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Examples of minimal diffeomorphisms on $\mathbb{T}^2$ semiconjugate to an ergodic translation

by

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Abstract. We prove that for every $\epsilon > 0$ there exists a minimal diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ of class $C^{3-\epsilon}$ and semiconjugate to an ergodic translation with the following properties: zero entropy, sensitivity to initial conditions, and Li–Yorke chaos. These examples are obtained through the holonomy of the unstable foliation of Mañé's example of a derived-from-Anosov diffeomorphism on $\mathbb{T}^3$.

1. Introduction. The beautiful theory of Poincaré about orientation preserving homeomorphisms of the circle states, in particular, that any homeomorphism f with irrational rotation number $\rho(f)$ is in fact semiconjugate to the rigid rotation of angle $\rho(f)$. Moreover, it says that such an f is indeed conjugate to the relevant irrational rotation or has a minimal Cantor set such that every interval in its complement is wandering.

The celebrated theorem of Denjoy [D] asserts that in case f is a C^2 diffeomorphism, it is indeed conjugate to the rotation. Moreover Denjoy gave examples of maps of class $C^{1+\alpha}$ which are semiconjugate but not conjugate to rigid rotations, today called Denjoy type maps.

Thus, we can distinguish two facts about the theory of circle diffeomorphisms:

(i) A purely topological fact: a circle homeomorphism with irrational rotation number is semiconjugate to the respective rotation. Furthermore, whenever it is not conjugate to the rotation, there exists a minimal Cantor set such that the intervals in the complement are wandering.

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(ii) A rigidity phenomenon: a C^2 diffeomorphism with irrational rotation number cannot have a wandering open set. Hence, it is conjugate to the respective rotation.

During the last decades extensions of Poincaré and Denjoy theory to higher dimensional tori have been the object of interest of many authors, in particular the case of dimension two. This has led to the notion of rotation set for homeomorphisms on the two-torus isotopic to the identity, and to several results concerning the structure of this set and the dynamics of f.

Particular attention has been paid to the so called pseudo-translations of the two-torus, when the rotation set consists of a single totally irrational vector; the situation here is much more subtle than for circle diffeomorphisms.

First of all, pseudo-translations are no longer semiconjugate to the respective ergodic translation, as can be seen in [J]. Further, even assuming the existence of the semiconjugacy between a pseudo-translation f and the translation, it is not necessarily true that f has a wandering domain. It could even have complicated features, such as positive topological entropy (see [R1, R2, BCL1]). Moreover, examples of Denjoy maps (i.e., homeomorphisms which are semiconjugate but not conjugate to ergodic translations) with wandering domains can have very different dynamical structures: one can consider a product of two Denjoy maps of the circle, a product of a Denjoy map of the circle with a rigid rotation, and the suspension of a Denjoy map of the circle, in order to obtain three cases with different topological structures. Indeed, in the first case the wandering domain has a unique component which is doubly essential, in the second case the wandering domain is the orbit of a wandering essential annulus, and in the third case it is an unbounded disk (see [JKP]).

On the other hand, finding analogues of the rigidity phenomenon (ii) in higher dimensions is an open problem which seems to be far from being solved. Even so, there exists a result in KAM theory which suggests that C^{n+1} is the regularity which would imply rigidity for the n-dimensional torus. This is a positive result under the assumptions of a Diophantine rotation vector and being close enough to the respective translation.

Among all the different kinds of Denjoy examples known, there is only one in which diffeomorphisms of class at least C^2 are considered. Moreover these examples can be regarded as C^r diffeomorphisms for every $r < 3$, which corresponds to the expected differentiability class that could guarantee rigidity, i.e. C^3. This family of examples was introduced by McSwiggen [McS], and its elements have the following dynamical structure: the wandering domain is given by the orbit of a wandering bounded disk, and its complement is a minimal set. On the other hand, the existing examples of Denjoy maps without wandering domains are only C^0 regular (see [R1, BCL1]).

In this article we introduce a new family of Denjoy maps of the two-torus which are minimal diffeomorphisms of $\mathbb{T}^2$, of class C^r with r arbitrarily close to three. This improves the state of the art in the theory.

The elements of the family have the following simple properties: they are sensitive to the initial conditions, point-distal and non-distal, uniquely ergodic and have zero entropy. Furthermore, the examples can be made to exhibit Li–Yorke chaos. To the best of our knowledge, no minimal diffeomorphism with Li–Yorke chaos has been known before.

Our construction implies in particular that we obtain a Diophantine rotation vector. Indeed it is an algebraic vector of degree three. This property makes the examples even more interesting since Diophantine vectors are associated to a stronger rigidity than Liouvillean vectors. In fact, the results in KAM theory make use of the combination of differentiability and Diophantine rotation vector, in order to obtain rigidity. One may ask if another method and starting with a Liouvillean translation can provide an example with the same features and with higher differentiability.

Before we state our result, let us recall some of the above topological notions. Let f be a homeomorphism (of a metric space).

- f is *non-distal* if there exist $x \neq y$ such that
$$\inf_{n\in\mathbb{Z}} \operatorname{dist}(f^n(x), f^n(y)) = 0,$$
and f is *point distal* if there exists x such that for any $y \neq x$,
$$\inf_{n\in\mathbb{Z}} \operatorname{dist}(f^n(x), f^n(y)) > 0;$$
- f is *sensitive to initial conditions* if there exists $\epsilon > 0$ so that for any $x \in \mathbb{T}^2$ and any neighborhood $U(x)$ there exist $y \in U$ and $n > 0$ such that $\operatorname{dist}(f^n(x), f^n(y)) > \epsilon$;
- f has *Li–Yorke chaos* if there exists an uncountable *scrambled set* any points $x \neq y$ of which satisfy
$$\liminf_n \operatorname{dist}(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \limsup_n \operatorname{dist}(f^n(x), f^n(y)) > 0.$$

We are now ready to state the main result:

Main Theorem 1.1. *For every $r \in [1,3)$ there exists a diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ of class C^r which is minimal, isotopic and semiconjugate* (*but not conjugate*) *to an ergodic translation. If we denote by h the semiconjugacy, then:*

- *Each fiber $h^{-1}(x)$ is either a point or an arc.*
- *f preserves a minimal and invariant foliation with one-dimensional C^1 leaves. Each fiber $h^{-1}(x)$ is contained in a leaf of this foliation.*
- *The set $\{x \in \mathbb{T}^2 : h^{-1}(x)$ is a point$\}$ has full Lebesgue measure.*

As a consequence, f has zero entropy, is sensitive to initial conditions, point distal, non-distal and uniquely ergodic. Furthermore, f can be constructed so that there are uncountably many points x such that $h^{-1}(x)$ is a nontrivial arc exhibiting Li–York chaos as well.

Although examples with much more dynamical complexity can be constructed by the techniques given in [R1, BCL1], as we remarked before these constructions are C^0 and it is not known if C^1 can be achieved.

On the other hand, while unfortunately our examples are simple from the ergodic point of view: they have just one invariant measure, i.e., are *uniquely ergodic*, Denjoy type maps with interesting measurable dynamics are constructed by a refinement of these purely topological techniques in [BCL2]. Nevertheless we ask: does there exist a minimal diffeomorphism semiconjugate to an ergodic translation but not uniquely ergodic?

The reader may notice the opposition between zero entropy and the presence of Li–Yorke chaos in the constructed examples. In this sense one can ask whether such a dynamic deserves to be called chaotic or not (see [O] for a discussion of the subject).

The proof of our theorem is inspired by [McS]. There, examples are constructed through the holonomy map from a cross section to itself of the unstable foliation of a derived-from-Anosov diffeomorphism obtained through a Hopf bifurcation. Indeed, the construction is as follows. Start with a linear Anosov map on $\mathbb{T}^3$ having one real eigenvalue λ_u of modulus greater than 1 and a complex eigenvalue λ_s of modulus smaller than 1 and then perform a modification around a fixed point (that can be thought of as going through a Hopf bifurcation) so that the fixed point becomes a repeller. The new map still has a partially hyperbolic structure of the form $E^{cs} \oplus E^u$. Next consider the holonomy map from a two-torus transverse to E^u along the (strong) unstable foliation (tangent to E^u). The wandering domain appears due to the fact that the fixed point is a repeller and so the strong unstable foliation cannot return near the fixed point.

In this paper we use instead Mañé's example of a derived-from-Anosov diffeomorphism ([M1]). The construction is as follows. Start with a linear Anosov map on $\mathbb{T}^3$ with three real distinct eigenvalues $\lambda_s, \lambda_c, \lambda_u, 0 < \lambda_s < \lambda_c < 1 < \lambda_u$. Then perform a modification around a fixed point so that the fixed point becomes a hyperbolic point of unstable index 2 (which can be thought of as going through a pitch-fork bifurcation). The new map has a partially hyperbolic structure of the form $E^s \oplus E^c \oplus E^u$. The interesting feature here is that the strong unstable foliation (tangent to E^u) is minimal, i.e., every leaf is dense. Thus, considering the holonomy map from a two-torus transverse to E^u to itself along the unstable foliation leads to a minimal homeomorphism on the two-torus with the topological features of our result. In order to obtain Li–Yorke chaos we still have to make a delicate perturbation of the partial hyperbolic diffeomorphism on $\mathbb{T}^3$.

However, there is a main difference with [McS] which has to do with the class of differentiability of the unstable foliation. To study the differentiability class of the unstable foliation one uses the C^r section theorem (see Section 4 for details). And the first thing to do is to check the conditions on the initial linear Anosov diffeomorphism (hoping that the modification performed does not change these conditions too much). These conditions for the linear Anosov map of McSweegen's example can be written as

$$\frac{|\lambda_s|}{\lambda_u}\left(\frac{1}{|\lambda_s|}\right)^r < 1$$

and so with r arbitrarily close to 3 the conditions hold.

In our case, these conditions can be written as

$$\frac{\lambda_c}{\lambda_u}\left(\frac{1}{\lambda_s}\right)^r < 1$$

and so r cannot be chosen arbitrarily close to 3. Thus, what we have to do is, for a given r close to 3, find afterwards a linear Anosov map to start with, so that the above condition holds.

The paper is organized as follows: in Section 2 we give our construction of Mañé's derived-from-Anosov diffeomorphism and we prove the minimality of the unstable foliation (see Section 2.2), and the minimality of the central foliation through the semiconjugacy with the linear Anosov map (see Section 2.3); in Section 3 we give the topological version of our main result and in Section 4 we prove the differentiability of the unstable foliation through the C^r section theorem ([HPS]).

2. On Mañé's derived-from-Anosov diffeomorphism. In [M1] R. Mañé constructs an example on $\mathbb{T}^3$ which is robustly transitive but not Anosov. This is known as *Mañé's derived-from-Anosov diffeomorphism* due to its construction: it begins with an Anosov linear map on $\mathbb{T}^3$ with partially hyperbolic structure $E^s \oplus E^c \oplus E^u$ and modifies it in a neighborhood of the fixed point in order to change its unstable index (preserving the partially hyperbolic structure). See Figure 1.

Let us be more precise. Let $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ be the three-dimensional torus, denote by $\pi : \mathbb{R}^3 \to \mathbb{T}^3$ the canonical projection, and set $p = \pi(0)$.

Consider $B \in \mathrm{SL}(3,\mathbb{Z})$ with eigenvalues $0 < \lambda_s < \lambda_c < 1 < \lambda_u$ and denote also by B the induced linear Anosov system on $\mathbb{T}^3$ with hyperbolic structure $T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u$ (corresponding to the eigenspaces associated to λ_s, λ_c and λ_u). For simplicity of calculations we will define a Euclidean metric on $\mathbb{R}^3$ so that E^s_B, E^c_B and E^u_B are mutually orthogonal.

