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A counterexample concerning products in the shape category

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Abstract. We exhibit a metric continuum X and a polyhedron P such that the Cartesian product $X \times P$ fails to be the product of X and P in the shape category of topological spaces.

1. Introduction. In every category $\mathcal{C}$ the product of two objects X and Y is defined as the triple consisting of an object $X \times Y$ and two morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ having the following universal property. For an arbitrary object Z and arbitrary morphisms $f : Z \to X$ and $g : Z \to Y$ there is a unique morphism $h : Z \to X \times Y$ such that $\pi_X h = f$ and $\pi_Y h = g$. If a product exists, it is unique up to natural isomorphism. It is well known that in the category Top of topological spaces and continuous mappings the products exist and are represented by the Cartesian product $X \times Y$. More precisely, the product consists of the space $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ and of the canonical projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$, given by $\pi_X(x, y) = x$ and $\pi_Y(x, y) = y$. Similarly, the Cartesian product $X \times Y$ and the homotopy classes $[\pi_X] : X \times Y \to X$ and $[\pi_Y] : X \times Y \to Y$ of the canonical projections π_X and π_Y form a product in the homotopy category H(Top) of topological spaces and homotopy classes of mappings. Since shape theory is a modification of homotopy theory, it is natural to ask whether products exist in the category Sh(Top) of topological spaces and shape morphisms. In particular, one can ask whether the Cartesian product $X \times Y$ of two spaces X and Y is a product in the shape category Sh(Top). More precisely, let $S : \mathrm{H(Top)} \to \mathrm{Sh(Top)}$ denote the shape functor. Does the Cartesian product $X \times Y$ together with the morphisms $S[\pi_X] : X \times Y \to X$ and $S[\pi_Y] : X \times Y \to Y$ form a product in Sh(Top)? Equivalently, does the shape functor S preserve products?

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The answer to the above question is positive when X and Y are polyhedra, because shape morphisms into spaces having the homotopy type of polyhedra reduce to homotopy classes of mappings and thus, the question reduces to the analogous question in the category H(Top). In 1974 J. E. Keesling [2] proved that the Cartesian product of two compact Hausdorff spaces is a product in Sh(Top). In the same paper he also exhibited a separable metric space X such that $X \times X$ is not a product in Sh(Top). In a recent paper S. Mardešić [6] proved that, for arbitrary topological spaces X, Y, the Cartesian product $X \times Y$ is a product in Sh(Top) provided $X \times P$ is a product in Sh(Top), for all polyhedra P. These facts show the importance of the following question. Is the Cartesian product $X \times P$ of a compact Hausdorff space X and a polyhedron P a product in the shape category Sh(Top)? In this paper we give a negative answer by proving the following theorem.

THEOREM. *The Cartesian product $X \times P$ of the dyadic solenoid X and the wedge $P = P_1 \vee P_2 \vee \cdots$ of a sequence of* 1*-spheres is not a product in the shape category of topological spaces* Sh(Top).

Since solenoids are not movable, the following problem remains open.

PROBLEM. *Is the Cartesian product $X \times P$ of a movable compactum X and a polyhedron P a product in the shape category* Sh(Top)*?*

A positive answer would imply a positive answer to a problem of Y. Kodama [3], raised in 1977. Kodama asked if a product in Sh(Top) exists for every movable compactum and every metric space. For information on movable compacta see [1], [7].

It is easy to show that every shape morphism $F\colon Z \to X$ of a space Z to the dyadic solenoid X is induced by a mapping $f\colon Z \to X$, i.e., $F = S[f]$ (apply the first two assertions of Lemma 1 below). Since P is a polyhedron, every shape morphism $G\colon Z \to P$ admits a mapping $g\colon Z \to P$ such that $G = S(g)$. It follows that the diagonal mapping $h = (f, g)\colon Z \to X \times P$ induces a shape morphism $H = S[h]\colon Z \to X \times P$ such that $S[\pi_X]H = S[\pi_X]S[h] = S[\pi_X h] = S[f] = F$ and analogously, $S[\pi_P]H = G$. This means that the existence part of the universal property defining a product of X and P is fulfilled. Therefore, to prove the Theorem we need a space Z and two different shape morphisms $H, H'\colon Z \to X \times P$ such that $S[\pi_X]H = S[\pi_X]H'$ and $S[\pi_P]H = S[\pi_P]H'$.

Actually, we will exhibit two mappings $h, h'\colon P \to X \times P$ such that

$$S[\pi_X]S[h] = S[\pi_X]S[h'], \tag{1}$$

$$S[\pi_P]S[h] = S[\pi_P]S[h'], \tag{2}$$

$$S[h] \neq S[h']. \tag{3}$$

2. The dyadic solenoid. Let $S^1 = \{\zeta = e^{2\pi it} \mid 0 \le t \le 1\}$ denote the unit circle in the complex plane and let $p\colon S^1 \to S^1$ be the mapping given by $p(\zeta) = \zeta^2$, $\zeta \in S^1$. Let $\boldsymbol{X} = (X_i, p_{ii'}, \mathbb{N})$, where $X_i = S^1$, $p_{ii'} = p^{i'-i}$, $i \le i'$, and $\mathbb{N} = \{1, 2, \ldots\}$. Then $\boldsymbol{X}$ is an inverse sequence whose limit space $X = \lim \boldsymbol{X}$ is the dyadic solenoid and the canonical projections $p_i\colon X \to X_i$ map $\xi = (\xi_1, \xi_2, \ldots) \in X$ to ξ_i, $i \in \mathbb{N}$. For completeness of exposition we prove the following elementary lemma.

LEMMA 1. *Let Z be a topological space and let $f_i\colon Z \to X_i$, $i \in \mathbb{N}$, be mappings such that $f_{i-1} \simeq p_{i-1\,i} f_i$. Then there exist mappings $f'_i\colon Z \to X_i$ such that $f_i \simeq f'_i$ and $f'_{i-1} = p_{i-1\,i} f'_i$. The unique mapping $f'\colon Z \to X = \lim \boldsymbol{X}$ such that $f'_i = p_i f'$, $i \in \mathbb{N}$, has the property that $f_i \simeq p_i f'$, $i \in \mathbb{N}$. If Z_0 is a subset of Z and $f_{i-1} \simeq p_{i-1\,i} f_i$ $(\operatorname{rel} Z_0)$, then one can achieve that $f_i \simeq f'_i$ $(\operatorname{rel} Z_0)$.*

Proof. We construct the mappings f'_i by induction on i. Put $f'_1 = f_1$. If we have already constructed $f'_{i-1}\colon Z \to X_{i-1}$, then $p_{i-1\,i} f_i \simeq f_{i-1} \simeq f'_{i-1}$. Since $p_{i-1\,i} = p$ is a covering mapping, we can lift the homotopy $H_{i-1}\colon Z \times I \to X_{i-1}$ realizing $p_{i-1\,i} f_i \simeq f'_{i-1}$ to a homotopy $H_i\colon Z \times I \to X_i$ whose initial stage is f_i. Then its terminal stage is a mapping $f'_i\colon Z \to X_i$ such that $f_i \simeq f'_i$ and $p_{i-1\,i} f'_i = f'_{i-1}$. If $p_{i-1\,i} f_i \simeq f_{i-1}$ $(\operatorname{rel} Z_0)$ and $f_{i-1} \simeq f'_{i-1}$ $(\operatorname{rel} Z_0)$, then one can assume that H_{i-1} is a homotopy $\operatorname{rel} Z_0$. Since the fibers of $p_{i-1\,i}$ are discrete, the lift H_i of H_{i-1} will also be a homotopy $\operatorname{rel} Z_0$. ∎

3. The mappings h and h'. Let $P = \bigvee_{i=1}^{\infty} P_i$ be the wedge of a sequence of copies of 1-spheres $P_i = S^1$, obtained from the coproduct $\bigsqcup_{i=1}^{\infty} P_i$ by identifying all the base points $1 \in S^1$ in the various summands P_i to a single base point $*$ of P.

