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# Definability of small puncture sets

by

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**Abstract.** We characterize the class of definable families of countable sets for which there is a single countable definable set intersecting every element of the family.

**1. Introduction.** A set $P$ *punctures* a family of sets $\mathscr{A}$ if its intersection with each set in $\mathscr{A}$ is non-empty. This notion was isolated by Spencer, and provides a uniform way of handling many results in finite combinatorics. The basic problem studied in [Spe74] is to find puncture sets of small cardinality for finite families of finite sets. The arguments are primarily probabilistic in nature. As the author remarks and as is often the case in finite combinatorics, the bounds obtained through this approach yield better results than those obtained through explicit constructions.

On the other hand, if a family of sets of bounded finite cardinality admits a finite puncture set, then there is such a set which is explicitly constructible, and therefore definable, from the family. This follows from the observation that the family of puncture sets of minimal cardinality is itself finite (by a straightforward induction), thus the union of all such sets is as desired. This fact is quite useful in the descriptive set-theoretic setting. For instance, it was an important component of the proof of a recent result of Clemens–Conley–Miller, generalizing the Glimm–Effros dichotomy from Polish spaces to quotients of Polish spaces by countable Borel equivalence relations (see [CCM11]). Other recent results of Marks–Miller and Miller rely on the same observation (see [MM11, Mil11b]), suggesting that such arguments are not isolated phenomena, and that there may be value in studying related notions.

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The picture changes significantly for families of infinite sets. Suppose that $\kappa$ is an infinite cardinal and $X$ is a set, and let $[X]^{<\kappa}$ denote the family of subsets of $X$ of cardinality strictly less than $\kappa$. It is clear that every infinite set $A \in [X]^{<\kappa}$ punctures the corresponding family $\mathscr{A} = \{B \in [X]^{<\kappa} \mid A \cap B \text{ is infinite}\}$. However, since the group of transpositions of $X$ acts transitively on $X$ and fixes $\mathscr{A}$ setwise, no proper subset of $X$ is in any reasonable sense definable from $\mathscr{A}$.

Nevertheless, it is possible to define countable puncture sets from appropriate witnesses to the definability of the families in question. Generalizing the perfect set theorem, Przymusiński has shown that an analytic family of finite subsets of a Polish space has a countable puncture set if and only if it does not have a pairwise disjoint subset of cardinality $\mathfrak{c}$ (see [Prz78, Theorem 1]), and Hohti has shown that this characterization continues to hold for analytic families of countable subsets of Polish spaces (see [Hoh88, Theorem 2.1]). Moreover, it is easy to extract definitions of the corresponding puncture sets from the proofs. Here we generalize these results beyond Polish spaces.

Before proceeding further, we must first introduce some terminology. For each set $X$, let $[X]^{\leq\aleph_0}$ denote the family of countable subsets of $X$. The *dual* of a pointclass $\varGamma$ is the pointclass $\check{\varGamma}$ of complements of sets in $\varGamma$. Given an equivalence relation $E$ on $X$ and a set $N$, a *lifting* of a sequence $x \in (X/E)^N$ is a sequence $y \in X^N$ such that $y(i) \in x(i)$ for all $i \in N$. Let $\ell(x)$ denote the set of all such liftings. We say that a set $A \subseteq (X/E)^N$ is *$\varGamma$-measurable* if its *lifting* $\bigcup\{\ell(x) \mid x \in A\}$ is in $\varGamma$, and we say that a family $\mathscr{A} \subseteq [X/E]^{\leq\aleph_0}$ is *$\varGamma$-measurable* if $\{x \in (X/E)^{\mathbb{N}} \mid \{x(n) \mid n \in \mathbb{N}\} \in \mathscr{A}\}$ is $\varGamma$-measurable.

Suppose that $\mathscr{X}$ is a class of Hausdorff spaces. In §2, we consider pointclasses $\varGamma$ which are *$\kappa$-chromatic-on-$\mathscr{X}$*, in the sense that every $\varGamma$-measurable $\aleph_0$-dimensional digraph on a space $X \in \mathscr{X}$ satisfies a weak analog of the Kechris–Solecki–Todorcevic dichotomy theorem characterizing the existence of Borel colorings (see [KST99, Theorem 6.4]) relative to $\kappa$. A number of such classes appear in the literature, beginning with work of Lecomte (see [Lec09, Theorem 1.6]). We show that all such pointclasses satisfy another somewhat technical generalization of the original Kechris–Solecki–Todorcevic dichotomy theorem.