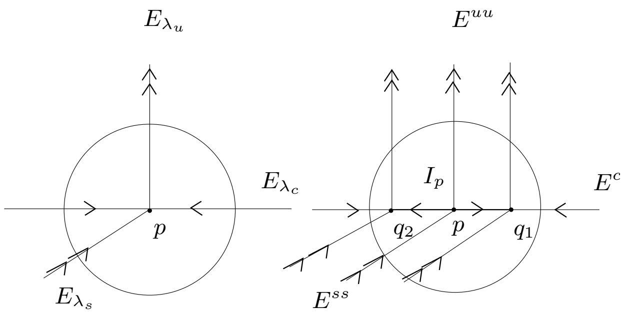

Fig. 1. Modification

Let ρ be small and consider the ball $B(p,\rho)$ centered at p. Let $Z:\mathbb{R}\to\mathbb{R}$ be a C^∞ bump function such that $Z(0)=1$, $\operatorname{supp}[Z]\subset(-\rho/2,\rho/2)$ (where $\operatorname{supp}[Z]$ is the support of Z) and $|Z'(z)|<4/\rho$ (see Figure 2).

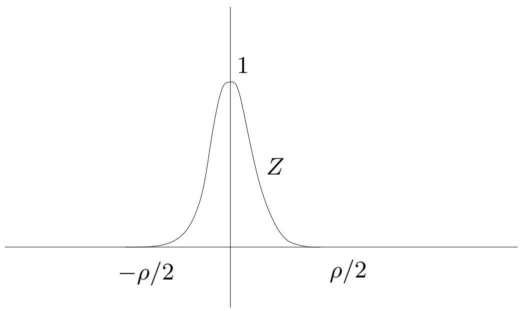

Fig. 2. The bump function Z

For our construction of Mañé's derived-from-Anosov diffeomorphism [1] we need an auxiliary function as in the next lemma.

LEMMA 2.1. *For all $k>0$ arbitrarily small there exists a function $\beta_k:[0,\infty)\to\mathbb{R}$ such that:*

(1) *β_k is C^∞, decreasing and such that $-k\le\beta_k'(t)t\le 0$.*
(2) *β_k is supported in $[0,k]$, i.e. $\operatorname{supp}[\beta_k]\subset[0,k]$.*
(3) *$\lambda_s+\beta_k(0)<1<\lambda_c+\beta_k(0)<1+k$.*

Proof. We may assume that $0<k<\lambda_c-\lambda_s$ and take b such that $1-\lambda_c<b<1-\lambda_c+k$. Let $r_0<k$. Since $\int_0^{r_0}(k/t)\,dt$ is divergent we may find a C^∞ non-negative function ψ with support in $[0,r_0]$ such that $\int_0^{r_0}\psi(t)\,dt=b$ and $\psi(t)\le k/t$ (in other words, the graph of ψ is below the graph of $h(t)=k/t$.)

[1] Our construction is slightly different because we need to keep control of the relation between E^s and E^c to obtain higher differentiability of the unstable foliation. In particular the central foliation is not kept unchanged.

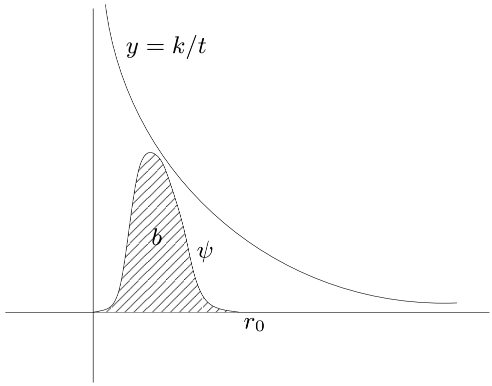

Fig. 3. The function ψ

Define

$$\beta_k(t) = b - \int_0^t \psi(s)\, ds.$$

This function is as desired. ∎

Finally, define $g_{B,k} : \mathbb{T}^3 \to \mathbb{T}^3$ by

$$g_{B,k}(\xi) = B(\xi) \quad \text{for } \xi \notin B(p,\rho), \tag{2.1}$$

and for $\xi \in B(p,\rho)$ in local coordinates with respect to $E^s_B \oplus E^c_B \oplus E^u_B$, $\xi = (x,y,z)$,

$$g_{B,k}(\xi) = (\lambda_s x, \lambda_c y, \lambda_u z) + Z(z)\beta_k(r)(x,y,0) \tag{2.2}$$

where $r = x^2 + y^2$.

Proposition 2.2. *If k is sufficiently small, then $g_{B,k} : \mathbb{T}^3 \to \mathbb{T}^3$ defined above is a diffeomorphism with partially hyperbolic structure* (²) *$T\mathbb{T}^3 = E^s_{g_{B,k}} \oplus E^c_{g_{B,k}} \oplus E^u_{g_{B,k}}$ where $E^s_{g_{B,k}}$ is uniformly contracting and $E^u_{g_{B,k}}$ is uniformly expanding. Moreover, given cones C^s, C^c and C^u around E^s_B, E^c_B and E^u_B respectively we have $E^s_{g_{B,k}} \in C^s$, $E^c_{g_{B,k}} \in C^c$ and $E^u_{g_{B,k}} \in C^u$. Furthermore, the same is true for any g in any sufficiently small C^1 neighborhood $\mathcal{U}$ of $g_{B,k}$.*

Proof. First of all, the C^0 distance between $g_{B,k}$ and B is smaller than $\sqrt{k}$ and hence (assuming k small) we conclude that $g_{B,k}$ is a differentiable homeomorphism. To ease notation, set $g = g_{B,k}$ for the time being.

For $\xi \notin B(p,\rho)$ we have $dg_\xi = B$. For $\xi \in B(p,\rho)$ we have (with respect to the decomposition $E^s \oplus E^c \oplus E^u$)

(²) We remark that it is not absolutely partially hyperbolic.

(2.3) $dg_\xi =$

$$\begin{pmatrix} \lambda_s + Z(z)(\beta(r)+\beta'(r)2x^2) & Z(z)\beta'(r)2xy & Z'(z)\beta(r)x \\ Z(z)\beta'(r)2xy & \lambda_c + Z(z)(\beta(r)+\beta'(r)2y^2) & Z'(z)\beta(r)y \\ 0 & 0 & \lambda_u \end{pmatrix}.$$

We may write $dg_\xi = A_\xi + M_\xi$ where (agreeing that Z and β are identically zero outside $B(p,\rho)$)

$$A_\xi = \begin{pmatrix} \lambda_s + Z(z)\beta(r) & 0 & 0 \\ 0 & \lambda_c + Z(z)\beta(r) & 0 \\ 0 & 0 & \lambda_u \end{pmatrix}, \tag{2.4}$$

$$M_\xi = \begin{pmatrix} Z(z)\beta'(r)2x^2 & Z(z)\beta'(r)2xy & Z'(z)\beta(r)x \\ Z(z)\beta'(r)2xy & Z(z)\beta'(r)2y^2 & Z'(z)\beta(r)y \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.5}$$

Since $|\beta'(r)r| \le k$ it is straightforward to check the inequality $\|M_\xi\| \le \max\{2k, 8\beta(0)\sqrt{k}/\rho\}$. Therefore, choosing k arbitrarily small we also get $\|M_\xi\|$ arbitrarily small. Since the co-norm ($= \|A_\xi^{-1}\|^{-1}$) of A_ξ is bounded away from zero we see that dg_ξ is an isomorphism and hence g is a diffeomorphism. On the other hand, $A_\xi(E_B^j) = E_B^j$, $j = s, c, u$, and

- $\lambda_s \le \|A_{\xi/E_B^s}\| \le \lambda_s + \beta_k(0) < 1$,
- $\lambda_c \le \|A_{\xi/E_B^c}\| \le \lambda_c + \beta_k(0) < 1 + k$,
- $\|A_{\xi/E_B^s}\|/\|A_{\xi/E_B^c}\| \le \lambda_s/\lambda_c < 1$,
- $\|A^{-1}_{\xi/E_B^u}\| \le \lambda_u^{-1}$.

From this it is easy to conclude the proof of the proposition, taking k sufficiently small (and so $\|M_\xi\|$ sufficiently small) and $\mathcal{U}$ sufficiently small. ∎

For $g_{B,k} : \mathbb{T}^3 \to \mathbb{T}^3$ with k small and $g \in \mathcal{U}(g_{B,k})$ so that the above proposition applies, we set

$$\lambda_s(g)(\xi) = \|dg_{\xi/E_g^s}\|, \qquad \lambda_s(g) = \max_{\xi\in\mathbb{T}^3} \lambda_s(g)(\xi),$$

$$\lambda_c(g)(\xi) = \|dg_{\xi/E_g^c}\|, \qquad \lambda_c(g) = \max_{\xi\in\mathbb{T}^3} \lambda_c(g)(\xi),$$

$$\lambda_u(g)(\xi) = \|dg_{\xi/E_g^u}\|, \qquad \lambda_u(g) = \min_{\xi\in\mathbb{T}^3} \lambda_u(g)(\xi).$$

Remark 2.3. Notice that, given $\epsilon > 0$ small, the following conditions hold for $g \in \mathcal{U}(g_{B,k})$ with k and $\mathcal{U}$ sufficiently small:

(1) $0 < \lambda_s(g)(\xi) < \lambda_c(g)(\xi) < \lambda_u(g)(\xi)$ for all $\xi \in \mathbb{T}^3$.
(2) $\lambda_c - \epsilon < \lambda_c(g)(\xi)$ for all $\xi \in \mathbb{T}^3$.
(3) $\lambda_s(g) < \lambda_s + \beta(0) + \epsilon < 1$.

(4) $\lambda_s(g)(\xi) < \lambda_s + \epsilon$ for $\xi \in \mathbb{T}^3 - B(p,\rho)$.
(5) $\lambda_c(g) < \lambda_c + \beta(0) + \epsilon$.
(6) $\lambda_u(g) > \lambda_u - \epsilon > 1$ and $\lambda_u(g) > \lambda_c + \beta(0) + \epsilon$.

Once we know that $g \in \mathcal{U}(g_{b,k})$ is partially hyperbolic, by well known results (see [HPS]) we deduce that the bundles E^s_g and E^u_g uniquely integrate to foliations $\mathcal{F}^s_g$ and $\mathcal{F}^u_g$ called the (strong) *stable* and *unstable foliations* respectively. These foliations have an interesting property of being *quasi-isometric* (i.e. the distance in $\mathbb{R}^3$ of two points in the same leaf lifted to the universal cover $\mathbb{R}^3$ is comparable with the distance measured along the leaf, see the beginning of Section 2.1). This property is obtained in [BBI].

In our case, since E^s_g and E^u_g are contained in tiny cones around E^s_B and E^u_B we can conclude directly that $\mathcal{F}^s_g$ and $\mathcal{F}^u_g$ are quasi-isometric (see [B]). However, since g is not absolutely partially hyperbolic, the result in [B] (see also [BBI]) does not apply to prove that E^c_g is uniquely integrable. Recently, R. Potrie [Po] has extended the results in [BBI] to the non-absolutely partially hyperbolic setting, and we can conclude that E^c_g is uniquely integrable. However, for our particular case we can give a direct proof of the unique integrability of E^c_g in the spirit of [B] (see Section 2.1). We denote by $\mathcal{F}^c_g$ this central foliation; consequently, the bundles $E^s_g \oplus E^c_g$ and $E^c_g \oplus E^u_g$ are uniquely integrable and lead to the central stable and central unstable foliations. We also remark that in the particular case $g = g_{B,k}$ we have $E^s_g \oplus E^c_g = E^s_B \oplus E^c_B$ and so the central stable foliation of $g_{B,k}$ coincides with the two-dimensional stable foliation of B.

Also, in the following subsections we are going to study the properties of invariant foliations and consequences of the semiconjugacy with the linear Anosov map. These results are fundamental for our purposes.

THEOREM 2.4. *For all k sufficiently small and $\mathcal{U}(g_{B,k})$ sufficiently small, the central bundle E^c_g uniquely integrates to an invariant foliation $\mathcal{F}^c_g$. Furthermore, the central and unstable foliations $\mathcal{F}^c_g, \mathcal{F}^u_g$ of $g \in \mathcal{U}(g_{B,k})$ are minimal, i.e., all leaves are dense.*

The minimality of $\mathcal{F}^u_g$ can be obtained from [PS], and the minimality of $\mathcal{F}^c_g$ will follow from the semiconjugacy with the linear Anosov map. We are going to give a complete proof of the theorem in Sections 2.1, 2.2 (see Theorem 2.7) and 2.3 (see Corollary 2.13).

