} + 3$, then there is an α-large set $I \subseteq J$ such that for increasing tuples $\overline{x}, y$ and $\overline{x}, z$ in J^{n+1}, $F(\overline{x}, y) = F(\overline{x}, z)$; i.e., the value depends only on the first n elements of the tuple.*

Theorem 2.12 yields the following version of the pigeonhole principle.

PROPOSITION 2.13 (Pigeonhole principle). *Let $F : J \to c$. If J is θ-large, where $\theta = \omega^{\alpha+1} + \omega^3 + \max\{c, \|\alpha\|\} + 3$, then there is an α-large set $I \subseteq J$ on which F is constant.*

Proof. For $x, y \in J$ such that $x < y$, let $G(x, y) = F(y)$. Theorem 2.12 yields an $(\alpha + 1)$-large set $I \subseteq J$ such that for pairs in I, the value of G depends only on the first component. For $x, y, y' \in I$, if $x < y, y'$, then

$$F(y) = G(x, y) = G(x, y') = F(y').$$

Let I' be the result of removing the first element from I. Then I' is α-large, and F is constant on I'. ∎

Ketonen and Solovay's pigeonhole principle gives a much better bound than Proposition 2.13 in the special case. If J is $\omega \cdot c$-large, they get an ω-large homogeneous set I.

LEMMA 2.14. *Let $n \ge 1$. If J is ω_{n+2}-large, with first element $\ge c$, then*

- *there exists $J' \subseteq J$ that is $(\omega_{n+1} + \omega^3 + c + 3)$-large,*
- *there exists $J' \subseteq J$ that is $(\omega^{(\omega_n+1)} + \omega^3 + c + 3)$-large.*

The proof of Lemma 2.14 uses various facts on α-largeness. Using the lemma, we get the following relatively simple, although wasteful, version of Theorem 2.12.

PROPOSITION 2.15 (Inductive lemma). *Suppose $F : J^{[n+1]} \to c$, where J is ω_{k+2}-large and $\min(J) \geq c$. Then there is an ω_k-large $I \subseteq J$ such that for increasing tuples $\overline{x}, y$ and $\overline{x}, z$ in J^{n+1}, $F(\overline{x}, y) = F(\overline{x}, z)$. There is also one that is $(\omega_k + 1)$-large.*

Similarly, we get the following simple but wasteful version of Proposition 2.13.

PROPOSITION 2.16 (Pigeonhole principle). *Suppose $F : J \to c$, where J is ω_{k+2}-large and $\min(J) \geq c$. Then there is an ω_k-large $I \subseteq J$ on which F is constant. There is also one that is $(\omega_k + 1)$-large.*

By combining Propositions 2.15 and 2.16, we obtain Theorem 2.11.

2.4. *Wainer functions.* We define the Wainer hierarchy as Ketonen and Solovay [8] did.

DEFINITION 6 (Wainer hierarchy). For $\alpha < \epsilon_0$, $F_\alpha(x)$ is defined as follows:

- $F_0(x) = x + 1$,
- $F_{\alpha+1}(x) = F_\alpha^{(x+1)}(x)$,
- for a limit ordinal α, $F_\alpha(x) = \max\{F_{\{\alpha\}(j)}(x) : j \leq x\}$.

Ketonen and Solovay related the notion of α-largeness to the functions of the Wainer hierarchy. They introduced the function

$$G_\alpha(x) = \mu y([x, y] \text{ is } \alpha\text{-large}),$$

and they proved the following.

THEOREM 2.17. *For any $\alpha < \epsilon_0$,*

$$F_\alpha(n) \leq G_{\omega^\alpha}(n + 1), \qquad G_{\omega^\alpha}(n) \leq F_\alpha(n + 1).$$

Sommer [20] proved Theorem 2.17 in $I\Delta_0$. (Of course, Sommer used his definitions, and Ketonen and Solovay used theirs.)

3. Diagonal indiscernibles. We use the following classification of formulas.

DEFINITION 7.

- The B_0 *formulas* are just the Δ_0-formulas.
- The Σ_{n+1} *formulas* have the form $(\exists \overline{u})\, \varphi$, where φ is a B_n formula.
- The B_{n+1} *formulas* are obtained from the Σ_{n+1} formulas by taking Boolean combinations and adding bounded quantifiers.

NOTATION. For each $n \in \mathbb{N}$, B_n^T denotes the set of triples $(\varphi, \overline{u}, \overline{x})$, where φ is a B_n formula and $\overline{u}$ and $\overline{x}$ are the free variables of φ, partitioned into two disjoint parts. We identify these triples with their codes. When we write $\varphi(\overline{u}, \overline{x})$, indicating a split of the variables, we are identifying the formula with the triple $(\varphi, \overline{u}, \overline{x})$, which is in B_n^T for some n.

NOTATION. We write $\overline{a} \le b$, $\overline{a} < b$, $b \le \overline{a}$, $b < \overline{a}$ to mean that all elements of $\overline{a}$ are $\le b$, $< b$, $\ge b$, $> b$, respectively.

In results of Paris and his co-authors [15], [17]–[19], and in McAloon's theorem, and other more recent results, we obtain a model of PA from a special set of indiscernibles.

DEFINITION 8. Let I be a subset of a model $\mathcal{A}$. We say that I is *diagonal indiscernible* for $\varphi(\overline{u}, \overline{x})$—identified with the triple $(\varphi, \overline{u}, \overline{x})$—if for all $i < \overline{j}, \overline{k}$ in I,

$$\mathcal{A} \models (\forall \overline{u} \le i)\,[\varphi(\overline{u}, \overline{j}) \leftrightarrow \varphi(\overline{u}, \overline{k})].$$

The next lemma says how a model of PA is obtained from a set that is diagonal indiscernible for all bounded formulas.

PROPOSITION 3.1. *Let $\mathcal{A}$ be a model of $I\Delta_0$. Suppose I has order type ω under the ordering of $\mathcal{A}$, and*

(i) *for $i, j \in I$, $\mathcal{A} \models i < j \to i^2 < j$,*
(ii) *I is diagonal indiscernible for all elements of B_0^T.*

If $\mathcal{B}$ is the downward closure of I, then $\mathcal{B}$ is a model of PA.