Suppose that $X$ is a Hausdorff space. We say that a set $Y \subseteq X$ is *weakly $\aleph_0$-universally Baire* if $\pi^{-1}(Y)$ has the Baire property for every continuous function $\pi\colon 2^{\mathbb{N}} \to X$. In §3, we establish the following:

THEOREM. *Suppose that $\kappa$ is an infinite cardinal, $\mathscr{X}$ is a class of Hausdorff spaces, $\varGamma$ is a $\kappa$-chromatic-on-$\mathscr{X}$ pointclass, $X \in \mathscr{X}$, $E$ is a $\check{\varGamma}$-measurable, weakly $\aleph_0$-universally Baire equivalence relation on $X$, and $\mathscr{A} \subseteq$*

$[X/E]^{\leq\aleph_0}$ *is $\Gamma$-measurable. Then at least one of the following holds:*

(1) *There is a set of cardinality strictly less than $\kappa$ puncturing $\mathscr{A}$.*
(2) *There is a pairwise disjoint subset of $\mathscr{A}$ of cardinality $\mathfrak{c}$.*

*Moreover, if there is no surjection from a cardinal strictly less than $\kappa$ to $\mathfrak{c}$, then exactly one of the above conditions holds.*

In §4, we use this result to establish refinements for several distinguished pointclasses, and we discuss the definability of the puncture sets obtained from the proofs.

Unless specified otherwise, all of our arguments take place in ZF.

**2. Chromatic pointclasses.** In this section, we establish technical generalizations of the Kechris–Solecki–Todorcevic dichotomy theorem characterizing analytic graphs of uncountable Borel chromatic number. Although our results can be proven via straightforward modifications of the known proofs of the Kechris–Solecki–Todorcevic dichotomy theorem in the different contexts we have in mind, we will instead show that a weak $\aleph_0$-dimensional analog of the Kechris–Solecki–Todorcevic dichotomy theorem abstractly implies the results we desire.

Suppose that $N$ is a set and $S \subseteq N^{<\mathbb{N}}$. We say that $S$ is *dense* if $\forall r \in N^{<\mathbb{N}}\ \exists s \in S\ r \sqsubseteq s$, and $S$ is *sparse* if $|N^n \cap S| \leq 1$ for all $n \in \mathbb{N}$. Underlying the main result of this section is the following observation:

PROPOSITION 1. *Suppose that $C \subseteq \mathbb{N}^\mathbb{N}$ is comeager, $R \subseteq 2^{<\mathbb{N}}$ is sparse, $S \subseteq \mathbb{N}^{<\mathbb{N}}$ is dense, and $f_i \colon R \to \mathbb{N}^{<\mathbb{N}}$ for $i < 2$. Then there is a continuous function $\phi \colon 2^\mathbb{N} \to C$ such that*

$$\forall r \in R\ \exists s \in S\ \forall c \in 2^\mathbb{N}\ \exists b \in \mathbb{N}^\mathbb{N}\ \forall i < 2\ \phi(r^\frown(i)^\frown c) = s^\frown f_i(r)^\frown b.$$

*Proof.* We use $s \sqsubseteq t$ to denote extension, $s \sqsubset t$ to denote strict extension, and both $s^\frown t$ and $\bigoplus_{i<k} s_i$ to denote concatenation of sequences. We also use $\mathcal{N}_s$ to denote the basic open set determined by $s$. In order to avoid confusion, throughout this argument we use ${}^X Y$ to denote the family of all functions from $X$ to $Y$.

Fix dense open sets $U_k \subseteq {}^\mathbb{N}\mathbb{N}$ such that $\bigcap_{k\in\mathbb{N}} U_k \subseteq C$. Clearly we can assume that for each $k \in \mathbb{N}$ there is a unique sequence $r_k \in R \cap {}^{<\mathbb{N}}2$. For each $k \in \mathbb{N}$, fix an enumeration $(r_{i,k})_{i<2^k}$ of ${}^k2$. We will recursively construct functions $\phi_k \colon {}^k2 \to {}^{<\mathbb{N}}\mathbb{N}$, beginning with the trivial function $\phi_0 \colon {}^02 \to {}^0\mathbb{N}$.

Granting that we have found $\phi_k \colon {}^k2 \to {}^{<\mathbb{N}}\mathbb{N}$, recursively construct an increasing sequence $(u_{i,k})_{i<2^k}$ of non-empty sequences such that

$$\forall i < 2^k\ \mathcal{N}_{\phi_k(r_{i,k})^\frown u_{i,k}} \subseteq U_k.$$

Set $u_k = u_{2^k-1,k}$, fix $s_k \in S$ such that $\phi_k(r_k)^\frown u_k \sqsubseteq s_k$, let $v_k$ denote the unique sequence with the property that $\phi_k(r_k)^\frown u_k{}^\frown v_k = s_k$, and define

$\phi_{k+1}\colon {}^{k+1}2 \to {}^{<\mathbb{N}}\mathbb{N}$ by

$$\phi_{k+1}(s^\frown(i)) = \phi_k(s)^\frown u_k{}^\frown v_k{}^\frown f_i(r_k)$$

for $i < 2$ and $s \in {}^k2$.