REMARK 2.5. If one is not interested in obtaining Li–York chaos, the map $g_{B,k}$ with an appropriate B and k is enough to prove our main theorem. In order to get Li–Yorke chaos we need to have a C^∞ diffeomorphism g arbitrarily close to $g_{B,k}$ with certain properties we state in the next corollary. The reader not interested in the Li–York chaos property may skip any reference to the neighborhood $\mathcal{U}(g_{B,k})$ and just stick to the map $g_{B,k}$.

Since every $g \in \mathcal{U}(g_{B,k})$ is transitive (this follows from the minimality of $\mathcal{F}^u_g$) and has periodic points of different indices, it follows that the set of diffeomorphisms having a non-hyperbolic periodic point is dense in $\mathcal{U}(g_{B,k})$ (see [M2], [A] and [H2]). We have the following

Corollary 2.6. *Let k and $\mathcal{U}(g_{B,k})$ be as in the above theorem. Then there exists $g \in \mathcal{U}(g_{B,k})$ of class C^∞ such that*

(1) *g has a transverse homoclinic point associated to a periodic point of unstable index* 2.
(2) *There exists a non-trivial arc J such that, for some $m > 0$, $g^m_{/J} = \mathrm{id}_{/J}$, that is, J consists of periodic points of g of the same period m.*

Proof. Note that for $g_{B,k}$ the fixed point $p = \pi(0)$ has unstable index 2 since $dg_{B,k/E^c} = \lambda_c + \beta(0) > 1$. On the other hand, since $\mathcal{F}^u_{g_{B,k}}(p)$ is dense (and hence accumulates on $\mathcal{F}^s_{g_{B,k}}(p)$) by Hayashi's connecting lemma (see [H1]) we can perturb $g_{B,k}$ (with support disjoint from a ball at p) and find g_1 satisfying condition (1). Furthermore, any diffeomorphism C^1 close to g_1 will also satisfy (1). Now, we can find a diffeomorphism arbitrarily C^1 close to g_1 having a non-hyperbolic periodic point q. This diffeomorphism can be constructed of class C^∞. Now, by another C^1 arbitrarily small perturbation (but of class C^∞ since it can be done with an appropriate bump function) we can transform this non-hyperbolic periodic point q into an arc J of periodic points and find g as in the statement. ∎

2.1. Unique integrability of the bundle E^c_g. We first recall that a foliation $\mathcal{F}$ in $\mathbb{R}^3$ is *quasi-isometric* if there exist positive numbers C, D such that if x, y belong to the same leaf of the foliation, i.e. $y \in \mathcal{F}(x) = \mathcal{F}(y)$, then

$$d(x,y) \geq C d_{\mathcal{F}}(x,y) - D$$

where $d_{\mathcal{F}}$ means the distance along the leaf of the foliation.

Denote by $\tilde{\mathcal{F}}^j_G$, $j = s, u$, the lifts to the universal cover $\mathbb{R}^3$ of the stable and unstable foliations $\mathcal{F}^j_g$, $j = s, u$, for $g \in \mathcal{U}(g_{B,k})$. These foliations are quasi-isometric, as we remarked before. In particular, this means that if we have two points x, y in the same unstable leaf, then by future iteration, the rate of growth of $d(G^n(x), G^n(y))$ is the same as $d_{\mathcal{F}^u_G}(G^n(x), G^n(y))$. And similarly in the past for points in the stable leaf.

Now, assume for contradiction that the central bundle E^c_g is not (locally) uniquely integrable (at some point, say x). This implies (see [B]) that there exist two points z, w such that (see Figure 4)

- z, w can be joined by a curve J^c always tangent to E^c_g.
- z, w can be joined by the union of two curves J^s, J^u always tangent to E^s_g and E^u_g respectively (of course, one of them could be trivial).

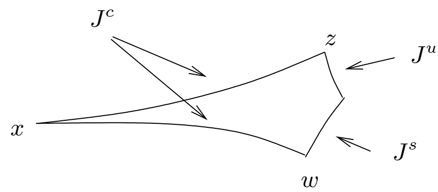

Fig. 4

The same holds in the universal cover and we will argue there. If the curve J^u is not trivial, then by future iteration we find that $d(G^n(z), G^n(w))$ grows at most with rate $\lambda_c(g)$ and on the other hand, by the quasi-isometric property of the unstable foliation, the rate of growth of $d(G^n(z), G^n(w))$ is the rate of growth of $G^n(J^u)$ (since the length of $G^n(J^s)$ decreases exponentially), which is at least $\lambda_u(g)$, which is greater than $\lambda_c(g)$ (see Remark 2.3), a contradiction.

If J^u is trivial, the argument is the same but more subtle and we need better estimates. Let ϵ and δ be small enough such that

$$\sigma := (\lambda_s + \epsilon)^{-1}(1 - \delta) > (\lambda_c - \epsilon)^{-1}$$

(recall that λ_s, λ_c are eigenvalues of B). Now, choose k, ρ and $\mathcal{U}(g_{B,k})$ small so that Remark 2.3 applies and such that for any curve tangent to E^s_g of length at least 1 the portion of it outside $B(p, \rho)$ is larger than $1 - \delta$.

Now we are ready to return to the points z, w. Since they are joined by a curve tangent to E^c_g, we have

$$d(G^{-n}(z), G^{-n}(w)) \leq (\lambda_c - \epsilon)^n \ell(J^c).$$

On the other hand, let n_0 be such that $G^{-n_0}(J^s)$ has length greater than 1. Then

$$\ell(G^{-n}(G^{-n_0}(J^s))) \geq (\lambda_s + \epsilon)^{-n}(1 - \delta)^n = \sigma^n.$$

For C and D the constant of the quasi-isometry of the stable foliation, for n large enough we have

$$C\sigma^n - D > (\lambda_c - \epsilon)^{n+n_0} \ell(J^c)$$

and so we get a contradiction:

$$\begin{aligned} d(G^{-n-n_0}(z), G^{-n-n_0}(w)) &\leq (\lambda_c - \epsilon)^{n+n_0} \ell(J^c) \\ &< C\sigma^n - D \leq C\ell(G^{-n}(G^{-n_0}(J^s))) - D \\ &\leq d(G^{-n-n_0}(z), G^{-n-n_0}(w)). \end{aligned}$$

Thus, we have finished the proof of the unique integrability of E^c, i.e., the first part of Theorem 2.4.

2.2. Minimality of the unstable foliation. In this subsection we will prove that $\mathcal{F}^u_g$ is minimal for $g \in \mathcal{U}(g_{B,k})$ for k and $\mathcal{U}$ small enough. The proof is based on the ideas and methods of [PS]:

THEOREM 2.7. *For all k sufficiently small and $\mathcal{U}(g_{B,k})$ sufficiently small, the unstable foliation $\mathcal{F}_g^u$ of $g \in \mathcal{U}(g_{B,k})$ is minimal, i.e., all leaves are dense.*

Proof. Recall that $0 < \lambda_s < \lambda_c < 1 < \lambda_u$ are the eigenvalues of B. Choose σ with $1 - (\lambda_c - \lambda_s) < \sigma < 1$. We may assume that ρ (the radius of the ball centered at p where the modification of B is performed) is small so that any arc I^s in $\mathcal{F}_B^s$ of length 1 has a subarc I_1^s of length at least $1/3$ with empty intersection with $B(p, 2\rho)$.

Let n_0 be such that

$$\sigma^{-n_0} > 3. \tag{2.6}$$

Let ϵ with $0 < \epsilon < \rho$ be such that $1 - (\lambda_c - \lambda_s) + \epsilon < \sigma$ and

$$\lambda := \lambda_c(1 + \epsilon)^{n_0} < 1. \tag{2.7}$$

Let us denote by $D_g^{cs}(x, \epsilon)$ the disk centered at x and of radius ϵ in the central stable leaf through x, $\mathcal{F}_g^{cs}(x)$.

Now, we may assume that k and $\mathcal{U}$ are so small that the following holds for all $g \in \mathcal{U}(g_{B,k})$:

(i) $\lambda_s(g) < \sigma$.
(ii) $\lambda_c(g) < 1 + \epsilon$.
(iii) $\|dg_{/E_g^{cs}(\xi)}\| \leq \lambda_c(1 + \epsilon)$ if $\xi \notin B(p, \rho)$.
(iv) Any arc I^s of $\mathcal{F}_g^s$ of length at least 1 has a subarc I_1^s of length at least $1/3$ with empty intersection with $B(p, 2\rho)$.
(v) Any leaf of $\mathcal{F}_g^u$ has non-empty intersection with $D_g^{cs}(x, \epsilon)$ for any x (since $\mathcal{F}_B^u$ is minimal and for k and $\mathcal{U}$ small the bundles E_B^u and E_g^u are close).

Given $x \in \mathbb{T}^3$ let $I^s(x)$ be an arc of length 1 such that $x \in I^s(x) \subset \mathcal{F}_g^s(x)$. We know that there exists a subarc I_1^s of length at least $1/3$ such that $I_1^s \cap B(p, 2\rho) = \emptyset$. Now, by (2.6), we conclude that $g^{-n_0}(I_1^s) \subset \mathcal{F}_g^s(g^{-n_0}(x))$ is an arc of length at least 1. Therefore, there exists a subarc $I_2^s \subset g^{-n_0}(I_1^s)$ of length at least $1/3$ such that $I_2^s \cap B(p, 2\rho) = \emptyset$. Arguing by induction, we conclude that for each $j \geq 1$ there exists $I_{j+1}^s \subset g^{-n_0}(I_j^s)$ such that $I_{j+1}^s \cap B(p, 2\rho) = \emptyset$.

Define

$$z_x = \bigcap_{j \geq 1} g^{jn_0}(I_{j+1}).$$

Notice that

$$z_x \in I^s(x) \quad \text{and} \quad g^{-jn_0}(z_x) \notin B(p, 2\rho) \quad \forall j \geq 0. \tag{2.8}$$

In other words, in any arc of length 1 on any leaf of $\mathcal{F}_g^s$ there exists a point whose g^{n_0}-backward orbit never meets $B(p, 2\rho)$. Let $z = z_x$ be such a point and let $j \geq 1$. Then

$$D_g^{cs}(g^{-jn_0}(z), \epsilon) \cap B(p, \rho) = \emptyset$$

and so, for any $y \in D_g^{cs}(g^{-jn_0}(z), \epsilon)$ we have, by (2.7), (ii) and (iii),

$$\|dg_y^{n_0}\| \leq \lambda_c(1+\epsilon)^{n_0} = \lambda < 1$$

and therefore

$$g^{n_0}(D_g^{cs}(g^{-jn_0}(z), \epsilon)) \subset D_g^{cs}(g^{-(j-1)n_0}(z), \lambda\epsilon) \tag{2.9}$$

and so for any $1 \leq m \leq j$ we have

$$g^{mn_0}(D_g^{cs}(g^{-jn_0}(z), \epsilon)) \subset D_g^{cs}(g^{-(j-m)n_0}(z), \lambda^m\epsilon).$$