For any fixed point $x \in X$, we define $h^x\colon P \to X \times P$ by putting

$$h^x(t) = (x, t), \qquad t \in P. \tag{4}$$

LEMMA 2. *For an arbitrary choice of points $x, x' \in X$, the mappings $h = h^x$, $h' = h^{x'}$ satisfy conditions* (1) *and* (2).

Proof. Since X is a continuum, any two constant mappings into X induce the same shape morphism. In particular, for $x, x' \in X$, $S[x] = S[x']$. Furthermore, for $t \in P$, $\pi_X h^x(t) = \pi_X(x, t) = x$ and $\pi_X h^{x'}(t) = \pi_X(x', t) = x'$. Consequently, $S[\pi_X]S[h^x] = S[\pi_X h^x] = S[x] = S[x'] = S[\pi_X h^{x'}] = S[\pi_X]S[h^{x'}]$ so that h and h' satisfy condition (1). Furthermore, $\pi_P h^x(t) = \pi_P(x, t) = t$, i.e., $\pi_P h^x$ is the identity mapping $1_P\colon P \to P$. Analogously, $\pi_P h^{x'} = 1_P$ so that h and h' also satisfy condition (2). ∎

The Theorem is an immediate consequence of Lemma 2 and the following Lemma 3.

Lemma 3. *There exist points* $x, x' \in X$ *such that* $h = h^x$, $h' = h^{x'}$ *satisfy condition* (3).

To prove Lemma 3, it suffices to exhibit points $x, x' \in X$, a CW-complex Q and a mapping $q\colon X \times P \to Q$ such that the mappings qh^x and $qh^{x'}$ are not homotopic. Indeed, in that case we cannot have $S[h^x] = S[h^{x'}]$, because this would imply $S[qh^x] = S[q]S[h^x] = S[q]S[h^{x'}] = S[qh^{x'}]$. However, since Q has the homotopy type of a polyhedron, $S[qh^x] = S[qh^{x'}]$ would imply $qh^x \simeq qh^{x'}$, which contradicts the assumption.

4. The CW**-complex** Q **and the mapping** $q\colon X \times P \to Q$**.** Consider the space

$$\widetilde{Q} = (X_1 \times *) \sqcup \bigsqcup_{i=1}^{\infty} (X_i \times P_i) \tag{5}$$

and consider the equivalence relation $\sim$ on $\widetilde{Q}$ generated by the requirement that

$$(p_{1i}(\zeta), *) \sim (\zeta, *), \qquad \zeta \in X_i,\ i \in \mathbb{N}. \tag{6}$$

Note that $(p_{ii'}(\zeta), *) \sim (\zeta, *)$ for $i < i'$ and $\zeta \in X_{i'}$. This is so because $(p_{ii'}(\zeta), *) \sim (p_{1i}p_{ii'}(\zeta), *) = (p_{1i'}(\zeta), *) \sim (\zeta, *)$. Put $Q = \widetilde{Q}/{\sim}$ and let $\phi\colon \widetilde{Q} \to Q$ be the corresponding quotient mapping. Note that for $(\zeta, *), (\zeta', *) \in X_1 \times *$ one has $(\zeta, *) \sim (\zeta', *)$ if and only if $\zeta = \zeta'$. Therefore, $X_1 \times *$ can be identified with $\phi(X_1 \times *)$ and can be viewed as a subspace of Q.

In order to define q, we first define mappings $q_i\colon X \times P_i \to Q$, $i \in \mathbb{N}$, by putting

$$q_i(\xi, t) = \phi(p_i(\xi), t), \qquad \xi \in X,\ t \in P_i,\ i \in \mathbb{N}. \tag{7}$$

Note that, for $\xi = (\xi_1, \xi_2, \ldots) \in X$ and the base point $* \in P_i$, (7) yields the value

$$q_i(\xi, *) = \phi(p_i(\xi), *) = \phi(p_{1i}p_i(\xi), *) = \phi(p_1(\xi), *) = (\xi_1, *) \in X_1 \times *. \tag{8}$$

Since this value does not depend on i, there is a well-defined mapping $q\colon X \times P \to Q$ such that $q|X \times P_i = q_i$, $i \in \mathbb{N}$.

It is now clear that to prove Lemma 3, we only need to prove that Q is a CW-complex (this will be accomplished in Lemma 10) and that the following lemma holds.

Lemma 4. *There exist points* $x, x' \in X$ *such that, for the above described mapping* $q\colon X \times P \to Q$,

$$qh^x \not\simeq qh^{x'}. \tag{9}$$

Remark 1. The space Q and the mapping q were suggested by the standard HPol-resolution for the general case of a product $X \times P$ of a compact

Hausdorff space and a polyhedron (see [5]). (For information on resolutions see [7] or [4].)

5. Points x, x' and paths u_i and c_i. Let $\boldsymbol{i} = (i_0 = 0 < i_1 < \cdots < i_k < \cdots)$ be a sequence of integers and let $x = (x_1, x_2, \ldots)$ be an arbitrary point in X. We will associate with $\boldsymbol{i}$ and x a point $x' = (x'_1, x'_2, \ldots) \in X$. We first construct a sequence of paths $u_i \colon I = [0,1] \to X_i$ with initial points $x_i = u_i(0)$. Then we define x'_i to be the terminal points of these paths, $x'_i = u_i(1)$. To describe the paths u_i we also need some loops $a_i \colon I \to X_i$ in X_i, based at x_i. By definition,

$$a_i(t) = x_i e^{2\pi i t}, \qquad t \in I. \tag{10}$$

The paths u_i are completely determined by the next lemma.

LEMMA 5. *Let $u_1 \colon I \to X_1$ be an arbitrary path in X_1 whose initial point is $u_1(0) = x_1$. Then there exists a unique sequence of paths u_i in X_i beginning at $u_i(0) = x_i$, $i \in \mathbb{N}$, and having the following properties*:

(i) *For $i \neq i_k + 1$, $k \in \mathbb{N}$, the path u_i is a lift of the path u_{i-1} with respect to the mapping $p_{i-1\,i} = p \colon X_i \to X_{i-1}$, i.e., $pu_i = u_{i-1}$*;

(ii) *For $i = i_k + 1$, $k \in \mathbb{N}$, u_i is a lift of the concatenation of paths $a_{i_k} \cdot u_{i_k}$.*

The terminal points $x'_i = u_i(1)$, $i \in \mathbb{N}$, form a point $x' \in X$.