Proof. Condition (i) guarantees that $\mathcal{B}$ is closed under addition and multiplication, so it is a model of $I\Delta_0$. Condition (ii) lets us convert arbitrary formulas into bounded formulas, using the following lemma.

LEMMA 3.2. *For each formula $\varphi(\overline{u})$, there is a bounded formula $\varphi^*(\overline{u}, \overline{v})$ such that if $k, \overline{i}$ is strictly increasing in I, and $\overline{b} \le k$, then $\mathcal{A} \models \varphi(\overline{b}) \leftrightarrow \varphi^*(\overline{b}, \overline{i})$.*

Idea of proof. We illustrate in an example. Suppose

$$\varphi(\overline{u}) = (\forall x)\,(\exists y)\,\delta(\overline{u}, x, y),$$

where $\delta(\overline{u}, x, y)$ is quantifier-free. We take $\varphi^*(\overline{u}, v, v')$ to be

$$(\forall x \le v)\,(\exists y \le v')\,\delta(\overline{u}, x, y).$$

If $\overline{b} \le k < i < j$, where $k, i, j \in I$, then we have

$$\mathcal{B} \models (\forall x)\,(\exists y)\,\delta(\overline{b}, x, y) \quad \text{iff} \quad \mathcal{B} \models (\forall x \le i)\,(\exists y \le j)\,\delta(\overline{b}, x, y). \ \blacksquare$$

Using Lemma 3.2, we can show that $\mathcal{B}$ satisfies induction for all formulas. Suppose $\mathcal{B} \models \varphi(\overline{b}, 0)$ and $\mathcal{B} \models (\forall y)\,[\varphi(\overline{b}, y) \to \varphi(\overline{b}, y + 1)]$. We must show that $\mathcal{B} \models \varphi(\overline{b}, c)$ for all c. Let $\varphi^*(\overline{u}, x, \overline{v})$ be as in Lemma 3.2, and take $k, \overline{i}$, increasing in I, with $\overline{b}, c \leq k$. Then

$$\mathcal{B} \models \varphi^*(\overline{b}, 0, \overline{i}) \quad \text{and} \quad \mathcal{B} \models (\forall y < k)\,[\varphi^*(\overline{b}, y, \overline{i}) \to \varphi^*(\overline{b}, y+1, \overline{i})].$$

Therefore, $\mathcal{B} \models \varphi^*(\overline{b}, c, \overline{i})$, so $\mathcal{B} \models \varphi(\overline{b}, c)$. ∎

The lemma below gives existence of diagonal indiscernibles in the standard model $\mathbb{N}$.

LEMMA 3.3. *If $I \subseteq \mathbb{N}$ is infinite, then for any formula $\varphi(\overline{u}, \overline{x})$, there is an infinite set $J \subseteq I$ that is diagonal indiscernible for $\varphi(\overline{u}, \overline{x})$. The same is true for any finite set of formulas.*

Proof. Suppose $\overline{u}$ has length m and $\overline{x}$ has length n. For any standard c, and any infinite set $S \subseteq I$, we partition the increasing n-tuples in S such that tuples $\overline{b}$ and $\overline{b}'$ lie in the same class provided that for all m-tuples $\overline{a} \leq c$, $\mathcal{A} \models \varphi(\overline{a}, \overline{b}) \leftrightarrow \varphi(\overline{a}, \overline{b}')$. Theorem 2.1 yields an infinite set $S' \subseteq S$ such that all n-tuples in S' lie in the same class in the partition. We iterate this to produce a nested sequence $(S_k)_{k\in\omega}$ of infinite sets, where $S_0 = I$, and S_{k+1} is obtained as above with $c = k$ and $S = S_k$. Now, we choose an increasing sequence $(a_k)_{k\in\omega}$ of numbers such that $a_0 \in S_0$, and $a_{k+1} \in S_{a_k}$, with $a_k < a_{k+1}$. Then $J = \{a_k : k \in \omega\}$ is the desired set of diagonal indiscernibles for $\varphi(\overline{u}, \overline{x})$. ∎

The next lemma is similar to Lemma 3.3, except that $\mathbb{N}$ is replaced by an arbitrary model of PA, and the sets of indiscernibles that we obtain are finite.

LEMMA 3.4. *Let $\mathcal{A}$ be a model of PA, and let I be a cofinal definable set. For any finite r and any finite set Γ of formulas (with free variables split), there is a set $J \subseteq I$ of size at least r that is diagonal indiscernible for all $\varphi(\overline{u}, \overline{x}) \in \Gamma$.*

Proof. Say the elements of Γ are $\varphi_i(\overline{u}_i, \overline{x}_i)$ for $1 \leq i \leq K$. For any c and any cofinal definable set $S \subseteq I$, we partition the increasing n_i-tuples in S so that tuples $\overline{b}$ and $\overline{b}'$ lie in the same class provided that for all m_i-tuples $\overline{a} \leq c$, $\mathcal{A} \models \varphi(\overline{a}, \overline{b}) \leftrightarrow \varphi(\overline{a}, \overline{b}')$. Theorem 2.2 yields a cofinal set $S' \subseteq S$ such that all n-tuples in S' lie in the same class in the partition. We say that S' is *homogeneous for $\varphi_i(\overline{u}, \overline{x})$ over c*. Let a be first in I. Applying the procedure above K times, we get a cofinal definable set $J_1 \subseteq I$ homogeneous for all $\varphi_i(\overline{u}_i, \overline{x}_i)$ over a_0. Let a_1 be first in J_1 greater than a_0. Applying the procedure above K more times, we get a cofinal definable set $J_2 \subseteq J_1$ homogeneous for all $\varphi_i(\overline{u}_i, \overline{x}_i)$ over a_1. Let a_2 be first in J_2 greater than a_1. We continue until we have $a_1, \ldots, a_r$. This is the desired set of diagonal indiscernibles. ∎

Lemma 3.3 is based on Theorem 2.1, while Lemma 3.4 is proved using Theorem 2.2. Using ideas from the proof of the MacDowell–Specker theorem (see [13]) instead of Theorem 2.2, we could obtain the following stronger statement. (We do not actually use this result.)