Now, we are ready to conclude the proof of the minimality of $\mathcal{F}_g^u$ (for the argument see Figure 5). Let $\xi \in \mathbb{T}^3$ and let U be some open set in $\mathbb{T}^3$. We want to prove that

$$\mathcal{F}_g^u(\xi) \cap U \neq \emptyset.$$

Let $y \in U$, and consider an arc $J_y \subset F_g^s(y)$, $J_y \subset U$. There exists m_0 so that $g^{-m_0}(J_y)$ has length greater than 1. Let $z \in g^{-m_0}(J_y)$ be the point constructed above, and let μ be such that

$$g^{m_0}(D_g^{cs}(z, \mu)) \subset U. \tag{2.10}$$

Let m_1 be such that $\lambda^{m_1}\epsilon < \mu$. From (2.9) we conclude that

$$g^{m_1n_0}(D_g^{cs}(g^{-m_1n_0}(z), \epsilon)) \subset D_g^{cs}(z, \mu).$$

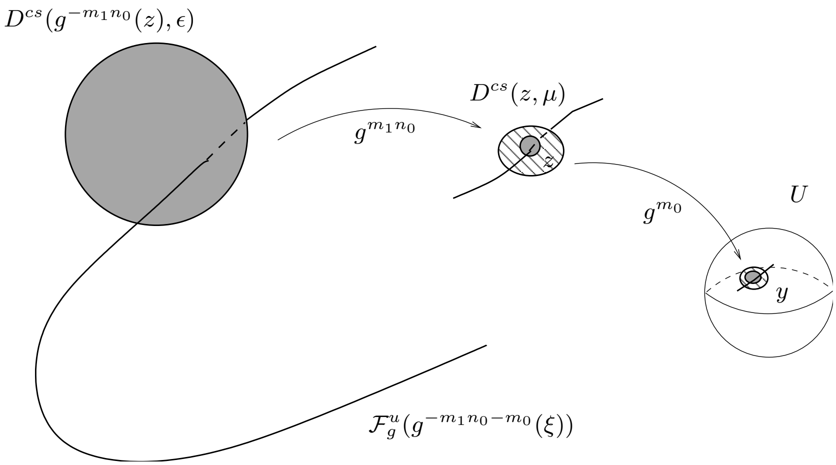

Fig. 5

On the other hand, from (v) we know that

$$\mathcal{F}_g^u(g^{-m_1n_0-m_0}(\xi)) \cap D_g^{cs}(g^{-m_1n_0}(z), \epsilon) \neq \emptyset.$$

Using (2.10), iterating $m_1 n_0 + m_0$ times we conclude that

$$\mathcal{F}_g^u(\xi) \cap U \neq \emptyset$$

as desired. This completes the proof of the minimality of $\mathcal{F}_g^u$ for $g \in \mathcal{U}(g_{B,k})$ with k and $\mathcal{U}$ small enough. ∎

2.3. Semiconjugacy to the linear Anosov system. In this subsection we establish a well known result about the semiconjugacy of any map isotopic to an Anosov map on the torus (see for instance [S]) and also we derive some consequence of it. Indeed, we establish it in the universal cover $\mathbb{R}^3$.

THEOREM 2.8. *Let $B : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear hyperbolic isomorphism. Then there exists $C > 0$ such that if $G : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism such that* $\sup\{\|G(x) - Bx\| : x \in \mathbb{R}^3\} = K < \infty$ *then there exists $H : \mathbb{R}^3 \to \mathbb{R}^3$ continuous and onto such that:*

(1) *$B \circ H = H \circ G$.*
(2) *$\|H(x) - x\| \le CK$ for all $x \in \mathbb{R}^3$.*
(3) *$H(x)$ is characterized as the unique point y such that*

$$\|B^n(y) - G^n(x)\| \le CK \qquad \forall n \in \mathbb{Z}.$$

(4) *$H(x) = H(y)$ if and only if $\|G^n(x) - G^n(y)\| \le 2CK$ for all $n \in \mathbb{Z}$ and if and only if $\sup_{n\in\mathbb{Z}} \|G^n(x) - G^n(y)\| < \infty$.*
(5) *If $B \in \mathrm{SL}(3,\mathbb{Z})$ and G is the lift of $g : \mathbb{T}^3 \to \mathbb{T}^3$ then H induces $h : \mathbb{T}^3 \to \mathbb{T}^3$ continuous and onto such that $B \circ h = h \circ g$ and* $\mathrm{dist}_{C^0}(h, \mathrm{id}) \le C\, \mathrm{dist}_{C^0}(B, g)$.

We will prove some consequence of the above theorem for our $B \in \mathrm{SL}(3,\mathbb{Z})$ and our construction of Mañé's derived-from-Anosov diffeomorphism $g_{B,k} : \mathbb{T}^3 \to \mathbb{T}^3$ and any $g \in \mathcal{U}(g_{B,k})$. Let $G : \mathbb{R}^3 \to \mathbb{R}^3$ be the lift of g such that $\sup\{\|G(x) - Bx\| : x \in \mathbb{R}^3\} = \mathrm{dist}_{C^0}(B, g)$ (which we may assume to be less than $\sqrt{k}$). Denote by $\tilde{\mathcal{F}}^j$, $j = s, c, u, cs, cu$, the lift of the stable, central, unstable, central stable and central unstable foliation respectively.

THEOREM 2.9. *With the above notations we have:*

(1) *$H(\tilde{\mathcal{F}}_G^{cu}(x)) = \tilde{\mathcal{F}}_B^{cu}(H(x))$ and $H(\tilde{\mathcal{F}}_G^{cs}(x)) = \tilde{\mathcal{F}}_B^{cs}(H(x))$*
(2) *$H(\tilde{\mathcal{F}}_G^{c}(x)) = \tilde{\mathcal{F}}_B^{c}(H(x))$.*
(3) *$H(\tilde{\mathcal{F}}_G^{u}(x)) = \tilde{\mathcal{F}}_B^{u}(H(x)) = H(x) + E_B^u$ and $H : \tilde{\mathcal{F}}_G^{u}(x) \to \tilde{\mathcal{F}}_B^{u}(H(x))$ is a homeomorphism.*
(4) *For any $x, y \in \mathbb{R}^3$,*

$$\#\{\tilde{\mathcal{F}}_G^{cs}(x) \cap \tilde{\mathcal{F}}_G^{u}(y)\} = 1 \quad \text{and} \quad \#\{\tilde{\mathcal{F}}_G^{cu}(x) \cap \tilde{\mathcal{F}}_G^{s}(y)\} = 1.$$

Proof. For the first item we only prove the first equality; the other one is similar. Let us prove first that $H(\tilde{\mathcal{F}}_G^{cu}(x)) \subset \mathcal{F}_B^{cu}(H(x)) = H(x) + E_B^{cu}$.

For contradiction, assume that there exists $y \in \tilde{\mathcal{F}}^{cu}_G(x)$ such that $H(y) \notin \tilde{\mathcal{F}}^{cu}_B(H(x))$ and let $z = \tilde{\mathcal{F}}^s_B(H(y)) \cap \tilde{\mathcal{F}}^{cu}_B(H(x))$. By backward iteration we have

$$\begin{aligned}
&\|B^{-n}(H(y)) - B^{-n}(H(x))\| \\
&\qquad \geq \|B^{-n}(H(y)) - B^{-n}(H(z))\| - \|B^{-n}(H(z)) - B^{-n}(H(x))\| \\
&\qquad \geq \lambda_s^{-n}\|H(y) - H(z)\| - \lambda_c^{-n}\|H(z) - H(x)\|.
\end{aligned}$$

On the other hand, since $y \in \tilde{\mathcal{F}}^{cu}_G(x)$ and (for k and $\mathcal{U}$ small) $\|dG^{-1}_{/E^{cu}_G}\| \leq (\lambda_c - \epsilon)^{-1}$ we have

$$\begin{aligned}
\|B^{-n}(H(x)) - B^{-n}(H(y))\| &\leq \|B^{-n}(H(x)) - G^{-n}(x)\| + \|G^{-n}(x) - G^{-n}(y)\| \\
&\quad + \|B^{-n}(H(y)) - G^{-n}(y)\| \\
&\leq 2C\sqrt{k} + (\lambda_c - \epsilon)^{-n}\operatorname{dist}_{\tilde{\mathcal{F}}^{cu}_G(x)}(x, y).
\end{aligned}$$

For n large enough we arrive at a contradiction with the previous inequality.

Now, since $\|H - \mathrm{id}\| \leq C\sqrt{k}$ we have:

- $\tilde{\mathcal{F}}^{cu}_G(x) \subset \{z : \operatorname{dist}_{\mathbb{R}^3}(z, H(x) + E^{cu}_B) \leq C\sqrt{k}\}$ (that is, roughly speaking, $\tilde{\mathcal{F}}^{cu}_G$ is a surface in a sandwich of size $C\sqrt{k}$ with central slice the plane $H(x) + E^{cu}_B$ (see Figure 6),
- $\tilde{\mathcal{F}}^{cu}_G(x)$ is transverse to E^s_B,
- $\tilde{\mathcal{F}}^{cu}_G(x)$ is a complete manifold,

and it is not difficult to see that $\tilde{\mathcal{F}}^{cu}_G(x)$ is the graph of a map $E^{cu}_B \to E^s_B$. Then, since $\|H - \mathrm{id}\| \leq C\rho$ it follows that $H : \tilde{\mathcal{F}}^{cu}_G(x) \to \tilde{\mathcal{F}}^{cu}_B(H(x))$ is onto.

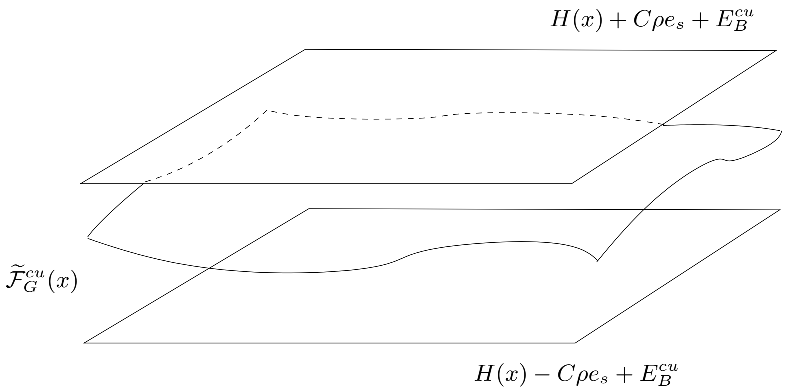

Fig. 6. The $\mathcal{F}^{cu}_G$ leaf

Let us prove the second item. From the first one it follows that

$$H(\tilde{\mathcal{F}}^c_G(x)) = H(\tilde{\mathcal{F}}^{cs}_G(x) \cap \tilde{\mathcal{F}}^{cu}_G(x)) \subset \tilde{\mathcal{F}}^{cs}_B(H(x)) \cap \tilde{\mathcal{F}}^{cu}_B(H(x)) = \tilde{\mathcal{F}}^c_B(H(x)).$$

Since $\|H-\mathrm{id}\| \le C\sqrt{k}$ we see that $\tilde{\mathcal{F}}^c_G(x)$ is in a cylinder of radius $C\sqrt{k}$ with axis $H(x)+E^c_B = \tilde{\mathcal{F}}^c_B(H(x))$. Since E^c_G is in a tiny cone around E^c_B we may assume that E^c_B is always transverse to $E^s_B \oplus E^u_B$ and moreover $\tilde{\mathcal{F}}^c_G(x)$ is the graph of a map $E^c_B \to E^s_B \oplus E^u_B$. Using again $\|H-\mathrm{id}\| \le C\sqrt{k}$ we conclude that $H : \tilde{\mathcal{F}}^c_G(x) \to \tilde{\mathcal{F}}^c_B(H(x))$ is onto.

For the third item observe also that $H(\tilde{\mathcal{F}}^u_G(x)) \subset \tilde{\mathcal{F}}^u_B(H(x))$ since for $y \in \tilde{\mathcal{F}}^u_G(x)$ we have $\|G^n(y)-G^n(x)\| \to 0$ as $n \to -\infty$ and hence the distance between $H(G^n(y)) = B^n(H(y))$ and $H(G^n(x)) = B^n(H(x))$ is bounded for $n \le 0$, which implies that $H(y) \in H(x)+E^u_B$. By similar arguments to those in the previous item we find that $H : \tilde{\mathcal{F}}^u_G(x) \to \tilde{\mathcal{F}}^u_B(H(x))$ is onto. On the other hand, $H_{/\tilde{\mathcal{F}}^u_G(x)}$ is injective: otherwise, if $H(z) = H(y)$ for some $z, y \in \tilde{\mathcal{F}}^u_G(x)$ then by forward iteration $\|G^n(y)-G^n(z)\|$ goes to infinity (recall that $\tilde{\mathcal{F}}^u_G$ is quasi-isometric) and so $\|H(G^n(y)) - H(G^n(z))\|$ also goes to infinity by forward iteration, which is impossible since $H(G^n(y)) = B^n(H(y)) = B^n(H(z)) = H(G^n(z))$.