Recall that the concatenation $\eta_1 \cdot \eta_2$ of two paths is defined by the formula

$$(\eta_1 \cdot \eta_2)(t) = \begin{cases} \eta_1(2t), & 0 \le t \le 1/2, \\ \eta_2(2t-1), & 1/2 \le t \le 1. \end{cases} \tag{11}$$

By definition, $\eta_1 \cdot \eta_2 \cdots \eta_n = (\eta_1 \cdot \eta_2 \cdots \eta_{n-1}) \cdot \eta_n$. It is well known that concatenation of paths is associative up to homotopy rel ∂I. Every path η determines its reversed path η^{-1}. By definition, $\eta^{-1}(t) = \eta(1-t)$. It is well known that $\eta \cdot \eta^{-1} \simeq \eta(0)$ and $\eta^{-1} \cdot \eta \simeq \eta(1)$.

Proof of Lemma 5. Since $p \colon S^1 \to S^1$ is a covering mapping, the initial point x_i and the path u_{i-1} completely determine the path u_i. If $i \neq i_k + 1$, then $p(x'_i) = (pu_i)(1) = u_{i-1}(1) = x'_{i-1}$. If $i = i_k + 1$, then $p(x'_i) = p(x'_{i_k+1}) = (pu_{i_k+1})(1) = (a_{i_k} \cdot u_{i_k})(1) = u_{i_k}(1) = u_{i-1}(1) = x'_{i-1}$. Therefore, $x' = (x'_1, x'_2, \ldots) \in X$. ■

In the proof of Lemma 4 we will impose additional conditions on the sequence $\boldsymbol{i}$ (see Section 10).

With every path η in $\widetilde{Q}$ the mapping ϕ associates a path $\phi_\#(\eta)$ in Q defined by $(\phi_\#(\eta))(t) = (\phi\eta)(t)$, $t \in I$. In particular, since $u_i \times *$ is a path in $X_i \times * \subseteq \widetilde{Q}$ given by $(u_i \times *)(t) = (u_i(t), *)$, we see that

$$c_i = \phi_\#(u_i \times *), \qquad i \in \mathbb{N}, \tag{12}$$

is a path in $\phi(X_i \times *) \subseteq X_1 \times * \subseteq Q$. It connects the point $\phi(u_i(0), *) = \phi(x_i, *) = \phi(p_{1i}(x_i), *) = \phi(x_1, *) = (x_1, *)$ to the point $\phi(u_i(1), *) = \phi(x'_i, *) = (x'_1, *)$.

In the next lemma we will give explicit formulae determining c_i up to homotopy of paths, i.e., homotopy rel ∂I. To be able to write the formulae in a concise way, we associate with every sequence $\boldsymbol{i} = (i_0 = 0 < i_1 < \cdots < i_k < \cdots)$ of integers an integral-valued function m whose domain consists of the integers $i \geq i_1 + 1$. By definition,

$$m(i) = 2^{i_1-1} + 2^{i_2-1} + \cdots + 2^{i_k-1}, \quad i_k + 1 \leq i \leq i_{k+1},\ k \in \mathbb{N}. \tag{13}$$

Lemma 6.

$$\begin{aligned} c_i &= c_1, && 1 \leq i \leq i_1, \\ c_i &\simeq (a_1^{m(i)} \times *) \cdot c_1, && i_1 + 1 \leq i. \end{aligned} \tag{14}$$

Proof. It suffices to prove the following two formulae:

$$c_i = c_{i-1}, \quad i_k + 1 < i \leq i_{k+1},\ k \in \{0, 1, \ldots\}, \tag{15}$$

$$c_{i_k+1} \simeq (a_1^{2^{i_k-1}} \times *) \cdot c_{i_k}, \quad k \in \mathbb{N}. \tag{16}$$

Proof of (15):

$$c_i = \phi_\#(u_i \times *) = \phi_\#(p_{i-1\,i} u_i \times *), \tag{17}$$

because $(u_i(\zeta), *) \sim (p_{i-1\,i} u_i(\zeta), *)$ for $\zeta \in X_i$. However, $p_{i-1\,i} = p$ and $pu_i = u_{i-1}$ for $i_k + 1 < i \leq i_{k+1}$ and $k \in \{0, 1, \ldots\}$. Consequently, $c_i = \phi_\#(u_{i-1} \times *) = c_{i-1}$.

Proof of (16): By (12),

$$c_{i_k+1} = \phi_\#(u_{i_k+1} \times *) = \phi_\#(p_{i_k i_k+1} u_{i_k+1} \times *). \tag{18}$$

Since $p_{i_k i_k+1} = p$ and $pu_{i_k+1} = a_{i_k} \cdot u_{i_k}$, it follows that

$$\begin{aligned} c_{i_k+1} &= \phi_\#(a_{i_k} \cdot u_{i_k} \times *) = \phi_\#((a_{i_k} \times *) \cdot (u_{i_k} \times *)) \\ &= \phi_\#(a_{i_k} \times *) \cdot \phi_\#(u_{i_k} \times *). \end{aligned} \tag{19}$$

Since by (12), $\phi_\#(u_{i_k} \times *) = c_{i_k}$, it remains to show that

$$\phi_\#(a_{i_k} \times *) \simeq a_1^{2^{i_k-1}} \times *. \tag{20}$$

This is a special case of the formula

$$\phi_\#(a_i \times *) \simeq a_1^{2^{i-1}} \times *, \tag{21}$$

valid for all $i \in \mathbb{N}$. Since

$$\phi_\#(a_i \times *) = \phi(a_i \times *) = \phi(p_{1i} a_i \times *) = \phi(p^{i-1} a_i \times *), \tag{22}$$

to prove (21), it suffices to show that

$$p^{i-1} a_i \simeq a_1^{2^{i-1}}, \tag{23}$$

because then

$$\phi(p^{i-1}a_i \times *) \simeq \phi(a_1^{2^{i-1}} \times *) = a_1^{2^{i-1}} \times *. \tag{24}$$

Formula (23) follows by induction on i, using the formula

$$pa_i = a_{i-1}^2. \tag{25}$$

Indeed, if (23) holds, we see, by (25) for $i+1$, that

$$p^i a_{i+1} = p^{i-1}pa_{i+1} = p^{i-1}a_i^2 = p^{i-1}a_i \cdot p^{i-1}a_i \simeq a_1^{2^{i-1}} \cdot a_1^{2^{i-1}} \simeq a_1^{2^i}. \tag{26}$$

To verify (25), note that, for $t \in I$,

$$pa_i(t) = p(x_i e^{2\pi it}) = p(x_i)p(e^{2\pi it}) = x_{i-1}e^{4\pi it}. \tag{27}$$

On the other hand,

$$(a_{i-1}^2)(t) = \begin{cases} a_{i-1}(2t) = x_{i-1}e^{4\pi it}, & 0 \le t \le 1/2, \\ a_{i-1}(2t-1) = x_{i-1}e^{2\pi i(2t-1)}, & 1/2 \le t \le 1. \end{cases} \tag{28}$$

Since $e^{2\pi i(2t-1)} = e^{4\pi it}$, we conclude that also $(a_{i-1}^2)(t) = x_{i-1}e^{4\pi it}$ for $t \in I$. ■