Clearly $\phi_k(s) \sqsubset \phi_{k+1}(s^\frown(i))$ for $i < 2$, $k \in \mathbb{N}$, and $s \in {}^k2$, so we obtain a continuous map $\phi\colon {}^{\mathbb{N}}2 \to {}^{\mathbb{N}}\mathbb{N}$ by setting $\phi(c) = \bigcup_{k\in\mathbb{N}} \phi_k(c\restriction k)$. Note that if $c \in {}^{\mathbb{N}}2$, then $\mathcal{N}_{\phi_{k+1}(c\restriction(k+1))} \subseteq U_k$ for all $k \in \mathbb{N}$, so

$$\phi(c) \in \bigcap_{k\in\mathbb{N}} \mathcal{N}_{\phi_k(c\restriction k)} \subseteq \bigcap_{k\in\mathbb{N}} U_k \subseteq C,$$

thus $\phi[{}^{\mathbb{N}}2] \subseteq C$. Finally, note that if $c \in {}^{\mathbb{N}}2$, $i < 2$, and $k \in \mathbb{N}$, then $\phi(r_k{}^\frown(i)^\frown c) = s_k{}^\frown f_i(r_k)^\frown b$, where $b = \bigoplus_{j>k} u_j{}^\frown v_j{}^\frown f_{c(j-k-1)}(r_j)$. ∎

Suppose that $X$ is a set and $\kappa$ is a cardinal. An $\aleph_0$-*dimensional digraph* on $X$ is a set $G \subseteq X^{\mathbb{N}}$ of non-constant sequences. The *restriction* of $G$ to a set $Y \subseteq X$ is the $\aleph_0$-dimensional digraph $G\restriction Y$ on $Y$ given by $G\restriction Y = G \cap Y^{\mathbb{N}}$. We say that $Y$ is $G$-*independent* if $G\restriction Y = \emptyset$. A $\kappa$-*coloring* of $G$ is a map $c\colon X \to \kappa$ such that $c^{-1}(\{\alpha\})$ is $G$-independent for all $\alpha < \kappa$. A $({<}\kappa)$-*coloring* is a $\lambda$-coloring for some $\lambda < \kappa$. More generally, a *homomorphism* from an $\aleph_0$-dimensional digraph $G$ on $X$ to an $\aleph_0$-dimensional digraph $H$ on $Y$ is a function $\phi\colon X \to Y$ sending sequences in $G$ to sequences in $H$.

Fix sequences $s_k \in \mathbb{N}^k$ which are together dense in $\mathbb{N}^{<\mathbb{N}}$, and define

$$G_0(\mathbb{N}^{\mathbb{N}}) = \{(s_k{}^\frown(i)^\frown c)_{i\in\mathbb{N}} \mid c \in \mathbb{N}^{\mathbb{N}} \text{ and } k \in \mathbb{N}\}.$$

Suppose that $\kappa$ is a cardinal, $\mathscr{X}$ is a class of Hausdorff spaces, and $\Gamma$ is a pointclass of subsets of Hausdorff spaces. We say that $\Gamma$ is $\kappa$-*chromatic-on-*$\mathscr{X}$ if every $\Gamma$-measurable $\aleph_0$-dimensional digraph $H$ on a space $X \in \mathscr{X}$ satisfies the following weak form of the $\aleph_0$-dimensional analog of the Kechris–Solecki–Todorcevic dichotomy theorem: if there is no $({<}\kappa)$-coloring of $H$, then there is a comeager set $C \subseteq \mathbb{N}^{\mathbb{N}}$ for which there is a continuous homomorphism from $G_0(\mathbb{N}^{\mathbb{N}})\restriction C$ to $H$.

A number of instances of this notion have appeared in the literature, but we defer their mention until §4.

A *digraph* on $X$ is an irreflexive set $G \subseteq X \times X$. The notions of *restriction*, *independence*, *coloring*, and *homomorphism* are defined for digraphs exactly as for $\aleph_0$-dimensional digraphs. A *mixed* $\kappa$-*coloring* of a sequence $\mathbf{G} = (G_n)_{n\in\mathbb{N}}$ of digraphs on $X$ is a function $c\colon X \to \kappa$ such that for all $\alpha < \kappa$ there exists $n \in \mathbb{N}$ for which $c^{-1}(\{\alpha\})$ is $G_n$-independent. A *mixed* $({<}\kappa)$-*coloring* is a mixed $\lambda$-coloring for some $\lambda < \kappa$. A *homomorphism* from a sequence $\mathbf{G} = (G_n)_{n\in\mathbb{N}}$ of digraphs on $X$ to a sequence $\mathbf{H} = (H_n)_{n\in\mathbb{N}}$ of digraphs on $Y$ is a function $\phi\colon X \to Y$ sending pairs in $G_n$ to pairs in $H_n$ for all $n \in \mathbb{N}$.

Fix sequences $r_k \in 2^k$ which are together dense in $2^{<\mathbb{N}}$. Then there are natural numbers $n_k$ such that

$$\forall n \in \mathbb{N}\ \{r_k \mid k \in \mathbb{N} \text{ and } n_k = n\} \text{ is dense.}$$

Let $\mathbf{G}_0 = (G_{0,n})_{n\in\mathbb{N}}$ denote the sequence of digraphs on $2^{\mathbb{N}}$ given by

$$G_{0,n} = \{(r_k{}^\frown(0)^\frown c, r_k{}^\frown(1)^\frown c) \mid c \in 2^{\mathbb{N}},\ k \in \mathbb{N}, \text{ and } n_k = n\}.$$

We say that $\Gamma$ is *mixed $\kappa$-chromatic-on-$\mathscr{X}$* if every sequence $\mathbf{H} = (H_n)_{n\in\mathbb{N}}$ of $\Gamma$-measurable digraphs on a space $X \in \mathscr{X}$ satisfies the following version of the Kechris–Solecki–Todorcevic dichotomy theorem: if there is no mixed $(<\kappa)$-coloring of $\mathbf{H}$, then there is a continuous homomorphism from $\mathbf{G}_0$ to $\mathbf{H}$.