For the fourth and last item observe that

$$\#\{\tilde{\mathcal{F}}^{cs}_G(x) \cap \tilde{\mathcal{F}}^u_G(y)\} \le 1.$$

Otherwise, let $z, w \in \tilde{\mathcal{F}}^{cs}_G(x) \cap \tilde{\mathcal{F}}^u_G(y)$ and iterating forward we see (since $\tilde{\mathcal{F}}^u$ is quasi-isometric) that $\|G^n(z) - G^n(w)\| \sim \mathrm{dist}_{\tilde{\mathcal{F}}^u}(G^n(z), G^n(w))$, which grows with exponential rate $\sim \lambda_u$. On the other hand, since $z, w \in \tilde{\mathcal{F}}^{cs}$ the distance can grow at most with rate $\lambda_c(g) < 1+\epsilon < \lambda_u$, and we get a contradiction.

To see the intersection is non-empty just recall that $\tilde{\mathcal{F}}^{cs}(x)$ is the graph of a (bounded) map $E^{cs}_B \to E^u_B$ and $\tilde{\mathcal{F}}^u(y)$ is the graph of a (bounded) map $E^u_B \to E^{cs}_B$.

The second part of this item is very similar to what we have already done. Nevertheless (for the very last argument) it is worth mentioning that it is not true in general that $H(\tilde{\mathcal{F}}^s_G(x)) = \tilde{\mathcal{F}}^s_B(H(x))$, and so we cannot be sure that $\tilde{\mathcal{F}}^s_G(x)$ is at a bounded distance from $H(x)+E^s_B$ but still it is not difficult to see (since E^s_G is in a tiny cone around E^s_B) that $\tilde{\mathcal{F}}^s_G(x)$ is the graph of a map $E^s_B \to E^{cu}_B$. ∎

COROLLARY 2.10. *With the above notations, assume that* $H(x) = H(y)$. *Then* x, y *belong to the same central leaf* $\tilde{\mathcal{F}}^c_G(x) = \tilde{\mathcal{F}}^c_G(y)$. *Moreover, if we denote by* $[x,y]_c$ *the central arc in* $\tilde{\mathcal{F}}^c_G(x)$ *with ends* x *and* y *then* $H([x,y]_c) = H(x) = H(y)$ *and the diameter of* $[x,y]_c$ *is bounded by* $2C\sqrt{k}$. *In particular, for any* z, $H^{-1}(z)$ *is either a point or an arc.*

Proof. Let x, y be such that $H(x) = H(y)$. We claim that $y \in \tilde{\mathcal{F}}^{cs}_G(x)$. Otherwise, from the last theorem we may consider $z = \tilde{\mathcal{F}}^{cs}_G(x) \cap \tilde{\mathcal{F}}^u_G(y)$. By similar arguments to those before, since by forward iteration the distance between $G^n(z)$ and $G^n(y)$ grows with a rate much higher than the one between $G^n(z)$ and $G^n(x)$ could attain, we conclude that

$$\|G^n(x) - G^n(y)\| \xrightarrow[n\to\infty]{} \infty.$$

This is impossible due to $H(G^n(y)) = H(G^n(x))$, and so $G^n(z)$ and $G^n(y)$ are at a bounded distance for every n.

In a similar way we prove that $y \in \tilde{\mathcal{F}}^{cu}_G(x)$. Therefore

$$y \in \tilde{\mathcal{F}}^{cs}_G(x) \cap \tilde{\mathcal{F}}^{cu}_G(x) = \tilde{\mathcal{F}}^{c}_G(x).$$

Now, recall that $\tilde{\mathcal{F}}^c(z)$ is the graph of a map $H(z) + E^c_B \to H(z) + E^s_B \oplus E^u_B$ and bounded by $C\sqrt{k}$ (in particular $\tilde{\mathcal{F}}^c(z)$ is quasi-isometric) for any z. We shall denote by $\Pi^{su} : \mathbb{R}^3 \to E^c_B$ the projection along $E^s_B \oplus E^u_B$.

Now, if $w \in [x, y]_c$ it follows that for any n,

$$\Pi^{su}(G^n(x)) < \Pi^{su}(G^n(w)) < \Pi^{su}(G^n(y)).$$

Hence $\sup_{n\in\mathbb{Z}} \|G^n(x) - G^n(y)\| < \infty$ and so $H(x) = H(w)$. Finally, if $H(w) = H(z)$ then $\|z - w\| \le 2C\sqrt{k}$. ∎

For $x \in \mathbb{R}^3$ set $[x] = \{y \in \mathbb{R}^3 : H(y) = H(x)\} = H^{-1}(H(x))$. In other words $[x]$ is the equivalence class of the equivalence relation $x \sim y$ if and only if $H(x) = H(y)$. From the above lemma we know that $[x]$ is a point or an arc contained in the central leaf $\mathcal{F}^c_G(x)$. In particular, as $H : \tilde{F}^u_G(x) \to \tilde{F}^u_B(H(x))$ is a homeomorphism, we have (see Figure 7):

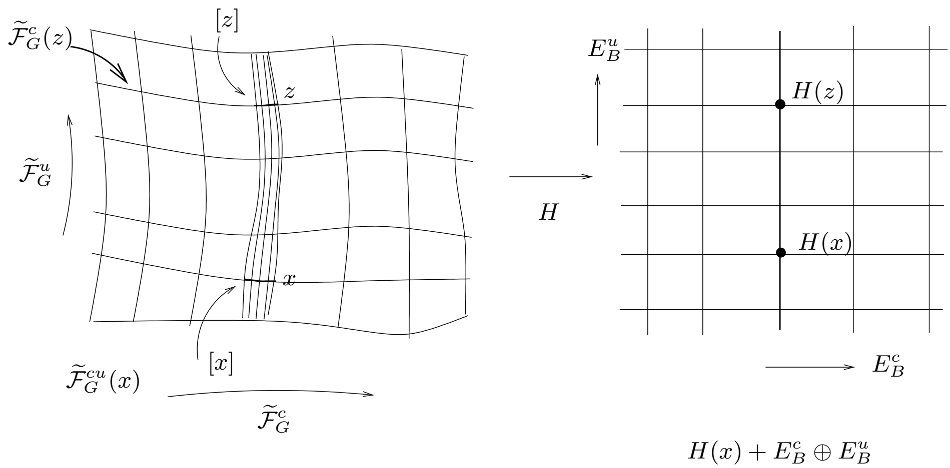

Fig. 7

Corollary 2.11. *Let $x \in \mathbb{R}^3$ and let $z \in \mathcal{F}^u_G(x)$. Then*

$$[z] = \Big(\bigcup_{y\in[x]} \tilde{\mathcal{F}}^u_G(y)\Big) \cap \tilde{\mathcal{F}}^c_G(z). \tag{2.11}$$

Now, going back to the linear Anosov diffeomorphism on the 3-torus induced by $B \in \mathrm{SL}(3,\mathbb{Z})$ and the Mañé's DA $g \in \mathcal{U}(g_{B,k})$ and applying the previous results we get the following

THEOREM 2.12. *There exists $h : \mathbb{T}^3 \to \mathbb{T}^3$ continuous and onto such that*

(1) $B \circ h = h \circ g$.
(2) $\mathrm{dist}_{C^0}(h, \mathrm{id}) \le C\sqrt{k}$.
(3) $h(\mathcal{F}^j_g(x)) = \mathcal{F}^j_B(h(x))$ *where* $j = cs, cu, c, u$ *and* $h : \mathcal{F}^u_g(x) \to \mathcal{F}^u_B(h(x))$ *is a homeomorphism.*
(4) *If* $h(x) = h(y)$ *then* $y \in \mathcal{F}^c_g(x)$.
(5) $h^{-1}(z)$ *is either a point or an arc contained in a central leaf* (*with diameter less than* $2C\sqrt{k}$).
(6) *If we set* $[x] = h^{-1}(h(x)) = \{y \in \mathbb{T}^3 : h(x) = h(y)\}$ *then, for* $z \in \mathcal{F}^u_g(x)$,
$$[z] = \Big(\bigcup_{y\in[x]} \mathcal{F}^u_g(y)\Big) \cap \mathcal{F}^c_g(z).$$

COROLLARY 2.13. *Let $g \in \mathcal{U}(g_{B,k})$ be as above. Then $\mathcal{F}^c_g$ is minimal, i.e., every leaf is dense in $\mathbb{T}^3$.*

Proof. Let $x \in \mathbb{T}^3$ and let $U \subset \mathbb{T}^3$ be an open set. We want to prove that $\mathcal{F}^c_g(x) \cap U \neq \emptyset$. Consider a small two-dimensional disk $S \subset U$ transverse to E^c_g. We know that $h_{/S}$ is injective and hence $h(S)$ is a two-dimensional topological manifold transverse to E^c_B. Since $\mathcal{F}^c_B$ is minimal, $\mathcal{F}^c_B(h(x)) \cap h(S) \neq \emptyset$, that is, there exists $y \in S$ such that $h(y) \in \mathcal{F}^c_B(h(x)) = h(\mathcal{F}^c_g(x))$. Therefore $y \in \mathcal{F}^c_g(x)$ and so $\mathcal{F}^c_g(x) \cap U \neq \emptyset$. ∎

COROLLARY 2.14. *Let $g \in \mathcal{U}(g_{B,k})$ be as above. Then $\mathcal{A} = \{z \in \mathbb{T}^3 : h^{-1}(z)$ is a point$\}$ has full Lebesgue measure.*

Proof. Since the preimage by h of a central leaf of B is a central leaf of g, the set of points x whose preimage is a non-trivial arc is countable in each central leaf of B and hence it has measure zero on each central leaf. On the other hand the central foliation of B is by lines and we can apply the Fubini Theorem to conclude that the Lebesgue measure of the complement of $\mathcal{A}$ is zero. ∎

2.4. Further analysis of semiconjugacy. In this section we give more details on the semiconjugacy with the linear Anosov diffeomorphism B and on the equivalence classes $[x] = h^{-1}(h(x)) = \{y : h(y) = h(x)\}$. This section is needed to get uncountably many non-trivial fibers and to get Li–York chaos in our example. Let us begin with the following

LEMMA 2.15. *For $g \in \mathcal{U}(g_{B,k})$ as above, if*

$$\liminf_{n\to-\infty} \frac{1}{n} \log \|dg^n_{/E^c_g(x)}\| > 0$$

then $[x] = h^{-1}(h(x)) \supsetneq \{x\}$.

Proof. Let γ be such that

$$\liminf_{n\to-\infty} \frac{1}{n} \log \|dg^n_{/E^c_g(x)}\| > \gamma > 0.$$

Then for n large enough we have

$$\|Dg^{-n}_{E^c_g(x)}\| \le e^{-\gamma} n$$

and therefore, by standard arguments, there exists a central arc I_c containing x such that the length of $g^{-n}(I_c)$ is uniformly bounded for $n \ge 0$ (indeed, $I_c \subset W^u(x)$). We claim that $g^n(I_c)$ has bounded length for $n \ge 0$. We will denote by $\ell(I)$ the length of I.

We may assume that ρ is small (recall that the support of the modification of B is in $B(p,\rho)$) so that if J_c is a central arc such that $4\rho \le \ell(J_c) \le 6\rho$ then $J_c \cap B(p,\rho)$ has at most one connected component of length at most 2ρ. Recall also that $\lambda_c(g) < 1+\epsilon$ where ϵ is small (taking k small) (for instance, $\epsilon < 1-\lambda_c$ and $\epsilon < 1/2$).