6. Loops b_i and b_i'. For every $i \in \mathbb{N}$ we now define two loops $\widetilde{b}_i, \widetilde{b}_i' \colon I \to X_i \times P_i \subseteq \widetilde{Q}$ by putting

$$\widetilde{b}_i(t) = (x_i, e^{2\pi it}), \qquad \widetilde{b}_i'(t) = (x_i', e^{2\pi it}). \tag{29}$$

Note that these loops are based at the points $(x_i, *)$ and $(x_i', *)$, respectively. Next put

$$b_i = \phi_\#(\widetilde{b}_i), \qquad b_i' = \phi_\#(\widetilde{b}_i'). \tag{30}$$

The loop b_i is based at $\phi(x_i, *) = \phi(p_{1i}(x_i), *) = \phi(x_1, *) = (x_1, *)$ and b_i' is based at $\phi(x_i', *) = (x_1', *)$. Recall that c_i is a path in Q connecting $(x_1, *)$ to $(x_1', *)$. Denoting by c_i^{-1} the inverse path of c_i, i.e., the path given by $c_i^{-1}(t) = c_i(1-t)$, we conclude that $c_i^{-1} \cdot b_i \cdot c_i$ is a well-defined loop in Q, based at $(x_1', *)$. The next lemma plays an important role in the proof of the Theorem.

LEMMA 7. *In Q the following homotopy of loops based at $(x_1', *)$ holds:*

$$c_i^{-1} \cdot b_i \cdot c_i \simeq b_i', \qquad i \in \mathbb{N}. \tag{31}$$

Proof. We first prove the analogous formula in $\widetilde{Q}$, which reads as follows:

$$(u_i^{-1} \times *) \cdot \widetilde{b}_i \cdot (u_i \times *) \simeq \widetilde{b}_i', \qquad i \in \mathbb{N}. \tag{32}$$

Since $X_i \times P_i \subseteq \widetilde{Q}$, it suffices to exhibit a homotopy $H_i \colon I \times I \to X_i \times P_i$ (rel ∂I) which connects the left side of (32) to its right side. Using the product structure of $X_i \times P_i$, one readily obtains a homotopy $H'_i\colon I \times I \to X_i \times P_i$ (rel ∂I) connecting the left side of (32) to the concatenation $x'_i \cdot \widetilde{b}'_i \cdot x'_i$, where x'_i is now viewed as a constant path. Indeed, it suffices to put

$$H'_i(s,t) = ((u_{is}^{-1} \times *) \cdot \widetilde{b}_{is} \cdot (u_{is} \times *))(t), \tag{33}$$

where $u_{is}^{-1}(t) = u_i(1 - t(1-s))$, $\widetilde{b}_{is}(t) = (u_i(s), e^{2\pi it})$ and $u_{is}(t) = u_i(s + t(1-s))$. It is easy to find a homotopy $H''_i\colon I \times I \to X_i \times P_i$ (rel ∂I) which connects $x'_i \cdot \widetilde{b}'_i \cdot x'_i$ to $\widetilde{b}'_i$. Then the concatenation $H = H' \cdot H''$ is a homotopy which realizes (32).

To obtain (31), it now suffices to apply $\phi_\#$ to (32). Indeed, by (12), $\phi_\#(u_i \times *) = c_i$. Similarly, $\phi_\#(u_i^{-1} \times *) = c_i^{-1}$, because $\phi(u_i^{-1}(t), *) = \phi(u_i(1-t), *) = (\phi_\#(u_i \times *))(1-t) = c_i(1-t) = c_i^{-1}(t)$. Finally, $\phi(\widetilde{b}_i) = b_i$ and $\phi(\widetilde{b}'_i) = b'_i$. ∎

7. A consequence of the assumption $qh^x \simeq qh^{x'}$**.** The next lemma shows that the assumption $qh^x \simeq qh^{x'}$ has an important consequence for the loops b_i.

LEMMA 8. *If for some points $x, x' \in X$ the mappings $qh^x, qh^{x'}\colon P \to Q$ are homotopic, then there exists a path $l\colon I \to Q$ which connects the points $(x_1, *)$ and $(x'_1, *)$ and is such that, for all $i \in \mathbb{N}$,*

$$l^{-1} \cdot b_i \cdot l \simeq b'_i. \tag{34}$$

Proof. Choose a homotopy $L\colon P \times I \to Q$ which connects the mappings qh^x and $qh^{x'}$. Let $l\colon I \to Q$ be the path in Q given by the restriction of L to $* \times I$, i.e., let $l(s) = L(*, s)$ for $s \in I$. Note that l connects $l(0) = L(*, 0) = qh^x(*) = q(x, *) = (x_1, *)$ and $l(1) = L(*, 1) = qh^{x'}(*) = q(x', *) = (x'_1, *)$ and it does not depend on $i \in \mathbb{N}$. Denote by $\omega_i\colon I \to P_i \subseteq P$ the loop given by the formula

$$\omega_i(t) = e^{2\pi it}, \quad t \in I. \tag{35}$$

Then, by (4), (7), (28) and (29),

$$qh^x\omega_i(t) = q(x, e^{2\pi it}) = \phi(p_i(x), e^{2\pi it}) = \phi(x_i, e^{2\pi it}) = \phi(\widetilde{b}_i(t)) = b_i(t) \tag{36}$$

and thus,

$$L(\omega_i(t), 0) = b_i(t), \quad t \in I. \tag{37}$$

Analogously,

$$L(\omega_i(t), 1) = b'_i(t), \quad t \in I. \tag{38}$$

Now consider the product $P \times I$ and the path $\lambda\colon I \to P \times I$ given by the formula $\lambda(s) = (*, s)$. Also consider the loops $\omega_i \times 0, \omega_i \times 1\colon I \to P \times I$ for $i \in \mathbb{N}$. Let us first note that in $P_i \times I \subseteq P \times I$ the following homotopy of paths holds:

$$\lambda^{-1} \cdot (\omega_i \times 0) \cdot \lambda \simeq \omega_i \times 1. \tag{39}$$

To verify (39), consider the homotopy $G'_i \colon I \times I \to P_i \times I$ (rel ∂I) given by the formula

$$G'_i(s,t) = \lambda_s^{-1} \cdot (\omega_i \times s) \cdot \lambda_s, \tag{40}$$

where $\lambda_s^{-1}(t) = (*, 1-t(1-s))$ and $\lambda_s(t) = (*, s+t(1-s))$. It readily follows that G'_i connects the left side of (39) to the concatenation $(* \times 1) \cdot (\omega_i \times 1) \cdot (* \times 1)$, where we view $* \times 1$ as a constant path. Let $G = G'_i \cdot G''_i$, where G''_i is a homotopy which connects $(* \times 1) \cdot (\omega_i \times 1) \cdot (* \times 1)$ to $\omega_i \times 1$. Then G is a homotopy which realizes (39).

Now apply the homotopy L to (39). Note that $L\lambda = l$, because $L\lambda(s) = L(*, s) = l(s)$. Moreover, (37) and (38) show that $L(\omega_i \times 0) = b_i$ and $L(\omega_i \times 1) = b'_i$. Consequently, one obtains the desired formula (34). ∎

In the space Q choose a base point $*$ by putting $* = (x_1, *)$. Since $a_1 \times *, b_1, b_2, \ldots$ are loops in Q based at $*$, they determine elements $\alpha = [a_1 \times *], \beta_1 = [b_1], \beta_2 = [b_2], \ldots$ of the fundamental group $\pi_1(Q, *)$. The next lemma is crucial in our argument.