PROPOSITION 3.5. *Let $\mathcal{A}$ be a model of PA, and let I be a cofinal definable set. For any finite set Γ of formulas* (*with the free variables split*), *there is a cofinal definable set $J \subseteq I$ that is diagonal indiscernible for all $\varphi(\overline{u}, \overline{x}) \in \Gamma$.*

We give one more result on existence of diagonal indiscernibles. We need some further definitions.

DEFINITION 9. Let $\mathcal{A}$ be a model of PA. Let Γ be a finite set of formulas $\varphi(\overline{u}, \overline{x})$ with the free variables split into two parts, and let a be an element of $\mathcal{A}$. Say the formulas of Γ are $\varphi_i(\overline{u}_i, \overline{x}_i)$ for $1 \le i \le K$, where $\overline{u}_i$ has length m_i and $\overline{x}_i$ has length n_i.

- Let n^Γ be the greatest n_i.
- For a given Γ, $n = n^\Gamma$, and a, let $F^{\Gamma,a}$ be the partition of $\mathcal{A}^{[n]}$ such that tuples $\overline{x}$ and $\overline{y}$ lie in the same class if for all i and all m_i-tuples $\overline{u}_i \le a$, for all n_i-tuples $\overline{x}_i \subseteq \overline{x}$ and corresponding $\overline{y}_i \subseteq \overline{y}$,
$$\mathcal{A} \models \varphi_i(\overline{u}_i, \overline{x}_i) \leftrightarrow \varphi_i(\overline{u}_i, \overline{y}_i).$$

Note that for a given Γ, there is a function g, definable in PA, such that for all a, $g(a)$ bounds the number of equivalence classes under the partition $F^{\Gamma,a}$. We may let $g(a) = 2^{M(a)}$, where
$$M(a) = \prod_{i=1}^{K} \binom{a+1}{m_i} \cdot \binom{n}{n_i}.$$

NOTATION. Let g^Γ be the fixed function g described above.

PROPOSITION 3.6. *Let $\mathcal{A}$ be a model of PA. Let Γ be a finite set of formulas with the free variables split, and let n^Γ and g^Γ be as above. Let r be a standard number. There is a standard number m such that if I is $(\omega_m + 1)$-large, and for $i, j \in I$,*
$$\mathcal{A} \models i < j \to g^\Gamma(i) < j,$$
then there is a subset of I_r of size r that is diagonal indiscernible for all elements of Γ.

Proof. Let $n = n^\Gamma$ and let $g = g^\Gamma$. Let $m_1, \ldots, m_r$ be a decreasing sequence of standard numbers such that $m_{r-1} = 1$ and if J is ω_{m_i}-large and $F : J^{[n]} \to c$, where $\min(J) \ge c$, then there is a homogeneous set $J' \subseteq J$ that is $(\omega_{m_{i+1}} + 1)$-large. We pass from m_{i+1} to m_i by applying Theorem 2.11. Let $m = m_1$, and let I be $(\omega_m + 1)$-large. Let a be the first element of I. Let I_1 be the result of removing a from I. Then I_1 is ω_{m_1}-large. We restrict to $I_1^{[n]}$ the partition $F^{\Gamma,a}$ described above. Then $g(a)$ bounds the number of equivalence classes. Let $J_1 \subseteq I_1$ be a homogeneous set that is $(\omega_{m_2}+1)$-large, and let a_1 be the first element. Let I_2 be the result of removing a_1 from J_1. Then I_2 is ω_{m_2}-large. We restrict the partition F^{Γ,a_1} to $I_2^{[n]}$. Then $g(a_1)$ bounds the number of equivalence classes. Let $J_2 \subseteq I_2$ be a homogeneous set that is $(\omega_{m_3}+1)$-large, and let a_2 be the first element. Let I_3 be the result of removing a_2 from J_2. Then I_3 is ω_{m_3}-large. We restrict to $I_3^{[n]}$ the partition F^{Γ,a_2}. Then $g(a_2)$ bounds the number of equivalence classes. Let $J_3 \subseteq I_3$ be a homogeneous set that is $(\omega_{m_4}+1)$-large, and let a_3 be the first element. We continue in this way until we come to J_{r-1} that is $(\omega+1)$-large, and we let a_{r-1} and a_r be the first two elements. Then $\{a_1,\ldots,a_r\}$ is the desired subset of J that is diagonal indiscernible for all elements of Γ. ∎

4. Initial segments satisfying PA. Here is the original result of McAloon [12].

THEOREM 4.1 (McAloon). *Let $\mathcal{A}$ be a nonstandard model of $I\Delta_0$. Then there is an initial segment $\mathcal{B}$ that is a nonstandard model of PA.*

Proof. We may suppose that $\mathbb{N}$ is an initial substructure of $\mathcal{A}$. Let b be a nonstandard element. Let J be an infinite subset of $\mathbb{N}$ such that for $i,j\in J$, if $i<j$, then $i^2<j$. For any finite $\Gamma \subseteq B_0^T$ and any r, we can apply Lemma 3.3 to get an infinite set $I \subseteq J$ that is diagonal indiscernible for the elements of Γ. There are increasing sequences of elements of I of arbitrarily large finite length.

We have a bounded formula $\psi(b,u)$, with parameter b, saying that there is an increasing sequence σ of length u, with code $\le b$, such that if $i<j$ in $\mathrm{ran}(\sigma)$, then $i^2<j$, and σ is diagonal indiscernible for all $\varphi(\overline{u},\overline{x}) \in B_0^T$ such that $\varphi \le u$. For all standard n, $\mathcal{A} \models \psi(b,n)$. Then by overspill, there is some some nonstandard ν such that $\mathcal{A}\models\psi(b,\nu)$. Let σ be the witnessing sequence. The restriction of σ to standard terms yields a set I, ordered in type ω, that is diagonal indiscernible for all $\sigma \in B_0^T$. By Proposition 3.1, the downward closure of I is the desired nonstandard initial segment of $\mathcal{A}$ satisfying PA. ∎

Next, we summarize the known results saying which elements of a model $\mathcal{A}$ of $I\Delta_0$ can be included in an initial segment that is a nonstandard model of PA. One way to characterize these elements a is to say that a lies below an infinite set I, of order type ω, such that I is diagonal indiscernible for all elements of B_0^T. A second characterization says that there are finite approximations to such a set I, where these are all bounded in such a way that we can apply overspill to get an infinite set. Sommer [20] and D'Aquino [5] gave a third characterization, in terms of the Wainer functions [21]. These functions were used by Ketonen and Solovay [8] and Paris [15] in characterizing the provably recursive functions of PA and $I\Sigma_n$.