To prove the claim we may assume that $\ell(I_c) < 2\rho$ and arguing by contradiction, consider the case where the length of $g^n(I_c)$ is unbounded for $n \ge 0$. Let n_0 be the first time such that $\ell(g^n(I_c)) \ge 6\rho$. Since $4\rho\lambda_c(g) < 4\rho(1+\epsilon) < 6\rho$ it follows that

$$4\rho \le \ell(g^{n_0-1}(I_c)) < 6\rho.$$

Set $J_c = g^{n_0-1}(I_c)$. By the above condition on J_c and recalling that $\|dg_\xi\| = \|B\| = \lambda_c$ if $\xi \notin B(p,\rho)$, we get

$$6\rho \le \ell(g^{n_0}(I_c)) = \ell(g(J_c)) \le (1+\epsilon)\frac{\ell(J_c)}{2} + \lambda_c\frac{\ell(J_c)}{2} < \ell(J_c) < 6\rho,$$

a contradiction. Now since, $\ell(g^n(I_c))$ is bounded for all $n \in \mathbb{Z}$ we conclude that $h(I_c) = h(x)$ (this can be seen by lifting to $\mathbb{R}^3$ where it immediately follows that $\|G^n(x) - G^n(y)\|$ is bounded for all $n \in \mathbb{Z}$ and $y \in I_c$). ∎

The following lemma says that in any unstable leaf there is a point whose forward orbit never meets $B(p,2\rho)$; this is similar to what we have done in Section 2.2. Notice also that $\mathcal{F}^u_g$ is orientable and choose an orientation. For $x \in \mathbb{T}^3$ denote by $\mathcal{F}^{u,+}_g(x,t)$ an arc of length t in $\mathcal{F}^u_g(x)$ starting at x in the chosen orientation.

LEMMA 2.16. *Assume that $\lambda_u > 3$. Then for ρ, k and $\mathcal{U}$ small the following holds for each $g \in \mathcal{U}(g_{B,k})$: for any $x \in \mathbb{T}^3$ there exists a point $z_x \in \mathcal{F}^{u,+}_g(x,1)$ such that $g^n(z) \cap B(p,2\rho) = \emptyset$ for any $n \ge 0$.*

Proof. We may assume that ρ is so small that any segment I_u in $\mathcal{F}_B^u$ of length 1 has a subsegment I_{u_1} of length 1/3 such that $I_{u_1} \cap B(p, 2\rho) = \emptyset$. Now, if k and $\mathcal{U}(g_{B,k})$ are small we may assume that the same property holds for $g \in \mathcal{U}$, that is, any arc I_u in $\mathcal{F}_g^u$ of length 1 has a subarc I_{u_1} of length 1/3 such that $I_{u_1} \cap B(p, 2\rho) = \emptyset$. Moreover, we may assume that $\lambda^u(g) > 3$. Now, $g(I_{u_1})$ has length at least 1 and so it has a subarc I_{u_2} such that $I_{u_2} \cap B(p, 2\rho) = \emptyset$. By induction, for any n, $g(I_{u_n})$ contains $I_{u_{n+1}}$ such that $I_{u_{n+1}} \cap B(p, 2\rho) = \emptyset$. Therefore,

$$z_x \in \bigcap_{n \geq 0} g^{-n}(I_{u_{n+1}})$$

satisfies the conclusion of the lemma. ■

COROLLARY 2.17. *Let $g \in \mathcal{U}$ be as above and let $x \in \mathbb{T}^3$ be such that $[x] \supsetneq \{x\}$. Then given $\eta > 0$ there is a point $y \in \mathcal{F}_g^{u,+}(x)$ (the positive side of $\mathcal{F}_g^u$ in the chosen orientation) such that $\ell([y]) < \eta$.*

Proof. Recall that for $g \in \mathcal{U}$ we have $\|dg_{/E_g^c(\xi)}\| < \lambda_c(1+\epsilon) < 1$ if $\xi \notin B(p, \rho)$. Also, if k is small then $2C\sqrt{k} < \rho$. Let η be given and let n_0 be such that

$$(\lambda_c(1+\epsilon))^{n_0} 2C\sqrt{k} < \eta.$$

Consider x such that $[x] \supsetneq \{x\}$. From the above lemma, select $z \in \mathcal{F}_g^u(g^{-n_0}(x), 1)$ such that $g^n(z) \notin B(p, 2\rho)$ for any $n \geq 0$. Notice that, since $[g^{-n_0}(x)]$ is not trivial, the same is true for z. On the other hand, $[z]$ is a central segment of length at most $2C\sqrt{k}$. Therefore, $g^n([z]) \cap B(p, \rho) = \emptyset$ for $n \geq 0$. Therefore,

$$\ell(g^n[z]) \leq (\lambda_c(1+\epsilon))^n 2C\sqrt{k}.$$

Finally, setting $y = g^{n_0}(z) \in \mathcal{F}_g^{u,+}(x)$ we have

$$\ell([y]) = \ell(g^{n_0}[z]) \leq (\lambda_c(1+\epsilon))^{n_0} 2C\sqrt{k} < \eta. \ ■$$

The next result is fundamental for the behavior of the holonomy map along the unstable foliation. The main tool is the existence of a transverse homoclinic point (recall Corollary 2.6).

LEMMA 2.18. *Let $g \in \mathcal{U}(g_{B,k})$ have a transverse homoclinic point associated to the fixed point p of unstable index 2. There exist ϵ_0 and $z_p \in \mathcal{F}_g^{u,+}(p)$ such that*

$$\limsup_{n\to\infty} \ell(g^n([z_p])) > \epsilon_0.$$

Proof. Recall that $[p]$ is the central segment between q_1, q_2. Pick $\epsilon_0 < \min\{\ell[q_1, p]^c, \ell[p, q_2]^c\}$. Notice that

$$W^u(p) = \bigcup_{y \in (q_1, q_2)^c} \mathcal{F}_g^u(y).$$

Let z be a homoclinic point associated to p, that is, $z \in \mathcal{F}^s_g(p) \cap W^u(p)$. We know that

$$[z] = \Big(\bigcup_{y\in[p]} \mathcal{F}^u_g(y)\Big) \cap \mathcal{F}^c_g(z).$$

We may assume that the orientation in $\mathcal{F}^u_g$ is such that $z_p = [z] \cap \mathcal{F}^u_g(p) \in \mathcal{F}^{u,+}_g(p)$. Since $[z] = [z_p], z \in \mathcal{F}^s(p)$ and $[z] \subset \mathcal{F}^c_g(z) \subset \mathcal{F}^{cs}_g(p)$, by forward iteration $g^n([z])$ must approach $[q_1, p]$ or $[p, q_1]$ (see also Figure 8), and the lemma follows. ∎

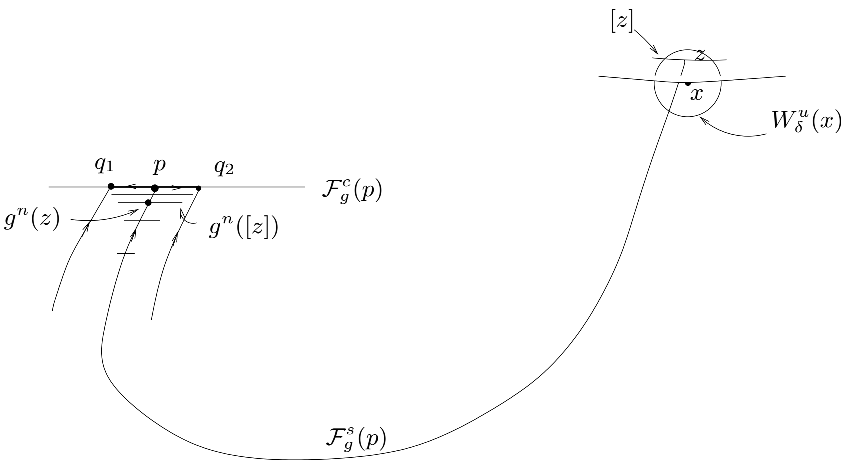

Fig. 8

Indeed, a more extensive result holds:

PROPOSITION 2.19. *Let $g \in \mathcal{U}(g_{B,k})$ have a transverse homoclinic point associated to the fixed point p of unstable index 2. Then there exists an uncountable set Λ_0 such that:*

(1) *If $x, y \in \Lambda_0$, $x \neq y$, then $\mathcal{F}^{cu}_g(x) \neq \mathcal{F}^{cu}_g(y)$.*
(2) *For any $x \in \Lambda_0$, $[x]$ is non-trivial.*
(3) *There exists ϵ_0 such that for any $x \in \Lambda_0$ and any $t > 0$ there exists $z_x \in \mathcal{F}^{u,+}_g(x) \setminus \mathcal{F}^u_g(x,t)$ such that $\ell([z_x]) > \epsilon_0$.*

Proof. From the existence of a transverse homoclinic point associated to p of index 2 we deduce the existence of a non-trivial hyperbolic compact invariant set Λ (of unstable index 2) and with local product structure. In particular, from Lemma 2.15 implies that for any $x \in \Lambda$, $[x]$ is non-trivial.

Notice that for $x \in \Lambda, W^u(x)$ is two-dimensional and contained in $\mathcal{F}^{cu}_g(x)$ and there exists $\delta > 0$ such that $W^u_\delta(x)$ has uniform size. We will denote by $W^{u,+}_\delta(x)$ the component of $W^u_\delta(x) \setminus \mathcal{F}^c_g(x)$, which is in the positive direction of $\mathcal{F}_g^u(x)$. Moreover, there are uncountably many disjoint unstable manifolds W^u. Furthermore, there is some L such that, setting $\mathcal{F}_g^s(p,L) = W_L^s(p)$, we have

$$\mathcal{F}_g^s(p,L) \cap W_\delta^u(x) \neq \emptyset \quad \forall x \in \Lambda.$$

Indeed, it is not difficult to see that if x is not in a central stable periodic leaf, then

$$\mathcal{F}_g^s(p,L) \cap W_\delta^{u,+}(x) \neq \emptyset.$$

Let us indicate a consequence of the above fact. Let $z \in \mathcal{F}_g^s(p,L) \cap W_\delta^{u,+}(x)$ and let $\epsilon_0 < \ell([p])/2$. Since $g^n(z) \to p$ we conclude that $\ell(g^n[z]) = \ell([g^n(z)]) > \epsilon_0$ for n large enough (see Figure 8). Indeed, $[z]$ is a central arc of uniform size and therefore, as moreover there exists m_0 such that $g^{m_0}(z) \in W^s_{\mathrm{loc}}(p)$, we infer that $g^{m_0}([z])$ is a central arc of uniform size in $\mathcal{F}^{cs}_{\mathrm{loc}}(p)$. Now, by forward iteration, we have $\ell(g^n([z])) > \epsilon_0$ for all $n \geq m_1$ for some m_1 (which is independent of x).