Lemma 9. *Let $\boldsymbol{i} = (0 < i_1 < \cdots < i_k < \cdots)$ be a sequence of integers and let $x, x' \in X$ be points chosen in accordance with Lemma 5. If $qh^x \simeq qh^{x'}$, then there exists an element $\kappa \in \pi_1(Q, *)$ such that*

$$(\alpha^{m(i)}\kappa)\beta_i = \beta_i(\alpha^{m(i)}\kappa) \tag{41}$$

for all $i > i_1$.

Proof. Since both l and c_1 are paths in Q from $* = (x_1, *)$ to $(x'_1, *)$, it follows that $k = c_1 \cdot l^{-1}$ is a well-defined loop in Q based at $*$. Then $\kappa = [k]$ is a well-defined element of the fundamental group $\pi_1(Q, *)$. Comparing (31) and (34), we conclude that

$$c_i^{-1} \cdot b_i \cdot c_i \simeq l^{-1} \cdot b_i \cdot l \tag{42}$$

for all $i \in \mathbb{N}$. Moreover, for $i > i_1$, (42) and Lemma 6 imply

$$c_1^{-1} \cdot (a_1^{m(i)} \times *)^{-1} \cdot b_i \cdot (a_1^{m(i)} \times *) \cdot c_1 \simeq l^{-1} \cdot b_i \cdot l. \tag{43}$$

Since $c_1 \cdot l^{-1} = k$, (43) assumes the form

$$(a_1^{m(i)} \times *) \cdot k \cdot b_i \simeq b_i \cdot (a_1^{m(i)} \times *) \cdot k. \tag{44}$$

Passing to homotopy classes, we conclude that κ satisfies (41) for all $i > i_1$. ∎

8. The fundamental group of Q. In this section we will prove the following lemma.

LEMMA 10. *Q is a connected CW-complex whose fundamental group $\pi_1(Q,*)$ has generators $\alpha,\beta_1,\beta_2,\ldots$ and relations*

$$\alpha^{2^{i-1}}\beta_i = \beta_i\alpha^{2^{i-1}}, \quad i\in\mathbb{N}. \tag{45}$$

Proof. First note that Q can also be obtained by attaching to the 1-sphere $X_1\times *$ the 2-tori $X_i\times P_i$, $i\in\mathbb{N}$, via the mappings $p_{1i}\times *\colon X_i\times *\to X_1\times *$. Next notice that the 2-torus $X_i\times P_i$ is obtained by attaching a 2-cell $D_i = I\times I$ to the wedge of two 1-spheres

$$W_i = (X_i\times *)\vee(x_i\times P_i) \tag{46}$$

via a mapping $\chi_i\colon \partial D_i\to W_i$ of the boundary $\partial D_i = (I\times\partial I)\cup(\partial I\times I)$. The mapping χ_i is given by the formulae

$$\chi_i|I\times 0 = \chi_i|I\times 1 = a_i\times *, \tag{47}$$

$$\chi_i|0\times I = \chi_i|1\times I = \widetilde{b}_i. \tag{48}$$

It is now clear that Q is a connected 2-dimensional CW-complex whose 0-skeleton is the base point $*$, the 1-skeleton is the wedge of 1-spheres

$$W = (X_1\times *)\vee\Big(\bigvee_{i=1}^{\infty}\phi(x_i\times P_i)\Big), \tag{49}$$

and Q is obtained by attaching to W the 2-cells D_i, $i\in\mathbb{N}$, via the mappings $\psi_i = \phi\chi_i\colon \partial D_i\to W$. Consequently, $\pi_1(W,*)$ is a free group whose generators are the homotopy classes of the loops $a_1\times *,\widetilde{b}_1,\widetilde{b}_2,\ldots$. The group $\pi_1(Q,*)$ is the quotient of $\pi_1(W,*)$ by the normal subgroup generated by the homotopy classes of the loops

$$\phi((a_i\times *)\cdot\widetilde{b}_i\cdot(a_i\times *)^{-1}\cdot\widetilde{b}_i^{-1}) = \phi(a_i\times *)\cdot\phi(\widetilde{b}_i)\cdot(\phi(a_i\times *))^{-1}\cdot(\phi(\widetilde{b}_i))^{-1}. \tag{50}$$

However, by (21), $\phi(a_i\times *)\simeq a_1^{2^{i-1}}\times * = (a_1\times *)^{2^{i-1}}$ and by (30), $\phi(\widetilde{b}_i) = b_i$. Therefore, the homotopy class of the loop (50) equals $\alpha^{2^{i-1}}\beta_i(\alpha^{2^{i-1}})^{-1}\beta_i^{-1}$. ∎

9. Two lemmas from group theory. In the proof of the Theorem we also need two lemmas on groups.

LEMMA 11. *Let $n\ge 2$ be an integer and let G be the group with two generators α,β and one relation $\alpha^n\beta=\beta\alpha^n$. If for some $m\in\mathbb{Z}$ the power α^m commutes with β, then m is divisible by n.*

Proof. Consider the symmetric group $S(n+1)$ of all permutations of the set $\{0,1,\ldots,n\}$. Let a be the permutation which keeps the point 0 fixed and permutes the set $\{1,\ldots,n\}$ cyclically, i.e., $a(0)=0$, $a(i)=i+1$ for $1\le i\le n-1$, and $a(n)=1$. Let b be the permutation which keeps n fixed and permutes $\{0,\ldots,n-1\}$ cyclically. Clearly, a^n equals the identity permutation 1 and therefore,

$$a^nb = ba^n. \tag{51}$$

On the other hand,

(52) $$a^k b \neq ba^k, \quad 1 \le k \le n-1.$$

Indeed, $a^k(0) = 0$ and therefore, $ba^k(0) = b(0) = 1$. For $1 \le k \le n-1$, $a^k b(0) = a^k(1) = k+1 \neq 1$, which establishes (52).

Now define a homomorphism φ of the free group F with basis $\{\alpha, \beta\}$ to $S(n+1)$ by putting $\varphi(\alpha) = a$ and $\varphi(\beta) = b$. Note that (51) implies that $\varphi(\alpha^n \beta \alpha^{-n} \beta^{-1}) = a^n b a^{-n} b^{-1} = 1$ and therefore, φ induces a homomorphism $\phi\colon G \to S(n+1)$ with $\phi(\alpha) = a$ and $\phi(\beta) = b$. It is now readily seen that the elements $\alpha, \alpha^2, \ldots, \alpha^{n-1}$ do not commute with β. Indeed, for $1 \le k \le n-1$, $\alpha^k \beta = \beta \alpha^k$ would imply $a^k b = ba^k$, contrary to (52). Now assume that a power α^m commutes with β. Note that there are integers l and r such that $m = ln + r$ and $0 \le r \le n-1$. Since $\alpha^{ln} = (\alpha^n)^l$ commutes with β, one concludes that also α^r commutes with β. By (52), one cannot have $1 \le r \le n-1$ and thus, $r = 0$, i.e., $m = ln$ is divisible by n. ■

The next lemma generalizes Lemma 11.