THEOREM 4.2 (McAloon, Sommer, D'Aquino, Paris, Dimitracopoulos). *Let $\mathcal{A}$ be a model of $I\Delta_0$, and let a be a nonstandard element. Then the following are equivalent*:

(i) *There is an initial segment $\mathcal{B}$ of $\mathcal{A}$ such that $a \in \mathcal{B}$ and $\mathcal{B}$ is a model of PA.*

(ii) *There is an infinite set I of order type ω, consisting of elements greater than a, such that if $i < j$ in I, then $\mathcal{A} \models i^2 < j$, and I is diagonal indiscernible for all $\varphi(\overline{u}, \overline{x})$ in B_0^T.*

(iii) *There exist b and c such that c codes satisfaction of bounded formulas by tuples $\leq b$, and for all finite r, there is a sequence I_r of size r, with $a < I_r < b$, such that if $i < j$ in I_r, then $\mathcal{A} \models i^2 < j$, and I_r is diagonal indiscernible for the first r elements of B_0^T.*

(iv) *There exists b such that for all $\alpha < \epsilon_0$, $F_\alpha(a) \downarrow < b$.*

Remarks on proof. It seems to us natural to try to prove Theorem 4.2 by showing (i)$\Rightarrow$(iv)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i). However, we have not found a published proof that proceeds in this way. We get (iii)$\Rightarrow$(ii) by applying overspill to a bounded formula $\psi(u, a, b, c)$ saying that there is an increasing sequence $\sigma < b$ of length u with first term $> a$, such that for successive terms i, j, $i^2 < j$, and σ is diagonal indiscernible for all $\varphi(\overline{u}, \overline{x}) \leq u$ in B_0^T. Proposition 3.1 gives the implication (ii)$\Rightarrow$(i). It is not difficult to show (i)$\Rightarrow$(iv). To complete the proof, it is enough to show (iv)$\Rightarrow$(iii). Sommer [20] and D'Aquino [5] showed, in a quite complicated proof, that (iv)$\Rightarrow$(i). It is not difficult to show (i)$\Rightarrow$(iii), so we get the implication (iv)$\Rightarrow$(iii). ∎

5. n-elementary initial segments satisfying PA. We turn to our main results, on n-elementary initial segments. Let $\mathcal{A}$ and $\mathcal{B}$ be structures for the language of arithmetic, where $\mathcal{B}$ is a substructure of $\mathcal{A}$.

DEFINITION 10. Let $\mathcal{A}$ and $\mathcal{B}$ be structures for the language of arithmetic. We say that $\mathcal{B}$ is an *n-elementary substructure* of $\mathcal{A}$, and we write $\mathcal{B} \leq_n \mathcal{A}$, if for all B_n formulas $\varphi(\overline{x})$ and all $\overline{b}$ in $\mathcal{B}$, $\mathcal{B} \models \varphi(\overline{b})$ iff $\mathcal{A} \models \varphi(\overline{b})$.

Note that if $\mathcal{B}$ is an initial substructure of $\mathcal{A}$, then $\mathcal{B} \leq_0 \mathcal{A}$.

The following is a version of the familiar Tarski criterion for n-elementary substructure.

LEMMA 5.1 (Tarski criterion). *Let $\mathcal{B} \leq_0 \mathcal{A}$, and let $n > 0$. Suppose that for all B_{n-1} formulas $\varphi(\overline{x}, \overline{u})$, and for all $\overline{b}$ in $\mathcal{B}$ (appropriate to substitute for $\overline{x}$), if there exists $\overline{d}$ such that $\mathcal{A} \models \varphi(\overline{b}, \overline{d})$, then there exists $\overline{d}'$ in $\mathcal{B}$ such that $\mathcal{A} \models \varphi(\overline{b}, \overline{d}')$. Then $\mathcal{B} \leq_n \mathcal{A}$.*

Proof. Let S be the set of formulas $\varphi(\overline{x})$ such that for $\overline{b}$ in $\mathcal{B}$, $\mathcal{B} \models \varphi(\overline{b})$ iff $\mathcal{A} \models \varphi(\overline{b})$. We show that S includes all B_n formulas. Clearly, S is closed under Boolean combinations and bounded quantifiers. It is straightforward to show by induction on $k \leq n$ that all B_k formulas are in S. ∎

Our goal is to produce initial segments $\mathcal{B}$ of a model $\mathcal{A}$ of $I\Delta_0$ such that $\mathcal{B} \leq_n \mathcal{A}$ and $\mathcal{B}$ satisfies full PA. Let $\mathcal{A}$ be a model of $I\Delta_0$. One way to obtain an n-elementary initial substructure is to take the downward closure of the set of elements definable by Σ_n formulas from a set of parameters (see [7, p. 135]). Alternatively, we may produce an n-elementary initial substructure by taking the downward closure of a set I of order type ω such that I "bounds witnesses" for B_{n-1} formulas. We give the definition below.

DEFINITION 11. Let $\mathcal{A}$ be a structure for the language of arithmetic and let $\varphi(\overline{u}, \overline{x})$ be a formula with the free variables separated into $\overline{u}$ and $\overline{x}$. We say that I *bounds witnesses for* $\varphi(\overline{u}, \overline{x})$ if for all $i, j \in I$ such that $\mathcal{A} \models i < j$, and all $\overline{a} \leq i$ in $\mathcal{A}$,

$$\mathcal{A} \models (\exists \overline{x})\, \varphi(\overline{a}, \overline{x}) \rightarrow (\exists \overline{x} < j)\, \varphi(\overline{a}, \overline{x}).$$

The lemma below is an extension of Proposition 3.1.