Now choose an uncountable set $\Lambda_0 \subset \Lambda$ such that for $x \neq y \in \Lambda_0$ we have $\mathcal{F}_g^{cu}(x) \neq \mathcal{F}_g^{cu}(y)$ and no $x \in \Lambda_0$ is in a periodic central stable leaf. It remains to prove (3). Let $x \in \Lambda_0$ and let $t > 0$, and choose $n_1 > m_1$ such that $g^{-n_1}(\mathcal{F}_g^{u,+}(x,t)) \subset W_\eta^u(g^{-n_1}(x))$ where η is such that $W_\eta^u(g^{-n_1}(x)) \cap \mathcal{F}_g^s(x,L) = \emptyset$. Let $w \in \mathcal{F}_g^s(p,L) \cap W_\delta^{u,+}(g^{-n_1}(x))$. It follows that $[w] \cap \mathcal{F}_g^{u,+}(g^{-n_1}(x)) \neq \emptyset$; let y be the point of intersection. Notice that on one hand $y \notin g^{-n_1}(\mathcal{F}_g^{u,+}(x,t))$ and therefore $z_x = g^{n_1}(y) \in \mathcal{F}_g^{u,+}(x) \setminus \mathcal{F}_g^{u,+}(x,t)$. On the other hand,

$$\ell([z_x]) = \ell([g^{n_1}(y)]) = \ell(g^{n_1}([y])) = \ell(g^{n_1}([z])) > \epsilon_0. \ \blacksquare$$

Finally, we will give a result for g as in Corollary 2.6, which is fundamental in order to get Li–Yorke chaos:

PROPOSITION 2.20. *Let g be as in Corollary 2.6 and let p_1, p_2 be any two distinct points in J. Then there exists $w \in J$ between p_1 and p_2, $z \in \mathcal{F}^{u,+}(w)$ and a non-trivial arc $I_c \subset [z]$ such that*

(1) $I_c \subset (\bigcup_{y \in [p_1,p_2]} \mathcal{F}_g^u(y)) \cap [z]$.
(2) $\lim_{k\to\infty} \ell(g^{km}(I_c)) > 0$ (*where m is given in Corollary* 2.6).
(3) $g^{km}(I_c) \subset \bigcup_{y \in [p_1,p_2]} \mathcal{F}_g^u(y)$ *for all k*.

Proof. Consider $D^{cs} = \bigcup_{x \in \mathrm{int}(J)} \mathcal{F}_g^s(x)$ which contains a disk in a central stable manifold. Now, let $w \in J$ lie between p_1 and p_2. Since the unstable foliation is minimal, we have $\mathcal{F}_g^u(w) \cap D^{cs} \neq \emptyset$. Let z be in this intersection. Then, since $\bigcup_{y \in [p_1,p_2]} \mathcal{F}_g^u(y) \cap [z]$ contains z in its interior (with respect to $[z]$) we may find I^c satisfying (1) of the proposition and such that $I^c \subset D^{cs}$. Since $g^m_{/J} \equiv \mathrm{id}$ we also deduce (2). Since all points of J are fixed by g^m, the

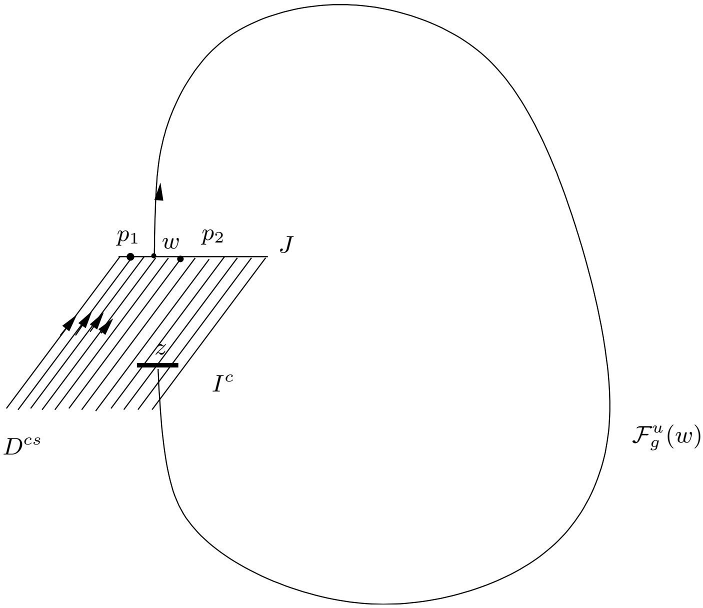

Fig. 9

unstable manifolds of points of J are invariant under g^m and so we obtain (3) as well. ∎

3. The induced holonomy map on $\mathbb{T}^2$. Let $B \in \mathrm{SL}(3,\mathbb{Z})$ (with eigenvalues $0 < \lambda_s < \lambda_c < 1 < \lambda_u$) and $g_{B,k}$ defined in (2.1) and (2.2), and let $g \in \mathcal{U}(g_{B,k})$ with k and $\mathcal{U}$ small so that the last section applies.

Consider a two-dimensional torus transverse to $\mathcal{F}^u_B$ and (assuming k and $\mathcal{U}$ small) also transverse to $\mathcal{F}^u_g$. In particular we may and do consider $\mathbb{T}^2 = (\mathbb{R}^2 \times \{0\})/(\mathbb{Z}^2 \times \{0\})$.

The foliations $\mathcal{F}^u_B$ and $\mathcal{F}^u_g$ are orientable and we choose similar orientations on both (that is, take unit vector fields $X_B = e^u$ and X_g close to X_B).

Definition 3.1. For g as above we define $f = f_g : \mathbb{T}^2 \to \mathbb{T}^2$ to be the holonomy map on $\mathbb{T}^2$ induced by the unstable foliation $\mathcal{F}^u_g$. In other words, $f(x)$ is the first return map of $\mathcal{F}^u_g(x)$ to $\mathbb{T}^2$ in the given orientation. Moreover, we can define $F : \mathbb{T}^3 \to \mathbb{T}^2$ as the first return to $\mathbb{T}^2$ of any $x \in \mathbb{T}^3$ along the positive orientation of $\mathcal{F}^u_g(x)$.

Remark 3.2. Notice that the induced map $f = f_g$ is a homeomorphism. Moreover, f is of class C^r if the unstable foliation $\mathcal{F}^u_g$ is of class C^r. Furthermore, $\mathcal{F}^u_g$ is of class C^r if the unstable bundle E^u_g is.

Moreover, if we consider the holonomy map $T_B : \mathbb{T}^2 \to \mathbb{T}^2$ induced by $\mathcal{F}^u_B$ we find that T_B is a minimal (and hence ergodic) translation. Moreover, $f = f_g$ and T_B are close as we wish if k is small.

If we apply the results of the previous section we obtain the topological version of our main result:

THEOREM 3.3. *For $g : \mathbb{T}^3 \to \mathbb{T}^3$ in $\mathcal{U}(g_{B,k})$ and $f = f_g : \mathbb{T}^2 \to \mathbb{T}^2$ and $T_B : \mathbb{T}^2 \to \mathbb{T}^2$ as above we have:*

(i) *f is minimal.*
(ii) *f is isotopic and semiconjugate to the ergodic translation T_B. If we denote by $\hat{h}$ the semiconjugacy, then $\hat{h}^{-1}(x)$ is either a point or an arc.*
(iii) *f preserves a minimal and invariant C^0 foliation with one-dimensional C^1 leaves. The fibers $\hat{h}^{-1}(x)$ are contained in the leaves of this foliation.*
(iv) *The set $\tilde{\mathcal{A}} = \{z \in \mathbb{T}^2 : \hat{h}^{-1}(z) \text{ is a point}\}$ has full Lebesgue measure.*

As a consequence:

(v) *f has zero entropy.*
(vi) *f is point-distal and non-distal.*
(vii) *f is sensitive to initial conditions.*
(viii) *f is uniquely ergodic.*

Furthermore, if g also satisfies the conditions of Corollary 2.6 *then:*

(ix) *f exhibits Li–York chaos.*
(x) *There are uncountably points x such that $\hat{h}^{-1}(x)$ is a non-trivial arc.*

Proof. (i) follows from the minimality of the unstable foliation $\mathcal{F}^u_g$ (see Section 2.2).

Let us prove (ii). Since f and T_B are C^0 close, they are isotopic. Recall that h is the semiconjugacy between $g = g_{B,k}$ and $B : \mathbb{T}^3 \to \mathbb{T}^3$ given in Section 2.3.

Since $\mathrm{dist}_{C^0}(h, \mathrm{id}) < C\sqrt{k}$ (which we may assume to be smaller than $1/4$), for every point in $h(\mathbb{T}^2)$ we can define a natural projection $P : h(\mathbb{T}^2) \to \mathbb{T}^2$ along the unstable foliation $\mathcal{F}^u_B$, that is, $P(h(x))$ is the closest point to $h(x)$ within $\mathcal{F}^u_B(h(x))$ in $\mathbb{T}^2$. Define

$$\hat{h} : \mathbb{T}^2 \to \mathbb{T}^2, \qquad \hat{h}(x) = P(h(x)).$$

Clearly, $\hat{h}$ is continuous and close to the identity (if k is small) and hence onto (and isotopic to the identity as well).

Now, the points $x \in \mathbb{T}^2$ and $f(x) \in \mathbb{T}^2$ are the ends of an arc $I^u \subset \mathcal{F}_g^u(x)$ and when lifted to $\mathbb{R}^3$ their coordinates have z-difference 1.

On the other hand, $h(I^u)$ is an arc (segment) of $\mathcal{F}_B^u(h(x))$ so that, when lifted to $\mathbb{R}^3$ the ends have coordinates whose z-difference is between $1-2C\sqrt{k}$ and $1+2C\sqrt{k}$. Therefore $P(h(f(x)) = T_B(\hat{h}(x))$, that is,

$$\hat{h} \circ f = T_B \circ \hat{h}.$$

Notice that:

- If $h^{-1}(x) = \{y\}$ then clearly $\hat{h}^{-1}(x)$ is a unique point.
- If $h^{-1}(x)$ is a non-trivial central arc, then its projection (by P) onto $\mathbb{T}^2$ is a non-trivial arc and equals $\hat{h}^{-1}(x)$.

This finishes the proof of (ii).

To prove (iii), for $x \in \mathbb{T}^2$ let $\mathcal{C}(x)$ be the connected component of $\mathcal{F}_g^{cu}(x) \cap \mathbb{T}^2$ that contains x. It follows that $\mathcal{C}$ is a continuous foliation with C^1 one-dimensional leaves (recall that $\mathcal{F}_g^{cu}(x)$ is a C^1 manifold) and obviously invariant under f, the holonomy map. Furthermore, since $h(\mathcal{F}_g^{cu})(x) = \mathcal{F}_B^{cu}(h(x))$ it follows that $\hat{h}(\mathcal{C}(x))$ is the connected component of $\mathcal{F}_B^{cu}(\hat{h}(x)) \cap \mathbb{T}^2$ that contains $\hat{h}(x)$. Since this foliation by lines on $\mathbb{T}^2$ is minimal we also conclude that $\mathcal{C}$ is minimal (proof similar to that of Corollary 2.13). Since $h^{-1}(x)$ live in a central unstable leaf, we see that $\hat{h}^{-1}$ lives in the leaves of this foliation.

To prove (iv), consider as in the statement the set

$$\tilde{\mathcal{A}} = \{x \in \mathbb{T}^2 : \hat{h}^{-1}(x) \text{ is a point}\}.$$

Observe that $\hat{h}^{-1}(x)$ is a point if and only if $h^{-1}(x)$ is. Moreover, if $h^{-1}(x)$ is a point, the same is true for any $y \in \mathcal{F}_B^u(x)$. By Corollary 2.14,

$$\mathcal{A} = \{x \in \mathbb{T}^3 : h^{-1}(x) \text{ is a point}\}$$

has full Lebesgue measure on $\mathbb{T}^3$ and therefore $\tilde{\mathcal{A}}$ has full Lebesgue measure on $\mathbb{T}^2$.

The proof of (v) is rather easy. Indeed, by Bowen's formula ([Bo]) we have

$$h_{\text{top}}(f) \le h_{\text{top}}(T_B) + \sup_{x \in \mathbb{T}^2} h_{\text{top}}(f, \hat{h}^{-1}(x))$$

where $h_{\text{top}}(f,K) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} n^{-1} \log N(\epsilon,n,f,K)$ and $N(\epsilon,n,f,K)$ is the minimum cardinality of an (n,ϵ)-separated set in K. Since for all x, $\hat{h}^{-1}(x)$ is either a point or an arc (with bounded length in the future and in the past), we have the result (see also [BFSV]). Notice also that if we happen to know that f is $C^{1+\alpha}$, the zero entropy follows from Katok's result [Ka].

Let us prove (vi). Recall that f is *point distal* if there exists $x \in \mathbb{T}^2$ such that for every $y \neq x$ there exists $r_y > 0$ so that $r_y \leq \inf\{\text{dist}(f^n(x), f^n(y)) : n \in \mathbb{Z}\}$, and f is *non-distal* if there exists a pair of points z, w such that $\inf\{\text{dist}(f^n(z), f^n(w)) : n \in \mathbb{Z}\} = 0$.