LEMMA 12. *Let $n_1, n_2, \ldots$ be a sequence of integers ≥ 2 and let G be the group with generators $\alpha, \beta_1, \beta_2, \ldots$ and relations $\alpha^{n_i} \beta_i = \beta_i \alpha^{n_i}$, $i \in \mathbb{N}$. If for some $j \in \mathbb{N}$ and some $m \in \mathbb{Z}$ the power α^m commutes with β_j, then m is divisible by n_j.*

Proof. If F is the free group with basis $\{\alpha, \beta_1, \beta_2, \ldots\}$ and $N \subseteq F$ is the normal subgroup generated by the elements $\alpha^{n_i} \beta_i \alpha^{-n_i} \beta_i^{-1}$, $i \in \mathbb{N}$, then $G = F/N$. Let F' be the free group with basis $\{a, b\}$ and let $N' \subseteq F'$ be the normal subgroup generated by the element $a^{n_j} b a^{-n_j} b^{-1}$. Then $G' = F'/N'$ is the group with generators a, b and with the only relation $a^{n_j} b = b a^{n_j}$. Consider the homomorphism $\varphi\colon F \to F'$ determined by putting $\varphi(\alpha) = a$, $\varphi(\beta_j) = b$ and $\varphi(\beta_i) = 1$ for $i \neq j$. Note that $\varphi(\alpha^{n_j} \beta_j \alpha^{-n_j} \beta_j^{-1}) = a^{n_j} b a^{-n_j} b^{-1}$ and $\varphi(\alpha^{n_i} \beta_i \alpha^{-n_i} \beta_i^{-1}) = a^{n_i} a^{-n_i} = 1$ for $i \neq j$ and thus, $\varphi(N) \subseteq N'$. Therefore, φ induces a homomorphism $\phi\colon G \to G'$ such that $\phi(\alpha) = a$, $\phi(\beta_j) = b$ and $\phi(\beta_i) = 1$ for $i \neq j$. Applying ϕ to $\alpha^m \beta_j = \beta_j \alpha^m$, one concludes that a^m commutes with b. We now apply Lemma 11 to n_j, to the group G' and to the power a^m and we conclude that m is divisible by n_j. ■

10. Proof of Lemma 4. With every sequence $\boldsymbol{i} = \{0 = i_0 < \cdots < i_k < \cdots\}$ we associate a sequence $\boldsymbol{s}$ of integers s_k, $k \ge 2$, defined by the formula

(53) $$s_k = 2^{i_k - 1} - m(i_k) = 2^{i_k - 1} - (2^{i_1 - 1} + 2^{i_2 - 1} + \cdots + 2^{i_{k-1} - 1}).$$

Clearly,

(54) $$s_{k+1} - s_k = 2^{i_{k+1} - 1} - 2^{i_k} = 2^{i_k}(2^{i_{k+1} - i_k - 1} - 1), \quad k \ge 2.$$

Since $i_k+1 \le i_{k+1}$, (53) and (54) imply that $0 < s_2 < \cdots < s_k < s_{k+1} < \cdots$, i.e., $\boldsymbol{s}$ is a strictly increasing sequence. Moreover, if for some $k \ge 2$,

$$i_k + 2 \le i_{k+1}, \tag{55}$$

then (54) implies that

$$s_{k+1} > s_{k+1} - s_k \ge 2^{i_k} \ge 2^k. \tag{56}$$

LEMMA 13. *Let $\boldsymbol{i} = \{0 = i_0 < \cdots < i_k < \cdots\}$ be a sequence of integers such that* (55) *holds for infinitely many integers $k \ge 2$. Then, for an arbitrary point $x \in X$ and for $x' \in X$ determined by $\boldsymbol{i}$ and x as in Lemma* 5, *the mappings qh^x and $qh^{x'}$ are not homotopic.*

Proof. Assume that $qh^x \simeq qh^{x'}$. Then Lemma 9 yields an element $\kappa \in \pi_1(Q,*)$ such that (41) holds for all $i > i_1$. By Lemma 10, $\pi_1(Q,*)$ is the quotient group $G = F/N$, where F is the free group with basis $\{\alpha, \beta_1, \beta_2, \ldots\}$ and $N \subseteq F$ is the normal subgroup generated by the elements $\alpha^{2^{i-1}}\beta_i\alpha^{-2^{i-1}}\beta_i^{-1}$, $i \in \mathbb{N}$. Since the elements $\alpha, \beta_1, \beta_2, \ldots$ generate G, κ is a product of the form $\kappa = \gamma_1\gamma_2\cdots\gamma_s$, where every γ_l, $1 \le l \le s$, is either a power of α or a power of one of the generators β_i. There are only finitely many such β_i, hence there is an integer $r \ge 1$ such that, for $1 \le l \le s$ and $i > r$, γ_l is not a power of β_i. Without loss of generality we can assume that $r \ge i_1$.

We now consider the group $G' = F'/N'$, where F' is the free group with basis $\{a, b_{r+1}, b_{r+2}, \ldots\}$ and $N' \subseteq F'$ is the normal subgroup generated by the elements $a^{2^{i-1}}b_ia^{-2^{i-1}}b_i^{-1}$, $i \ge r+1$. Let $\varphi\colon F \to F'$ be the homomorphism determined by putting $\varphi(\alpha) = a$, $\varphi(\beta_i) = 1$ for $1 \le i \le r$, and $\phi(\beta_i) = b_i$ for $i \ge r+1$. Note that $\varphi(\alpha^{2^{i-1}}\beta_i\alpha^{-2^{i-1}}\beta_i^{-1}) = 1 \in N'$ for $1 \le i \le r$, and $\varphi(\alpha^{2^{i-1}}\beta_i\alpha^{-2^{i-1}}\beta_i^{-1}) = a^{2^{i-1}}b_ia^{-2^{i-1}}b_i^{-1} \in N'$ for $i \ge r+1$, and thus, $\varphi(N) \subseteq N'$. Consequently, φ induces a homomorphism $\phi\colon G \to G'$ such that $\phi(\alpha) = a$, $\phi(\beta_i) = 1$ for $1 \le i \le r$, and $\phi(\beta_i) = b_i$ for $i \ge r+1$. Putting $\phi(\kappa) = c$ and applying ϕ to (41) for $i \ge r+1$, we conclude that c is an element of G' such that

$$(a^{m(i)}c)b_i = b_i(a^{m(i)}c), \qquad i \ge r+1. \tag{57}$$

Now note that $c = \phi(\gamma_1)\cdots\phi(\gamma_s)$. If for some l, $1 \le l \le s$, γ_l is a power of α, say, $\gamma_l = \alpha^n$, then $\phi(\gamma_l) = a^n$ is a power of a. If γ_l is a power of β_i, then $i \le r$ and thus $\phi(\gamma_l) = 1$. Consequently, $c \in G'$ is a power of a, say $c = a^M$, and thus, $a^{m(i)}c = a^{m(i)+M}$ is also a power of a. Moreover, by (57), for any $j \ge r+1$, $a^{m(j)+M}$ commutes with b_j. This enables us to apply Lemma 12 to the sequence of integers 2^{i-1}, $i \ge r+1$, to the group G' with generators a, b_i and relations $a^{2^{i-1}}b_ia^{-2^{i-1}}b_i^{-1}$ for $i \ge r+1$, and to the element $a^{m(j)+M}$. We conclude that, for $j \ge r+1$, $m(j)+M$ is divisible by 2^{j-1}.