PROPOSITION 2. *Suppose that $\kappa$ is a cardinal, $\mathscr{X}$ is a class of Hausdorff spaces, and $\Gamma$ is a $\kappa$-chromatic-on-$\mathscr{X}$ pointclass. Then $\Gamma$ is mixed $\kappa$-chromatic-on-$\mathscr{X}$.*

*Proof.* Suppose that $X \in \mathscr{X}$ and $\mathbf{H} = (H_n)_{n\in\mathbb{N}}$ is a sequence of $\Gamma$-measurable digraphs on $X$. Let $H$ denote the $\aleph_0$-dimensional digraph on $X$ given by

$$(x_n)_{n\in\mathbb{N}} \in H \iff \forall n \in \mathbb{N}\ x_{2n}\ H_n\ x_{2n+1}.$$

Note that a subset of $X$ is $H$-independent if and only if it is $H_n$-independent for some $n \in \mathbb{N}$. In particular, it follows that there is a $(<\kappa)$-coloring of $H$ if and only if there is a mixed $(<\kappa)$-coloring of $\mathbf{H}$, so we can assume that there is a comeager set $C \subseteq \mathbb{N}^{\mathbb{N}}$ for which there is a continuous homomorphism $\phi\colon C \to X$ from $G_0(\mathbb{N}^{\mathbb{N}})\restriction C$ to $H$. By Proposition 1, there is a continuous function $\psi\colon 2^{\mathbb{N}} \to C$ such that

$$\forall k \in \mathbb{N}\ \exists l \in \mathbb{N}\ \forall c \in 2^{\mathbb{N}}\ \exists b \in \mathbb{N}^{\mathbb{N}}\ \forall i < 2\ \phi(r_k{}^\frown(i)^\frown c) = s_l{}^\frown(2n_k + i)^\frown b.$$

Clearly $\phi \circ \psi$ is a continuous homomorphism from $\mathbf{G}_0$ to $\mathbf{H}$. ∎

**3. The existence of small puncture sets.** In the context of families of infinite sets, the existence of small puncture sets is easy to characterize. Although this can be seen directly, we will first note a somewhat more general fact.

From this point on, it will be convenient to work with sets of sequences $A \subseteq X^{<\kappa}$ in lieu of families of sets $\mathscr{A} \subseteq [X]^{<\kappa}$. Given a set $K \subseteq \kappa$ and a sequence $x \in X^{<\kappa}$, we use $x\restriction K$ to denote the restriction of $x$ to the intersection of $K$ with the domain of $x$, and we use $x[K]$ to denote the set of points of the form $x(\alpha)$, where $\alpha$ is in the domain of $x\restriction K$. Given a set $A \subseteq X^{<\kappa}$, we use $A\restriction K$ to denote the set given by $A\restriction K = \{x\restriction K \mid x \in A\}$. We say that a set $P$ *punctures* $A$ if $x[\kappa] \cap P \neq \emptyset$ for all $x \in A$.

Proposition 3 (ZFC). *Suppose that $\kappa$ is an infinite regular cardinal, $X$ is a set, and $A \subseteq X^{<\kappa}$. Then for every non-empty set $K \subseteq \kappa$, exactly one of the following holds:*

(1) *There is a set of cardinality strictly less than $\kappa$ puncturing $A \restriction K$.*
(2) *There is a sequence $(x_\alpha)_{\alpha<\kappa}$ of elements of $A$ such that*

$$\forall \alpha < \beta < \kappa \; x_\alpha[\kappa] \cap x_\beta[K] = \emptyset.$$

*Proof.* It is clear that conditions (1) and (2) are mutually exclusive, and the obvious transfinite recursion shows that at least one holds. ∎

As promised, Proposition 3 gives a characterization of the existence of small puncture sets:

Proposition 4 (ZFC). *Suppose that $\kappa$ is an infinite regular cardinal, $X$ is a set, and $\mathscr{A} \subseteq [X]^{<\kappa}$. Then exactly one of the following holds:*

(1) *There is a set of cardinality strictly less than $\kappa$ puncturing $\mathscr{A}$.*
(2) *There is a pairwise disjoint subset of $\mathscr{A}$ of cardinality $\kappa$.*

*Proof.* Set $A = \{x \in X^{<\kappa} \mid x[\kappa] \in \mathscr{A}\}$ and $K = \kappa$. Proposition 3 ensures that either there is a set of cardinality strictly less than $\kappa$ puncturing $A$, or there is a pairwise disjoint subset of $A$ of cardinality $\kappa$, which corresponds to conditions (1) and (2). ∎

Proposition 3 also yields a natural situation in which the underlying space $\bigcup \mathscr{A}$ is itself a small puncture set:

Proposition 5 (ZFC). *Suppose that $\kappa$ is an infinite regular cardinal, $X$ is a set, $\mathscr{A} \subseteq [X]^{<\kappa}$, and there is no sequence $(A_\alpha)_{\alpha<\kappa}$ of sets in $\mathscr{A}$ such that $\forall \alpha < \beta < \kappa \; A_\beta \nsubseteq A_\alpha$. Then $|\bigcup \mathscr{A}| < \kappa$.*

*Proof.* Set $A = \{x \in X^{<\kappa} \mid x[\kappa] \in \mathscr{A}\}$ and $K = \{0\}$. As $\bigcup \mathscr{A}$ is the projection of $A$ onto the 0th coordinate, Proposition 3 ensures that $|\bigcup \mathscr{A}| < \kappa$. ∎

Let $\perp$ denote incomparability with respect to containment. The following example shows that consistently the hypothesis of Proposition 5 cannot be replaced with a natural weakening:

Example 6. Recall that a $\kappa$-*Suslin tree* is a (set-theoretic) tree of height $\kappa$ with no chains of length $\kappa$ nor antichains of cardinality $\kappa$. If $T$ is such a tree and $\mathscr{A} = \{\{s \in T \mid s \leq t\} \mid t \in T\}$, then $|\bigcup \mathscr{A}| = \kappa$ but for no sequence $(A_\alpha)_{\alpha<\kappa}$ of sets in $\mathscr{A}$ is it the case that either $\forall \alpha < \beta < \kappa \; A_\alpha \subseteq A_\beta$ or $\forall \alpha < \beta < \kappa \; A_\alpha \perp A_\beta$.

Nevertheless, we will now establish strengthenings of these results in the descriptive set-theoretic setting.

Given an equivalence relation $E$ on $X$ and a sequence $x \in X^{\mathbb{N}}$, let $[x]_E$ denote the sequence in $(X/E)^{\mathbb{N}}$ whose $n$th entry is $[x(n)]_E$. For each set

$A \subseteq X^{\mathbb{N}}$, the *quotient* of $A$ by $E$ is the set $A/E$ of sequences obtained in this fashion.

THEOREM 7. *Suppose that $\kappa$ is an infinite cardinal, $\mathscr{X}$ is a class of Hausdorff spaces, $\Gamma$ is a mixed $\kappa$-chromatic-on-$\mathscr{X}$ pointclass, $X \in \mathscr{X}$, $E$ is a $\check{\Gamma}$-measurable, weakly $\aleph_0$-universally Baire equivalence relation on $X$, and $A \subseteq X^{\mathbb{N}}$ is $\Gamma$-measurable. Then for every non-empty set $N \subseteq \mathbb{N}$, at least one of the following holds:*

1. *There is a set of cardinality strictly less than $\kappa$ puncturing $(A \restriction N)/E$.*
2. *There is a non-empty perfect set $P \subseteq A$ such that for all distinct points $x, y \in P$, the sets $x[\mathbb{N}]/E$ and $y[N]/E$ are disjoint.*

*Moreover, if there is no surjection from a cardinal strictly less than $\kappa$ to $\mathfrak{c}$, then exactly one of the above conditions holds.*

*Proof.* If both (1) and (2) hold as witnessed by a puncture set $B$ and a perfect set $P$, then each point of $B$ intersects the range of at most one sequence in $(P \restriction N)/E$, thus there is a surjection from $B$ to $(P \restriction N)/E$, and therefore from a cardinal strictly less than $\kappa$ to $\mathfrak{c}$.

It remains to show that at least one of conditions (1) and (2) holds. Fix a sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers for which $N = \{k_n \mid n \in \mathbb{N}\}$, and let $\mathbf{H} = (H_n)_{n\in\mathbb{N}}$ denote the sequence of digraphs on $A$ given by

$$x \ H_n \ y \ \Leftrightarrow \ \neg \ x(k_n) \ E \ y(k_n).$$

Suppose that $c$ is a mixed $({<}\kappa)$-coloring of $\mathbf{H}$, and fix $\lambda < \kappa$ such that $c[A] \subseteq \lambda$. For each $\alpha < \lambda$, set $X_\alpha = c^{-1}(\{\alpha\})$ and $B_\alpha = \mathrm{proj}_{k_n}[X_\alpha]$, where $n \in N$ is least for which $X_\alpha$ is $H_n$-discrete. Then the quotient of the set $B = \bigcup_{\alpha<\lambda} B_\alpha$ by $E$ has cardinality at most $\lambda$ and punctures $(A \restriction N)/E$.