LEMMA 5.2. *Let $\mathcal{A}$ be a model of $I\Delta_0$, and let $n > 0$. Suppose $I \subseteq \mathcal{A}$ is a set of order type ω that is diagonal indiscernible for all elements of B_0^T and bounds witnesses for all elements of B_{n-1}^T. Let $\mathcal{B}$ be the downward closure of I. Then $\mathcal{B}$ is an n-elementary initial substructure of $\mathcal{A}$ satisfying full PA.*

Proof. Since I is cofinal in $\mathcal{B}$ and bounds witnesses for all elements of B_{n-1}^T, we can apply Lemma 5.1 to see that $\mathcal{B} \leq_n \mathcal{A}$. Among the bounded formulas is $u^2 = x$, so for $i, j \in I$, $\mathcal{A} \models i < j \rightarrow i^2 < j$. Since I is diagonal indiscernible for all elements of B_0^T, we can apply Proposition 3.1 to see that $\mathcal{B}$ is a model of PA. ∎

We begin by considering a model $\mathcal{A}$ of $I\Delta_0$ such that $\mathbb{N} \leq_n \mathcal{A}$. For $n = 0$, this is automatically true. For $n > 0$, however, it is a nontrivial assumption. We say, under this special assumption, when there is a *nonstandard* n-elementary substructure satisfying full PA.

LEMMA 5.3. *Suppose $\mathbb{N} \leq_n \mathcal{A}$. If $I \subseteq \omega$ is an infinite subset of $\mathbb{N}$, and $\beta(\overline{x}, \overline{u})$ is B_{n-1}, then there is an infinite set $J \subseteq I$ that bounds witnesses for $\beta(\overline{x}, \overline{u})$.*

Proof. Say $\overline{x}$ has length m and $\overline{u}$ has length n. We define a sequence $(j_0, j_1, j_2, \ldots)$ by induction. Let j_0 be an arbitrary element of I. Suppose we have determined j_k. There are finitely many m-tuples $\overline{a}$ in $\mathbb{N}$ such that $\overline{a} \leq j_k$. For each such $\overline{a}$, if $\mathcal{A} \models (\exists \overline{u})\, \beta(\overline{a}, \overline{u})$, then since $\mathbb{N} \leq_n \mathcal{A}$, there is some $\overline{b}$ in $\mathbb{N}$ such that $\mathbb{N} \models \beta(\overline{a}, \overline{b})$. We choose the first such $\overline{b}$. Let j_{n+1} be an element of I, greater than j_n, and greater than all of the chosen $\bar{b}$. Then $J = \{j_k : k \in \omega\}$ is the desired set bounding witnesses for $\beta(\overline{u}, \overline{x})$. ∎

THEOREM 5.4. *Suppose that $\mathcal{A}$ is a nonstandard model of $I\Delta_0$ such that $\mathbb{N} \leq_n \mathcal{A}$. Then the following are equivalent*:

(i) *There is a nonstandard initial segment $\mathcal{B}$ such that $\mathcal{B} \leq_n \mathcal{A}$ and $\mathcal{B}$ is a model of PA.*

(ii) *There exist b and c such that b is nonstandard and c codes satisfaction of Σ_n formulas in $\mathcal{A}$ by tuples $\overline{x} \leq b$.*

NOTE. For $n = 0$, statements (i) and (ii) are simply true, by Theorem 4.1 (McAloon's theorem). Even for larger n, we do not know of an example in which the statements are invalid.

Proof. To prove that (i)⇒(ii), suppose b is a nonstandard element of $\mathcal{B}$. There exists c in $\mathcal{B}$ coding satisfaction in $\mathcal{B}$ of Σ_n formulas by tuples bounded by b. Since $\mathcal{B} \leq_n \mathcal{A}$, c also codes satisfaction in $\mathcal{A}$. We must prove that (ii)⇒(i). We shall obtain the model $\mathcal{B}$ by applying Lemma 5.2. We need a set I of order type ω that is diagonal indiscernible for elements of B_0^T and bounds witnesses for elements of B_{n-1}^T. The outline is like that for Theorem 4.1. To obtain the required set I, we show that there are finite approximations in $\mathbb{N}$, and then apply overspill to an appropriate bounded formula.

We get the finite approximations to I by using Lemma 5.3 together with Lemma 3.3. We have a bounded formula $\psi(u, b, c)$, with parameters b and c, saying that there is a sequence $\sigma \leq b$ of length u such that σ is diagonal indiscernible for all elements of B_0^T bounded by u, and σ bounds witnesses for all elements of B_{n-1}^T bounded by u. The formula $\psi(u, b, c)$ is satisfied in $\mathcal{A}$ by all standard n. Therefore, by overspill, it is satisfied by some nonstandard ν. Let σ be a witness. Taking the restriction of σ to standard number inputs, we get a set I of order type ω that is diagonal indiscernible for all elements of B_0^T, and bounds witnesses for all elements of B_{n-1}^T. ∎

Now, we drop the assumption that $\mathbb{N} \leq_n \mathcal{A}$. The following result is analogous to Lemma 5.3. In what follows, we use it only for inspiration.

PROPOSITION 5.5. *Let $\mathcal{B}$ be a model of PA. If I is a cofinal definable set, and $\beta(\overline{u}, \overline{x})$ is a B_{n-1} formula, then there is a cofinal definable set $J \subseteq I$ that bounds witnesses for $\beta(\overline{u}, \overline{x})$.*

Proof. We have a definable function $G : \mathcal{B} \to I$ such that

- $G(0) = \min(I)$,
- $G(a + 1)$ is the first $b \in I$ such that $b > G(a)$ and for all $\overline{u} \leq G(a)$,

$$\mathcal{B} \models (\exists \bar{x})\, \beta(\overline{u}, \overline{x}) \to (\exists \bar{x} < b)\, \beta(\overline{u}, \overline{x}).$$

Let $J = \operatorname{ran}(G)$. ∎

We define a family of partial functions $F_{\Gamma,\alpha}$, for finite $\Gamma \subseteq B^T_{n-1}$ and $\alpha < \epsilon_0$, such that $F_{\Gamma,\alpha}(a)$ is a specific sequence C witnessing the existence of an α-large sequence J such that $a < J$ and J bounds witnesses for all elements of Γ. We identify ordinals with their codes.