Now choose an integer k_0 so large that, for $k \geq k_0$, $i_k \geq r+1$. Then, if we put $j = i_k$, the above assertion shows that there is an integer n_k such that

$$m(i_k) + M = n_k 2^{i_k - 1}, \quad k \geq k_0. \tag{58}$$

Let us first show that there is an integer $k_1 \geq \max\{3, k_0\}$ such that $k \geq k_1$ implies $n_k \geq 1$. Indeed, if $n_k \leq 0$, then $-M = m(i_k) - n_k 2^{i_k - 1} \geq m(i_k)$. However, by (13),

$$m(i_k) = 2^{i_1 - 1} + 2^{i_2 - 1} + \cdots + 2^{i_{k-1} - 1}, \quad k \geq 2, \tag{59}$$

and therefore, $k \geq 3$ implies $m(i_k) \geq 2k-3$. One cannot have infinitely many $k \geq \max\{3, k_0\}$ such that $n_k \leq 0$, because that would imply that there are infinitely many k such that $-M \geq 2k - 3$, which is obviously false. Consequently, there is an integer k_1 having the desired properties. Now assume that $k + 1 \geq k_1$ and thus, $n_{k+1} \geq 1$. Then, by (58) for $k + 1$,

$$M = n_{k+1} 2^{i_{k+1} - 1} - m(i_{k+1}) \geq 2^{i_{k+1} - 1} - m(i_{k+1}) = s_{k+1}. \tag{60}$$

However, by (56), for infinitely many k, one has $s_{k+1} > 2^k$ and thus, $M > 2^k$, which is obviously false. ∎

Proof of Lemma 4. Choose a sequence $\boldsymbol{i} = \{0 = i_0 < \ldots < i_k < \ldots\}$ such that (55) holds for infinitely many $k \geq 2$. Choose an arbitrary point $x \in X$ and determine $x' \in X$ by $\boldsymbol{i}$ and x as in Lemma 5. Then Lemma 13 shows that the mappings qh^x and $qh^{x'}$ are not homotopic. This completes the proofs of Lemma 4 and of the Theorem.

We will now state and prove a sharper version of Lemma 3.

PROPOSITION 1. *If two points $x, x' \in X$ belong to the same path component of X, then $h^x \simeq h^{x'}$ and thus, $S[h^x] = S[h^{x'}]$. Conversely, if $S[h^x] = S[h^{x'}]$, then x, x' belong to the same path component of X.*

Proof. First assume that x and x' belong to the same path component of X. Choose a path $\varphi\colon I \to X$ which connects x to x'. Then the formula $\phi(t, s) = (\varphi(s), t)$ defines a homotopy $\phi\colon P \times I \to X \times P$ such that $\phi(t, 0) = (\varphi(0), t) = (x, t) = h^x(t)$ and $\phi(t, 1) = (\varphi(1), t) = (x', t) = h^{x'}(t)$. Consequently, $h^x \simeq h^{x'}$ and thus, $S[h^x] = S[h^{x'}]$.

Now assume that $S[h^x] = S[h^{x'}]$. We first construct, by induction on $i \geq 1$, a sequence of paths u_i in X_i which begin at x_i and end at x'_i. For u_1 we choose an arbitrary path in X_1 which connects x_1 and x'_1. Assume that we have already constructed the path u_i in X_i, $i \geq 1$. Consider the lifts of u_i and $a_i \cdot u_i$ with respect to the mapping $p_{i\,i+1} = p\colon X_{i+1} \to X_i$, having x_{i+1} for its initial point. Clearly, either the first or the second of the lifted paths must have x'_{i+1} for its terminal point. Let that path be u_{i+1}.

If there are only finitely many integers $i \geq 1$ for which the second case occurs, then there is an $i_0 \in \mathbb{N}$ such that, for $i \geq i_0$, u_{i+1} is the lift of u_i, i.e., $pu_{i+1} = u_i$. In this case the paths $u_i\colon I \to X_i$, $i \geq i_0$, determine a path $u\colon I \to X$ such that $p_i u = u_i$. Since $p_i u(0) = u_i(0) = x_i = p_i(x)$, we conclude that $u(0) = x$. Analogously, $u(1) = x'$ and we see that the points x, x' are connected by the path u. Hence, they belong to the same path component.

Now consider the case where the integers $i \geq 1$ for which u_{i+1} lifts $a_i \cdot u_i$ form a sequence $i_1 < \cdots < i_k < \cdots$. Since u_{i_k+1} is a lift of $a_{i_k} \cdot u_{i_k}$, we see that the sequence $\boldsymbol{i} = \{0 = i_0 < i_1 < \cdots < i_k < \cdots\}$ of integers and the points x, x' have all the properties stated in Lemma 5. We claim that there are only finitely many integers $k \geq 1$ for which condition (55) holds. Indeed, in the opposite case, Lemma 13 would imply that $qh^x \not\simeq qh^{x'}$ and thus, $S[h^x] \neq S[h^{x'}]$, which contradicts the present assumption. Consequently, there is an integer $k_0 \geq 1$ such that for all $k \geq k_0$, $i_k + 1 = i_{k+1}$ and the sequence $\boldsymbol{i}$ is of the form $\boldsymbol{i} = \{0 < i_1 < \cdots < i_{k_0} < i_{k_0} + 1 < i_{k_0} + 2 < \cdots\}$, i.e., starting from the term i_{k_0}, it consists of consecutive integers. Therefore, for $i \geq i_{k_0}$, $pu_{i+1} = a_i \cdot u_i$. To complete the proof it now suffices to construct a sequence of paths v_i in X_i, $i \geq i_{k_0}$, such that v_i connects x'_i and x_i and v_{i+1} is a lift of v_i, i.e., $pv_{i+1} = v_i$. Indeed, such a sequence of paths determines a unique path $v\colon I \to X = \lim \boldsymbol{X}$ such that $p_i v = v_i$, $i \geq i_{k_0}$, and the endpoints of v are x' and x. Hence, x, x' again belong to the same path component of X.

To define the paths v_i consider the fundamental groupoid $\pi(X)$ of X. Let w_i be a representative of the homotopy class $[u_i]^{-1}[a_i] \in \pi(X)$. Then $w_i(0) = x'_i$, $w_i(1) = x_i$ and $[u_i][w_i] = [a_i]$. Therefore, for $i \geq i_{k_0}$, $[a_i]([u_i][pw_{i+1}]) = ([a_i][u_i])[pw_{i+1}] = [pu_{i+1}][pw_{i+1}] = [p(u_{i+1} \cdot w_{i+1})] = [pa_{i+1}] = [a_i \cdot a_i] = [a_i][a_i]$. Consequently, $[u_i][pw_{i+1}] = [a_i] = [u_i][w_i]$. However, this implies that $[pw_{i+1}] = [w_i]$, i.e., $pw_{i+1} \simeq w_i$ (rel ∂I). Applying Lemma 1, we see that there exist paths $v_i\colon I \to X$ such that $pv_{i+1} = v_i$ and $v_i \simeq w_i$ (rel ∂I). However, the latter relation implies that $v_i(0) = w_i(0) = x'_i$, $v_i(1) = w_i(1) = x_i$. ∎

REMARK 2. The restrictions of qh^x and $qh^{x'}$ to the wedge $P^r = P_1 \vee \cdots \vee P_r$ of finitely many summands P_i are homotopic. This is so because P^r is compact and therefore, the compact subsets $h^x(X \times P^r)$ and $h^{x'}(X \times P^r)$ of $X \times P$ must be contained in a product of the form $X \times P^{r'}$ for some $r' \in \mathbb{N}$. Since X and $P^{r'}$ are compact, $X \times P^{r'}$ is a product in the category Sh(Top). It follows that conditions (1) and (2), for the restrictions $h^x|P^r$ and $h^{x'}|P^r$, imply $S[h^x|P^r] = S[h^{x'}|P^r]$ and thus, $S[qh^x|P^r] = S[q]S[h^x|P^r] = S[q]S[h^{x'}|P^r] = S[qh^{x'}|P^r]$, which is equivalent to $qh^x|P^r \simeq qh^{x'}|P^r$.