Suppose, on the other hand, that there is a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $\mathbf{G}_0$ to $\mathbf{H}$. For $m, n \in \mathbb{N}$, define

$$R_{m,n} = \{(c, d) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} \mid \phi(c)(m) \ E \ \phi(d)(k_n)\}.$$

LEMMA 8. *Suppose that $m, n \in \mathbb{N}$. Then $R_{m,n}$ is meager.*

*Proof.* As $R_{m,n} = ((\mathrm{proj}_m \circ \phi) \times (\mathrm{proj}_{k_n} \circ \phi))^{-1}(E)$, it has the Baire property. Moreover, if $c \in 2^{\mathbb{N}}$ and $\psi\colon X \to X \times X$ is the continuous function given by $\psi(x) = (\phi(c)(m), x)$, then $[\phi(c)(m)]_E = \psi^{-1}(E)$, so the $c$th vertical section of $R_{m,n}$ also has the Baire property, since it can be expressed as

$$(R_{m,n})_c = (\mathrm{proj}_{k_n} \circ \phi)^{-1}([\phi(c)(m)]_E) = (\psi \circ \mathrm{proj}_{k_n} \circ \phi)^{-1}(E).$$

By the Kuratowski–Ulam theorem (see, for example, [Kech95, Theorem 8.41]), it is therefore sufficient to show that for no $c \in 2^{\mathbb{N}}$ does there exist $r \in 2^{<\mathbb{N}}$ for which the set $C = (R_{m,n})_c$ is comeager in $\mathcal{N}_r$. Towards this end, suppose that there is such an $r$, and fix $k \in \mathbb{N}$ such that $n = n_k$ and $r \sqsubseteq r_k$. Then there exists $d \in 2^{\mathbb{N}}$ for which the points of the form

$d_i = r_k{}^\frown(i)^\frown d$ for $i < 2$ are in $C$. As $d_0 \; G_{0,n} \; d_1$ and $\phi$ is a homomorphism, it follows that $\phi(d_0) \; H_n \; \phi(d_1)$. But the fact that $d_0, d_1 \in C$ ensures that $\phi(d_0)(k_n) \; E \; \phi(c)(m) \; E \; \phi(d_1)(k_n)$, the desired contradiction. ■

Lemma 8 ensures that the set $R = \bigcup_{m,n \in \mathbb{N}} R_{m,n}$ is meager. By Mycielski's theorem (see, for example, [Kech95, Theorem 19.1]), there is a continuous function $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that

$$\forall c, d \in 2^{\mathbb{N}} \; (\psi(c) \, R \, \psi(d) \Rightarrow c = d).$$

It follows that $(\phi \circ \psi)[2^{\mathbb{N}}]$ is the desired perfect set. ■

Theorem 7 gives a characterization of the existence of small puncture sets in the descriptive set-theoretic setting:

THEOREM 9. *Suppose that $\kappa$ is an infinite cardinal, $\mathscr{X}$ is a class of Hausdorff spaces, $\Gamma$ is a mixed $\kappa$-chromatic-on-$\mathscr{X}$ pointclass, $X \in \mathscr{X}$, $E$ is a $\breve{\Gamma}$-measurable, weakly $\aleph_0$-universally Baire equivalence relation on $X$, and $\mathscr{A} \subseteq [X/E]^{\leq \aleph_0}$ is $\Gamma$-measurable. Then at least one of the following holds:*

(1) *There is a set of cardinality strictly less than $\kappa$ puncturing $\mathscr{A}$.*
(2) *There is a pairwise disjoint subset of $\mathscr{A}$ of cardinality $\mathfrak{c}$.*

*Moreover, if there is no surjection from a cardinal strictly less than $\kappa$ to $\mathfrak{c}$, then exactly one of the above conditions holds.*

*Proof.* Set $A = \{x \in X^{\mathbb{N}} \mid x[\mathbb{N}] \in \mathscr{A}\}$ and $N = \mathbb{N}$. Theorem 7 ensures that either there is a set of cardinality strictly less than $\kappa$ puncturing $A/E$, or there is a perfect pairwise disjoint subset of $A$, which corresponds to conditions (1) and (2). ■

Theorem 7 also yields a natural situation in which the underlying space $\bigcup \mathscr{A}$ is itself a small puncture set:

THEOREM 10. *Suppose that $\kappa$ is an infinite cardinal, $\mathscr{X}$ is a class of Hausdorff spaces, $\Gamma$ is a mixed $\kappa$-chromatic-on-$\mathscr{X}$ pointclass, $X \in \mathscr{X}$, $E$ is a $\breve{\Gamma}$-measurable, weakly $\aleph_0$-universally Baire equivalence relation on $X$, and $\mathscr{A} \subseteq [X/E]^{\leq \aleph_0}$ is $\Gamma$-measurable and has no pairwise incomparable subsets of cardinality $\mathfrak{c}$. Then $|\bigcup \mathscr{A}| < \kappa$.*

*Proof.* Set $A = \{x \in X^{\mathbb{N}} \mid x[\mathbb{N}] \in \mathscr{A}\}$ and $N = \{0\}$. As the projection of $A$ onto the 0th coordinate is $\bigcup \mathscr{A}$, Theorem 7 ensures that $|\bigcup \mathscr{A}| < \kappa$. ■

The following example shows that the conclusion of Theorem 10 cannot be replaced with a natural strengthening:

EXAMPLE 11. Set $X = \mathbb{Q}$ and $\mathscr{A} = \{\{q \in \mathbb{Q} \mid q < r\} \mid r \in \mathbb{R} \setminus \mathbb{Q}\}$. Clearly $\mathscr{A}$ has no non-trivial pairwise incomparable subsets. As the group of order-preserving permutations of $\mathbb{Q}$ acts transitively on $X$ and fixes $\mathscr{A}$ setwise, no proper subset of $X$ is in any reasonable sense definable from $\mathscr{A}$.