Definition 12. Let Γ be a finite subset of B^T_{n-1} and let $\alpha < \epsilon_0$. Assuming that $F_{\Gamma,\alpha}(a)$ is defined, it is the sequence $(\alpha_0, x_0, \alpha_1, \ldots, \alpha_{r-1}, x_{r-1}, \alpha_r)$ with the following properties:

- $\alpha_0 = \alpha$,
- if $\alpha_i = 0$, then C has length $2i + 1$ (i.e., $r = i$),
- if $\alpha_i \neq 0$, then
  (a) if $i = 0$, then x_0 is the first $z > a$ such that for all $\varphi(\overline{u}, \overline{x}) \in \Gamma$,
  $$(\forall \overline{u} \leq a)\,(\exists \overline{x})\,\varphi(\overline{u}, \overline{x}) \rightarrow (\exists \overline{x} < z)\,\varphi(\overline{u}, \overline{x}),$$
  (b) if $i > 0$, then x_i is the first $z > x_{i-1}$ such that for all $\varphi(\overline{u}, \overline{x}) \in \Gamma$,
  $$(\forall \overline{u} \leq x_{i-1})\,(\exists \overline{x})\,\varphi(\overline{u}, \overline{x}) \rightarrow (\exists \overline{x} < z)\,\varphi(\overline{u}, \overline{x}),$$
- if $\alpha_i = \beta \neq 0$ and $x_i = z$, then $\alpha_{i+1} = \{\beta\}(z)$.

We have $F_{\Gamma,\alpha}(a) \downarrow$ provided that we can carry out all of these computations, and we come to some $\alpha_i = 0$.

The result below is the analogue of Theorem 4.2.

Theorem 5.6. *Let $\mathcal{A}$ be a nonstandard model of $I\Delta_0$, and let $n > 0$. Then the following are equivalent*:

(i) *there is a nonstandard n-elementary initial segment $\mathcal{B}$ satisfying PA*,
(ii) *there exists a set I, of order type ω, such that I is diagonal indiscernible for all elements of B^T_0 and bounds witnesses for all elements of B^T_{n-1}*,
(iii) *there exist b and c such that c codes satisfaction of Σ_n formulas by tuples $\leq b$, and for each finite r, there is a sequence I_r of length r, with code $< b$, such that I_r is diagonal indiscernible for the first r elements of B^T_0 and bounds witnesses for the first r elements of B^T_{n-1}*,
(iv) *there exist b and c such that c codes satisfaction of Σ_n formulas by tuples $\leq b$, and for all standard ordinals $\alpha < \epsilon_0$ and all finite $\Gamma \subseteq B^T_{n-1}$, $F_{\Gamma,\alpha}(0) \downarrow< b$.*

Note that if $\mathcal{A}$ is a nonstandard model of $I\Delta_0$ such that $\mathbb{N}$ is an initial segment but *not* an n-elementary initial segment, then for some B_{n-1} formula $\varphi(\overline{x})$ satisfied in $\mathcal{A}$, there is no standard witness (satisfying the formula), and any n-elementary initial segment must include such a witness. If $\mathcal{A}$ has an n-elementary initial segment satisfying PA, then any B_{n-1} formula that is satisfied has a first witness.

We will obtain Theorem 5.6 from the following stronger result, saying which elements can be included in an initial segment that is n-elementary and satisfies full PA.

THEOREM 5.7. *Suppose $\mathcal{A}$ is a model of $I\Delta_0$, and let $n > 0$. For an element a, the following are equivalent*:

(i) *a is contained in a nonstandard n-elementary initial segment $\mathcal{B}$ that is a model of PA*,
(ii) *there is a set I, of order type ω, such that $a < I$, and I is diagonal indiscernible for all elements of B_0^T and bounds witnesses for all elements of B_{n-1}^T*,
(iii) *there exist $b > a$ and c such that c codes satisfaction of Σ_n formulas by tuples $\leq b$, and for each finite r, there is a sequence I_r of length r, with code $< b$, such that $a < I_r$, and I_r is diagonal indiscernible for the first r elements of B_0^T and bounds witnesses for the first r elements of B_{n-1}^T*,
(iv) *there exist b and c such that c codes satisfaction of Σ_n formulas by tuples $\leq b$, and for all $\alpha < \epsilon_0$ and all finite $\Gamma \subseteq B_{n-1}^T$, $F_{\Gamma,\alpha}(a) \downarrow< b$.*

Before proving Theorem 5.7, we note that if we let $a = 0$ in the statement of Theorem 5.7, then we get Theorem 5.6. With $a = 0$, each of the numbered statements in Theorem 5.7 is easily seen to be equivalent to the corresponding statement in Theorem 5.6.

Proof of Theorem 5.7. Our plan is to show (iii)⇒(ii)⇒(i)⇒(iv) and then (iv)⇒(iii).

(iii)⇒(ii): We can write a bounded formula $\psi(u, a, b, c)$ saying that there exists an increasing sequence σ of length u such that

- $a < \sigma$ and σ has a code $< b$,
- σ bounds witnesses for elements of B_{n-1}^T with codes $< u$ and is diagonal indiscernible for elements of B_0^T with codes $< u$.

To talk about satisfaction in a bounded way, we use the parameter c. By (iii), $\psi(u, a, b, c)$ is satisfied in $\mathcal{A}$ by all standard u. Then by overspill, it is satisfied by some nonstandard u. Let σ be a witness, and let I be the sequence of standard terms. This set satisfies (ii).

(ii)⇒(i): This follows immediately from Lemma 5.2.

(i)⇒(iv): We work in the initial n-elementary substructure $\mathcal{B}$ that is a model of PA and contains the element a. We get the fact that $F_{\Gamma,\alpha}(a)$ is defined using the ideas from Lemma 3.3 and Proposition 5.5. We can do the calculations in $\mathcal{B}$, knowing that they are the same in $\mathcal{A}$. Let $\Lambda(a, x)$ be a computable set of formulas of bounded complexity saying $F_{\Gamma,\alpha}(a) \downarrow< x$ for all standard $\alpha < \epsilon_0$. Every finite subset of $\Lambda(a, x)$ is satisfied in $\mathcal{B}$. Therefore, the whole set is satisfied in $\mathcal{B}$ by some element b. We have c coding satisfaction in $\mathcal{B}$ of Σ_n formulas by tuples $\leq b$. Satisfaction of these formulas in $\mathcal{A}$ is the same.