In homotopy theory one studies *phantom mappings* (of the second kind), i.e., mappings between CW-complexes $f\colon X \to Y$ whose restrictions to all compact subsets of X are homotopically trivial. A phantom mapping is called *essential* if the mapping is homotopically nontrivial [8]. A generalization is the notion of *essential phantom pairs* of mappings. These are pairs of nonhomotopic mappings $f, g \colon X \to Y$ whose restrictions to every compact subset of X are homotopic. The above constructed pair of mappings $qh^x, qh^{x'} : P \to Q$ is an example of an essential phantom pair. Phantom pairs of the first kind (restrictions to all n-skeleta are homotopic) were introduced in [9].

11. Is $X \times P$ a product in the strong shape category?

QUESTION. *Is the Cartesian product $X \times P$ of the dyadic solenoid X and the wedge $P = P_1 \vee P_2 \vee \cdots$ of a sequence of 1-spheres a product in the strong shape category of topological spaces*, SSh(Top)?

The mappings $h^x, h^{x'} : P \to X \times P$ cannot be used to prove that $X \times P$ is not a product in the strong shape category SSh(Top), because of the following proposition, where $\overline{S} \colon \mathrm{H(Top)} \to \mathrm{SSh(Top)}$ denotes the strong shape functor.

PROPOSITION 2. *For arbitrary points $x = (x_1, x_2 \ldots)$, $x' = (x'_1, x'_2 \ldots)$ in X the mappings $h^x, h^{x'} : P \to X \times P$ satisfy the condition*

$$\overline{S}[\pi_X]\overline{S}[h^x] = \overline{S}[\pi_X]\overline{S}[h^{x'}] \tag{61}$$

if and only if the points $x = (x_1, x_2 \ldots)$ and $x' = (x'_1, x'_2 \ldots)$ belong to the same path component of X. In that case $\overline{S}[h^x] = \overline{S}[h^{x'}]$.

Proof. If x and x' belong to the same path component of X, then Proposition 1 implies that $[h^x] = [h^{x'}]$ and thus, $\overline{S}[h^x] = \overline{S}[h^{x'}]$ and (61) holds. Conversely, assume that (61) holds and hence, also $\overline{S}[\pi_X h^x] = \overline{S}[\pi_X h^{x'}]$. Recall that $\pi_X h^x = x$ and $\pi_X h^{x'} = x'$ are constant mappings $x, x' : P \to X$. Composing them with the inclusion $* \to P$, we obtain constant mappings $x, x' \colon \{*\} \to X$ for which $\overline{S}[x] = \overline{S}[x']$.

Denote by $f_i, f'_i \colon * \to X_i$ the constant mappings $f_i, f'_i \colon \{*\} \to X_i$, where $f_i(*) = x_i$, $f_i(*) = x'_i$. Since $p_{i-1\,i}(x_i) = x_{i-1}$, the mappings f_i form a mapping $\boldsymbol{f} = (f_i) \colon * \to \boldsymbol{X}$. Similarly, the mappings f'_i form a mapping $\boldsymbol{f}' = (f'_i) \colon * \to \boldsymbol{X}$. The induced coherent mapping $C(\boldsymbol{f}) \colon * \to \boldsymbol{X}$ consists of the mappings $f_{i_0 \ldots i_n} \colon * \times \Delta^n \to X_{i_0}$, $i_0 \le i_1 \le \cdots$, where $f_{i_0 \ldots i_n}(*, t) = f_{i_0}(*) = x_{i_0}$ (see [4, §1.4]). By the description of the strong shape functor $\overline{S}$ in terms of coherent mappings (see [4, §8.2]), $\overline{S}[x] = \overline{S}[x']$ implies the existence of a coherent homotopy $\boldsymbol{F} = (F_{i_0 \ldots i_n}) \colon * \to \boldsymbol{X}$ which connects $C(\boldsymbol{f})$ to $C(\boldsymbol{f}')$ (see [4, §2.1]). In particular, one has mappings $F_{i_0} \colon * \times I \to X_{i_0}$ such that, for $i_0 \in \mathbb{N}$,

$$F_{i_0}(*, 0) = f_{i_0}(*) = x_{i_0}, \qquad F_{i_0}(*, 1) = f'_{i_0}(*) = x'_{i_0}, \tag{62}$$

and one has mappings $F_{i_0 i_1}\colon * \times I \times \Delta^1 \to X_{i_0}$ such that, for $i_0 \le i_1$, $s \in I$ and $\tau \in \Delta^1$,

$$F_{i_0 i_1}(*, s, e_1) = p_{i_0 i_1} F_{i_1}(*, s), \qquad F_{i_0 i_1}(*, s, e_0) = F_{i_0}(*, s), \tag{63}$$

$$F_{i_0 i_1}(*, 0, \tau) = f_{i_0 i_1}(*, \tau) = x_{i_0}, \qquad F_{i_0 i_1}(*, 1, \tau) = f'_{i_0 i_1}(*, \tau) = x'_{i_0}. \tag{64}$$

Formulae (62)–(64) show that $u_i\colon I \to X_i$ defined by putting $u_i(s) = F_i(*, s)$ is a path in X_i which connects the point x_i to x'_i, while $u_{i-1\,i}\colon I \times I \to X_{i-1}$ defined by $u_{i-1\,i}(s,t) = F_{i-1\,i}(*, s, (1-t)e_0 + te_1)$ is a homotopy which connects the path u_{i-1} to $p_{i-1\,i}u_i$. Moreover, this homotopy is fixed for $s = 0$ and $s = 1$, i.e., $u_{i-1} \simeq p_{i-1\,i}u_i$ $(\mathrm{rel}\,\partial I)$. Indeed, $u_{i-1\,i}(0,t) = F_{i-1\,i}(*, 0, (1-t)e_0 + te_1) = x_{i_0}$ and $u_{i-1\,i}(1,t) = F_{i-1\,i}(*, 1, (1-t)e_0 + te_1) = x'_{i_0}$. We now apply Lemma 1 to the sequence of paths $u_i\colon I \to X_i$ and we obtain a new sequence of paths $v_i\colon I \to X_i$ which connect x_i and x'_i, and satisfy $p_{i-1\,i}v_i = v_{i-1}$ and $u_i \simeq v_i$ $(\mathrm{rel}\,\partial I)$. The paths v_i determine a unique path $v\colon I \to X$ such that $p_i v = v_i$. Moreover, v connects x and x'. ∎

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