**4. Applications.** We say that a subset of a Hausdorff space is *analytic* if it is the continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$.

THEOREM 12. *Suppose that $X$ is a Hausdorff space, $E$ is a co-analytic equivalence relation on $X$, and $\mathscr{A} \subseteq [X/E]^{\leq \aleph_0}$ is analytic. Then exactly one of the following holds:*

(1) *There is a countable set puncturing $\mathscr{A}$.*
(2) *There is a pairwise disjoint subset of $\mathscr{A}$ of cardinality $\mathfrak{c}$.*

*Proof.* By [Lec09, Theorem 1.6] (see also [Mil11a, Theorem 4]), the pointclass of analytic sets is $\aleph_1$-chromatic on Hausdorff spaces. Proposition 1 therefore implies that it is mixed $\aleph_1$-chromatic on Hausdorff spaces, and since the Luzin–Sierpiński theorem ensures that co-analytic subsets of $2^{\mathbb{N}}$ have the Baire property (see, for example, [Kech95, Theorem 21.6]), Theorem 9 yields the desired result. ∎

Recall that an *aleph* is an infinite cardinal $\kappa$ which can be well-ordered. We say that a subset of a Hausdorff space is *$\kappa$-Suslin* if it is the continuous image of a closed subset of $\kappa^{\mathbb{N}}$. We say that a set is *$(<\kappa)$-Suslin* if it is $\lambda$-Suslin for some $\lambda < \kappa$. Generalizing Theorem 12, we have the following:

THEOREM 13. *Suppose that $\kappa$ is an uncountable aleph, $X$ is a Hausdorff space, $E$ is a co-$(<\kappa)$-Suslin, weakly $\aleph_0$-universally Baire equivalence relation on $X$, and $\mathscr{A} \subseteq [X/E]^{\leq \aleph_0}$ is $(<\kappa)$-Suslin. Then at least one of the following holds:*

(1) *There is a set of cardinality strictly less than $\kappa$ puncturing $\mathscr{A}$.*
(2) *There is a pairwise disjoint subset of $\mathscr{A}$ of cardinality $\mathfrak{c}$.*

*Moreover, if there is no surjection from a cardinal strictly less than $\kappa$ to $\mathfrak{c}$, then exactly one of the above conditions holds.*

*Proof.* By [Mil11a, Theorem 4], the pointclass of $(<\kappa)$-Suslin sets is $\kappa$-chromatic on Hausdorff spaces. Proposition 1 therefore implies that it is mixed $\kappa$-chromatic on Hausdorff spaces, so Theorem 9 yields the desired result. ∎

REMARK 14. Recall that under `AD`, every subset of a Polish space has the Baire property, thus the assumption that $E$ is weakly $\aleph_0$-universally Baire is superfluous. Moreover, as `AD` also ensures that no uncountable subset of an analytic Hausdorff space can be well-ordered, it follows that if $E$ is trivial then condition (1) can be strengthened to the existence of a countable set puncturing $\mathscr{A}$.

The theory `AD`$^+$ is a strengthening due to Woodin of `AD` (for an introduction, see [CK11a] and the references therein). We say that $V$ is a *natural* *model of* $\mathtt{AD}^+$ if it is a model of $\mathtt{AD}^+$ and has the form $L(\mathcal{P}(\mathbb{R}))$ or $L(T,\mathbb{R})$ for some set $T$ of ordinals.

THEOREM 15 ($V$ is a natural model of $\mathtt{AD}^+$). *Suppose that $X$ is a set, $E$ is an equivalence relation on $X$, and $\mathscr{A} \subseteq [X/E]^{\leq\aleph_0}$. Then exactly one of the following holds:*

(1) *There is a well-orderable set puncturing $\mathscr{A}$.*
(2) *There is a pairwise disjoint subset of $\mathscr{A}$ of cardinality $\mathfrak{c}$.*

*Proof.* Given an aleph $\kappa$, we say that a subset of a topological space is *$\kappa$-Borel* if it belongs to the closure of the open sets under the operations of complementation and unions of length strictly less than $\kappa$. In the context of determinacy, it is desirable to work with a uniform version of this definition for Polish spaces, where a *code* (a set of ordinals) describing the construction of the set in terms of basic open sets is provided. There are different ways of formalizing this approach; see [CK11a] for details. In $\mathtt{ZF}$, the theory $\mathtt{DC}_{\mathbb{R}}+$"there is a fine $\sigma$-complete measure on $[\mathbb{R}]^{\leq\aleph_0}$" is a consequence of $\mathtt{AD}^+$. As shown in [CK11b], this theory implies that for all alephs $\kappa$ there is an aleph $\kappa^*$ such that the pointclass of $\kappa$-Borel sets is $\kappa^*$-chromatic on Polish spaces, and therefore mixed $\kappa^*$-chromatic on Polish spaces. As $V$ is a natural model of $\mathtt{AD}^+$, the desired result follows from Theorem 9 by a straightforward adaptation of the proof of [CK11a, Theorem 4.8]. ∎