(iv)$\Rightarrow$(iii): Proving that (iv)$\Rightarrow$(iii) will take several steps. Here is the outline.

Steps in proving that (iv)$\Rightarrow$(iii)

1. Give a bounded formula $\varphi(u, a, b, c)$ saying that there exists $b' < b$ such that for each ordinal α with code $\leq u$, there exist J_1, J_2, C_1, C_2 such that
   (a) J_1, J_2 bound witnesses for all $\varphi \leq u$ in B^T_{n-1},
   (b) C_i witnesses that J_i is α-large,
   (c) $a < J_1$ and J_1 has a code $< b'$,
   (d) $b' < J_2$, and C_1 and C_2 have codes $< b$.
2. Show that $\mathcal{A} \models \varphi(u, a, b, c)$ for all standard u.
3. Apply overspill to get a nonstandard u satisfying $\varphi(u, a, b, c)$. Then we get $b' < b$ such that for all standard $\alpha < \epsilon_0$, there exist J_1, J_2, C_1, C_2 such that
   (a) J_1, J_2 bound witnesses for all standard elements of B^T_{n-1},
   (b) C_i witnesses that J_i is α-large,
   (c) $a < J_1$ and J_1 has a code $< b'$,
   (d) $b' < J_2$ and C_1 and C_2 have codes $< b$.
4. Suppose $b' < b$, where for all standard α, there exist J and C such that
   (a) J bounds witnesses for all elements of B^T_0,
   (b) C witnesses that J is α-large,
   (c) $b' < J$ and C has a code $< b$.

   Show that $F_\alpha(b') \downarrow < b$ for all standard α. Then by Theorem 4.2, there is an initial segment $\mathcal{B}$ of $\mathcal{A}$ containing b' such that $\mathcal{B}$ is a model of PA.
5. Show that for each finite r, there exists a set I_r of size r such that
   (a) $a < I_r$ and I_r has a code $< b'$,
   (b) I_r bounds witnesses for the first r elements of B^T_{n-1} and is diagonal indiscernible for the first r elements of B^T_0.

We discuss the five steps in order.

STEP 1. It is not difficult to write a bounded formula $\varphi(u, a, b, c)$ with the desired meaning. We use c to talk about satisfaction of the formulas in B^T_{n-1}.

STEP 2. Take a standard number u. Let Γ be the finite set of elements of B^T_{n-1} with codes $\leq u$, and let $\alpha_1, \ldots, \alpha_k$ be the ordinals with codes $\leq u$.

We show that there exists $b' < b$ such that for each α_i, there exist $J_{1,i}$, $J_{2,i}$, $C_{1,i}$, $C_{2,i}$ with the following features:

- $J_{1,i}, J_{2,i}$ bound witnesses for all elements of Γ,
- $C_{1,i}$, $C_{2,i}$, witness that $J_{1,i}$, $J_{2,i}$ are α_i-large,
- $a < J_{1,i}$ and $J_{1,i}$ has a code $< b'$,
- $b' < J_{2,i}$ and $C_{2,i}$ has a code $< b$.

We may suppose $\alpha_1 < \cdots < \alpha_k$. Take the least m such that $\omega_m > \omega^{\alpha_k}$, and let

$$\alpha = \omega_m(\alpha_k) + \cdots + \omega_m(\alpha_1) + \omega_m + \omega^{\alpha_k} + \cdots + \omega^{\alpha_1}.$$

We are assuming statement (iv) (from Theorem 5.7), so there exist J and C such that

- J bounds witnesses for all elements of Γ,
- C witnesses that J is α-large,
- $a < J$ and C has a code $< b$.

By Lemma 2.5, we have

$$J = J_{1,1}\hat{}\ldots\hat{}J_{1,k}\hat{}J^*\hat{}J_{2,1}\hat{}\ldots\hat{}J_{2,k},$$

where $J_{1,i}$ is ω^{α_i}-large, J^* is ω_m-large, and $J_{2,i}$ is $\omega_m(\alpha_i)$-large. The elements of $J_{1,i}$ are smaller than those of $J_{1,i+1}$, those of $J_{1,k}$ are smaller than those of J^*, those of J^* are smaller than those of $J_{2,1}$, and those of $J_{2,i}$ are smaller than those of $J_{2,i+1}$. By Lemma 2.7, since $J_{1,i}$ is ω^{α_i}-large, it has a subsequence that is α_i-large. Similarly, since $J_{2,i}$ is $\omega_m(\alpha_i)$-large, it has a subsequence that is α_i-large. There are sequences $C_{1,i}$, C^*, and $C_{2,i}$ witnessing the largeness of the sets $J_{1,i}$, J^*, and $J_{2,i}$, where all of these are bounded by C. Since J^* is ω_m-large, it is nonempty. We let $b' \in J^*$. This completes Step 2.

Step 3. Having carried out Steps 1 and 2, we are in a position to apply overspill as in the description of Step 3.

Step 4. Recall that Theorem 2.17 connects the Wainer functions with largeness. Our assumption that J bounds witnesses for all bounded formulas simplifies both the statement and the proof of the result below.

Lemma 5.8. *Suppose C witnesses that J is α-large, where α is standard, J bounds witnesses for all standard elements of B_0^T, $b' < J$ (where b' is nonstandard) and C has a code $< b$. Then $F_\alpha(b') \downarrow < b$.*

Proof. Suppose

$$C = (\alpha_0, j_0, \alpha_1, j_1, \ldots, \alpha_{r-1}, j_{r-1}, \alpha_r).$$

Recall that if $J_k = (j_k, j_{k+1}, \ldots, j_r)$ is the part of J that appears after α_k in C, then J_k is α_k-large. Since α is standard and $b' < j_0$, where b' is nonstandard, the code for α is $< j_0$. We can show that for all k, the code for α_k is $< j_k$. For $k > 0$, we have a bounded formula saying how α_k is computed from α_{k-1} and j_{k-1}. Since J bounds witnesses for this formula, if $\alpha_{k-1} < j_{k-1}$, it follows that $\alpha_k < j_k$.