The above arguments ensure that if $\mathscr{A}$ admits a small puncture set, then such a set is constructible from an appropriate witness to the definability of $\mathscr{A}$. In many cases, much finer results can be obtained. This can be achieved, for example, by placing strong assumptions on the family in question:

THEOREM 16. *Suppose that $X$ is a Hausdorff space, $E$ is a co-analytic equivalence relation on $X$, and $\mathscr{A} \subseteq [X/E]^{\leq\aleph_0}$ is analytic and has no pairwise incomparable subsets of cardinality $\mathfrak{c}$. Then $\bigcup \mathscr{A}$ is countable.*

THEOREM 17. *Suppose that $\kappa$ is an uncountable aleph, $X$ is a Hausdorff space, $E$ is a co-$({<}\kappa)$-Suslin, weakly $\aleph_0$-universally Baire equivalence relation on $X$, and $\mathscr{A} \subseteq [X/E]^{\leq\aleph_0}$ is $({<}\kappa)$-Suslin and has no pairwise incomparable subsets of cardinality $\mathfrak{c}$. Then $|\bigcup \mathscr{A}| < \kappa$.*

THEOREM 18 ($V$ is a natural model of $\mathtt{AD}^+$). *Suppose that $X$ is a set, $E$ is an equivalence relation on $X$, and $\mathscr{A} \subseteq [X/E]^{\leq\aleph_0}$ has no pairwise incomparable subsets of cardinality $\mathfrak{c}$. Then $\bigcup \mathscr{A}$ is well-orderable.*

Theorems 16, 17, 18 are proved in a manner nearly identical to their counterparts Theorems 12, 13, 15, with the exception that Theorem 10 must be used in place of Theorem 9.

Finer definability of puncture sets can also be achieved by employing a strengthening of $\kappa$-chromaticity in which the $(<\kappa)$-coloring is required to be $\Delta$-measurable, where $\Delta = \Gamma \cap \breve{\Gamma}$. This stronger notion, which we refer to as *measurable $\kappa$-chromaticity*, holds for all of the pointclasses we have thus far utilized, and can be used to ensure that the puncture sets we construct are in $\Delta$. In particular, by applying [Lec09, Theorem 1.6], we obtain the following (see [Mos09] for the details of the effective theory):

THEOREM 19. *Suppose that $X$ is a recursively presented Polish space, $E$ is a $\Pi^1_1$ equivalence relation on $X$, and $\mathscr{A} \subseteq [X/E]^{\leq \aleph_0}$ is $\Sigma^1_1$. Then exactly one of the following holds:*

1. *There is a countable $\Delta^1_1$ set puncturing $\mathscr{A}$.*
2. *There is a pairwise disjoint subset of $\mathscr{A}$ of cardinality $\mathfrak{c}$.*

Yet another approach is to use a parametrized version of $\kappa$-chromaticity (which itself follows from $\kappa$-chromaticity). We omit the details, but note one result obtainable in this fashion:

THEOREM 20 ($V$ is a natural model of $\mathtt{AD}^+$). *Suppose that $X$ and $Y$ are sets, $E$ is an equivalence relation on $X$, and $\mathscr{A} \subseteq [X/E]^{\leq \aleph_0} \times Y$. Then exactly one of the following holds:*

1. *There is a set $R \subseteq ((X/E) \times (X/E)) \times Y$ such that $R^y$ is a well-ordering of a set puncturing $\mathscr{A}^y$ for all $y \in Y$.*
2. *For some $y \in Y$ there is a pairwise disjoint subset of $\mathscr{A}^y$ of cardinality $\mathfrak{c}$.*

Combining the latter two approaches, one can use a parametrized form of measurable $\kappa$-chromaticity to obtain the following:

THEOREM 21. *Suppose that $X$ and $Y$ are Hausdorff spaces, $E$ is a co-analytic equivalence relation on $X$, and $\mathscr{A} \subseteq [X/E]^{\leq \aleph_0} \times Y$ is analytic. Then exactly one of the following holds:*

1. *There is a Borel set $B \subseteq X \times Y$ such that $B^y/E$ is a countable set puncturing $\mathscr{A}^y$ for all $y \in Y$.*
2. *For some $y \in Y$, there is a pairwise disjoint subset of $\mathscr{A}^y$ of cardinality $\mathfrak{c}$.*

THEOREM 22. *Suppose that $\kappa$ is an uncountable aleph, $X$ and $Y$ are Hausdorff spaces, $E$ is a co-$(<\kappa)$-Suslin, weakly $\aleph_0$-universally Baire equivalence relation on $X$, and $\mathscr{A} \subseteq [X/E]^{\leq \aleph_0} \times Y$ is $(<\kappa)$-Suslin. Then exactly one of the following holds:*

1. *There is a $\kappa$-Borel set $B \subseteq X \times Y$ such that $B^y/E$ is a set of cardinality strictly less than $\kappa$ puncturing $\mathscr{A}^y$ for all $y \in Y$.*

(2) *For some $y \in Y$, there is a pairwise disjoint subset of $\mathscr{A}^y$ of cardinality $\mathfrak{c}$.*

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