DEFINITION 13. For any x and any β, we define the *x-unwinding* of β to be the sequence $(\beta_0, \ldots, \beta_k)$, where $\beta = \beta_0$, for $i < k$, $\beta_i \neq 0$ and $\beta_{i+1} = \{\beta_i\}(x)$, and $\beta_k = 0$.

It follows from Proposition 2.6 that for $x \leq j_k$, the terms of the x-unwinding of α_k appear in C. Moreover, the code for the unwinding exists, since it is defined by recursion using a bounded formula, with C bounding everything we need. Since the code for α_k is $< j_k$, if $x \leq j_k$, then the code for the unwinding is $< j_{k+1}$. To prove the lemma, we show the following.

CLAIM. *For all $k < r$, for all $x \leq j_k$, $F_{\alpha_k}(x) \downarrow$.*

Proof of Claim. We proceed by induction on the ordinals in C starting with α_{r-1}, which we may suppose to be 1, and working our way up to $\alpha_0 = \alpha$. For all $x \leq j_{r-1}$, $F_1(x) = F_0^{x+1}(x) = 2x + 1$. Suppose the Claim holds for α_{k+1}, i.e., for all $x \leq j_{k+1}$, $F_{\alpha_{k+1}}(x) \downarrow$. We must show that the claim holds for α_k, i.e., for all $x \leq j_k$, $F_{\alpha_k}(x) \downarrow$. There are two cases.

CASE 1. Suppose α_k is a successor, and let $x \leq j_k$. By definition,

$$F_{\alpha_k}(x) = F_{\{\alpha_k\}(x)}^{x+1}(x),$$

where $\{\alpha_k\}(x) = \alpha_{k+1}$. We show by induction on $y \leq x + 1$ that $F_{\alpha_{k+1}}^y(x) \downarrow$. First, note that $F_{\alpha_{k+1}}^1(x) \downarrow$, by our inductive hypothesis (on the ordinals). Supposing that $F_{\alpha_{k+1}}^y(x) \downarrow$, where $y \leq x$, we show that $F_{\alpha_{k+1}}^{y+1}(x) \downarrow$. Since $\alpha_{k+1} = \{\alpha_k\}(x)$, where the code for α_k is $\leq j_k$, we have $F_{\alpha_{k+1}}^y(x) < j_{k+1}$ (this is defined by a bounded formula in terms of x, y, and α_k). Then $F_{\alpha_{k+1}}(F_{\alpha_{k+1}}^y(x)) \downarrow$, by our inductive hypothesis (on the ordinals). It follows that $F_\alpha(x) \downarrow$.

CASE 2. Suppose α_k is a limit ordinal, and let $x \leq j_k$. By definition,

$$F_{\alpha_k}(x) = \sup_{z \leq x} F_{\{\alpha_k\}(z)}(x).$$

For each $z \leq x$, $\{\alpha_k\}(z)$ occurs in the z-unwinding of α_k, so it is α_j for some $j > k$. By our inductive hypothesis, $F_{\alpha_j}(x) \downarrow$. Since $F_{\alpha_j}(x)$ is defined by a bounded formula in terms of z, x, and α_k, where z, x, and the codes for α_k are all $\leq j_k$, we have $F_{\alpha_j}(x) < j_{k+1}$. So, we get $F_{\alpha_k}(x) \downarrow < j_{k+1}$. ∎

We have proved the Claim, and this clearly gives the Lemma. ∎

Since $F_\alpha(b') \downarrow < b$, for all standard $\alpha < \epsilon_0$, we can apply Theorem 4.2 to get an initial segment $\mathcal{B}$ of $\mathcal{A}$ such that $b' \in \mathcal{B}$ and $\mathcal{B}$ is a model of PA.

STEP 5. We want to show that for each standard r, there is a set I_r of size r such that

- $a < I_r$ and I_r has a code $< b'$,
- I_r bounds witnesses for the first r elements of B^T_{b-1} and is diagonal indiscernible for the first r elements of B^T_0.

We work in the model $\mathcal{B}$ of PA that was obtained in Step 4. We shall use Proposition 3.6. Let Γ consist of the first r elements of B^T_0. Let n^Γ and g^Γ be as described just before Proposition 3.6. Let $\alpha = \omega_{1+2n^\Gamma(r-1)+1}$. In Step 3, we obtained an α-large sequence J such that $a < J$ and J bounds witnesses for the elements of Γ. Moreover, the code for J is $< b'$, so J is an element of $\mathcal{B}$. We need to be sure that J is still α-large when looked at in $\mathcal{B}$. Say $J = (j_0, j_1, \ldots, j_{r-1})$. The sequence

$$C = (\alpha, j_0, \alpha_1, j_1, \ldots, \alpha_{r-1}, j_{r-1}, 0)$$

witnessing that J is α-large in $\mathcal{A}$ is defined by recursion. In particular, the ordinals satisfy the relation $\alpha_{k+1} = \{\alpha_k\}(j_k)$. Calculating in $\mathcal{B}$, we arrive at the same ordinals, and we see that $\{\alpha_{r-1}\}(j_{r-1}) = 0$, so we find that J is α-large in $\mathcal{B}$. We are in a position to apply Proposition 3.6, and we get the required set I_r.

This completes the proof that (iv)$\Rightarrow$(iii), which was all that remained in the proof of Theorem 5.7. ∎

6. Problems

PROBLEM 1. *Suppose $\mathcal{A}$ is a nonstandard model of $I\Delta_0$ such that $\mathbb{N} \leq_n \mathcal{A}$. Must there exist b and c such that b is nonstandard, and c codes satisfaction in $\mathcal{A}$ of Σ_n formulas by tuples $\overline{x} \leq b$?*

PROBLEM 2. *Give conditions under which a nonstandard model of $I\Delta_0$ has a nonstandard m-elementary initial segment that is a model of $I\Sigma_n$, and say which elements can be included in such an initial segment.*

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