We first show that f is point distal. Let $x \in \mathbb{T}^2$ with $\hat{h}^{-1}(\hat{h}(x)) = \{x\}$ and consider any $y \in \mathbb{T}^2$. Let $\alpha = \text{dist}(\hat{h}(x), \hat{h}(y))$. By the (uniform) continuity of $\hat{h}$, there exists r such that if $\text{dist}(z, w) < r$ then $\text{dist}(\hat{h}(z), \hat{h}(w)) < \alpha$ for any $z, w \in \mathbb{T}^2$. We claim that $\inf\{\text{dist}(f^n(x), f^n(y)) : n \in \mathbb{Z}\} \geq r > 0$. Otherwise, if for some n we have $\text{dist}(f^n(x), f^n(y)) < r$ then (since T_B is an isometry)

$$\begin{aligned}\alpha > {} & \text{dist}(\hat{h}(f^n(x)), \hat{h}(f^n(y))) = \text{dist}(T_B^n(\hat{h}(x)), T_B(\hat{h}(y))) \\ & = \text{dist}(\hat{h}(x), \hat{h}(y)) = \alpha.\end{aligned}$$

Now, we prove that f is non-distal. Let x be such that $I_x = \hat{h}^{-1}(\hat{h}(x))$ is a non-trivial arc. It follows that $\liminf_{n\to\infty} f^n(I_x) = 0$ since the orbit of I_x is dense and must approach points with trivial equivalence class (fiber), and the equivalence classes vary lower semicontinuously. Finally, if we take $z \neq w \in I_x$ we conclude that $\inf\{\text{dist}(f^n(z), f^n(w)) : n \in \mathbb{Z}\} = 0$, i.e., f is non-distal.

For the proof of (vii), recall that f is sensitive to initial conditions if there exists some ϵ_2 such that for any $x \in \mathbb{T}^2$ and any open set U containing x there exist $y \in U$ and $n > 0$ such that $\text{dist}(f^n(x), f^n(y)) \geq \epsilon_2$. So, given ϵ_1, let ϵ_2 be such that any arc in $\mathcal{C}$ of length ϵ_1 has endpoints at distance at least $2\epsilon_2$. Let x and U be given. Assume first that $\hat{h}^{-1}(\hat{h}(x)) = \{x\}$, which is the same as $[x] = \{x\}$. Since f is minimal, there is m_k such that $f^{m_k}(p) \to_k x$. We claim that for k large enough, $f^{m_k}(I_p) \subset U$. Indeed, $\ell(f^{m_k}(I_p)) \to 0$, as otherwise $[x] \neq \{x\}$ (the equivalence classes are lower semicontinuous). Thus, choose some m so that $f^m(I_p) \subset U$. Since $\limsup \ell(f^n(I_p)) \geq \epsilon_1$ we get the result taking y as the appropriate endpoint of $f^m(I_p)$. Now, if $[x]$ is non-trivial we can argue as before, since in U there are points z such that $[z]$ is trivial and so for some m we have $f^m(I_p) \subset U$.

To prove (viii), denote by $\mathcal{M}_f$ the set of invariant probabilities of f. Given $\mu \in \mathcal{M}_f$ we may define a measure $\nu \in \mathcal{M}_{T_B}$ by $\nu(A) = \mu(h^{-1}(A))$. Since T_B is uniquely ergodic, $\nu = m$ (the Lebesgue measure on $\mathbb{T}^2$). That is, for every Borel set D and $\mu \in \mathcal{M}_f$ we have $\mu(h^{-1}(D)) = m(D)$. Therefore, for every $\mu \in \mathcal{M}_f$, setting $\mathcal{D} = \hat{h}^{-1}(\tilde{\mathcal{A}})$ where $\tilde{\mathcal{A}}$ is as in (iv), we have

$$\mu(\mathcal{D}) = \mu(\hat{h}^{-1}(\tilde{\mathcal{A}})) = m(\tilde{\mathcal{A}}) = 1.$$

Observe that for any Borel set A we have $A \cap \mathcal{D} = \hat{h}^{-1}(\hat{h}(A \cap \mathcal{D}))$.

Given $\mu_1, \mu_2 \in \mathcal{M}_f$ and any Borel set A we have

$$\mu_1(A) = \mu_1(A \cap \mathcal{D}) = \mu_1(\hat{h}^{-1}(\hat{h}(A \cap \mathcal{D}))) = m(\hat{h}(A \cap \mathcal{D}))$$
$$= \mu_2(\hat{h}^{-1}(h(A \cap \mathcal{D}))) = \mu_2(A \cap \mathcal{D}) = \mu_2(A).$$

Thus f is uniquely ergodic.

We now prove (ix) in case g is as in Corollary 2.6. Consider the arc J as in Corollary 2.6. Notice that for any $x \in J$, $[x] \supseteq J$. If $J \cap \mathbb{T}^2 = \emptyset$ let $\tilde{J} = F(J)$ where F is the first return map to $\mathbb{T}^2$ along the unstable foliation $\mathcal{F}^u_g$; otherwise, set $\tilde{J} = P(J)$. As above, $\liminf_{n\to\infty} f^n(\tilde{J}) = 0$. On the other hand, given any two points p_1, p_2 in $\tilde{J}$ and applying Proposition 2.20 we conclude that $\limsup_{n\to\infty} d(f^n(p_1), f^n(p_2)) > 0$.

Finally, by Proposition 2.19, there are uncountably many points x such that $\hat{h}^{-1}(x)$ is a non-trivial arc. This proves (x). ∎

Remark 3.4. If f were of class C^2 and the leaves of the foliation $\mathcal{C}$ also were of class C^2 one would be tempted to use Schwarz's argument ([Sch]) to show that non-trivial fibers of $\hat{h}^{-1}$ are not possible. However, in our case there is an extra difficulty: we do not know *a priori* that the sum of the lengths of the iterates of a non-trivial fiber (if any) does converge. In our examples, this sum does not converge!

Let us point out as well that with our method, the differentiability of the system and of the foliation are like the dishes on a balance. More differentiability for the system implies less for the foliation.

4. On the smoothness of E^u_g. From Theorem 3.3 and Remark 3.2 the only thing left to prove for our Main Theorem is the following: given $r \in [1, 3)$ there exists g so that the unstable bundle E^u_g is of class C^r.

In order to establish the differentiability class of E^u_g we recall a classical result from [HPS] that is very useful for this type of problem.

Theorem 4.1 (C^r-section theorem). *Let M be a compact C^r manifold and $g : M \to M$ a C^r diffeomorphism. Let $\pi : L \to M$ be a finite-dimensional Finsler vector bundle and let D be the disk subbundle with $\pi(D) = M$. Let $F : D \to D$ be a homeomorphism such that $F(L_\xi) = L_{g(\xi)}$, and let $l_\xi = l_\xi(F, g)$ be the Lipschitz constant of $F|_{L_\xi}$ for $\xi \in M$.*

If $l_\xi < 1$ for every $\xi \in M$, then there exists a unique continuous section $\sigma : M \to L$ such that $F \circ \sigma = \sigma \circ g$ (an invariant section).

Moreover, if $\pi : L \to M$ is a C^r vector bundle (with some structure which is compatible with the Finsler structure), F is C^r and setting $\tau_\xi = \tau_\xi(g) = \|(dg_\xi)^{-1}\|$ we have $l_\xi \tau^r_\xi < 1$, then the invariant section $\sigma : M \to L$ is C^r.

Let $B \in \mathrm{SL}(3, \mathbb{Z})$ be a linear transformation with eigenvalues $0 < \lambda_s < \lambda_c < 1 < \lambda_u$ and invariant hyperbolic structure $E^s_B \oplus E^c_B \oplus E^u_B$ as considered above and a Euclidean metric on $\mathbb{R}^3$ such that the above spaces are mutually orthogonal. Consider the vector space $\mathcal{L}(E^u_B, E^s_B \oplus E^c_B)$ of linear maps $t : E^u_B \to E^s_B \oplus E^c_B$ with the natural norm.

Consider the (trivial) vector bundle

$$L = \{(\xi, t) : \xi \in \mathbb{T}^3,\, t \in \mathcal{L}(E^u_B, E^s_B \oplus E^c_B)\}. \tag{4.1}$$

Then $\pi : L \to M$ given by $\pi(\xi, t) = \xi$ is a (finite-dimensional) C^∞ Finsler vector bundle.

Now, for $g = g_{B,k} : \mathbb{T}^3 \to \mathbb{T}^3$ we define the associated vector bundle map $F = F_{B,g} : L \to L$ as follows: for $(\xi, t) \in L$,

$$\begin{aligned} &F(\xi, t) = (g(\xi), s), \quad s \in \mathcal{L}(E^u_B, E^s_B \oplus E^c_B) \\ &\text{such that } \operatorname{graph}(s)) = dg_\xi(\operatorname{graph}(t)). \end{aligned} \tag{4.2}$$

Recall that $E^s_B \oplus E^c_B$ is invariant under dg_ξ for any $\xi \in \mathbb{T}^3$ and so F is a well defined vector bundle homeomorphism. Nevertheless, for g close to $g_{B,k}$ the associated map $F : L \to L$ may not be well defined on the whole L. To overcome this difficulty just set

$$D = \{(\xi, t) : \xi \in \mathbb{T}^3,\, t \in \mathcal{L}(E^u_B, E^s_B \oplus E^c_B),\, \|t\| \le 1\}.$$

Then from the above theorem we have

Corollary 4.2. *Assume that for some r, B and k we have*

$$l_\xi(F, g_{B,k}) < 1, \qquad l_\xi(F, g_{B,k})\tau(g_{B,k})^r < 1.$$

Then there exists $\mathcal{U}(g_{B,k})$ such that for any $g \in \mathcal{U}(g_{B,k})$ of class C^∞ the associated map $F_g : D \to D$ is well defined, $l_\xi(F) < 1$ and $l_\xi(F)\tau(g)^r < 1$. In particular, there exists a unique invariant section for F_g in D and it is of class C^r.

Remark 4.3. Observe that if $\sigma : \mathbb{T}^3 \to L$ is an invariant section for F, i.e., $F \circ \sigma = \sigma \circ g$, then $\operatorname{graph}(\sigma(\xi)) = E^u_g(\xi)$. So, in order to find the differentiability class we will apply the C^r section theorem to our $F : L \to L$ over g.

Remark 4.4. If we use the C^r section theorem to calculate the differentiability of the unstable vector bundle of the Anosov system induced by B, then we will have differentiability less than C^3: if $r = 3$, then

$$l_\xi \tau_\xi^r = \frac{\lambda_c}{\lambda_u}\frac{1}{\lambda_s^3} = \frac{\lambda_c^2}{\lambda_s^2} > 1.$$

Moreover, the last estimate shows that in order to have proximity to C^3 differentiability we must find linear Anosov systems with λ_s close to λ_c. This will be done in Section 4.1.

Throughout the rest of this subsection, to ease notation we set $g = g_{B,k}$. We want to estimate $l_\xi(F, g)$ and $\tau_\xi(g)$ for the graph transform F associated to $g = g_{B,k}$. Recall that the differential of g in the decomposition $E^s_B \oplus E^c_B \oplus E^u_B$ is given by

$$dg_\xi = \begin{pmatrix} \lambda_s & 0 & 0 \\ 0 & \lambda_c & 0 \\ 0 & 0 & \lambda_u \end{pmatrix} \qquad \text{for } \xi \in \mathbb{T}^3 \setminus B(p,\rho)$$

and

$$dg_\xi = \begin{pmatrix} \lambda_s + Z(z)(\beta(r)+\beta'(r)2x^2) & Z(z)\beta'(r)2xy & Z'(z)\beta(r)x \\ Z(z)\beta'(r)2xy & \lambda_c + Z(z)(\beta(r)+\beta'(r)2y^2) & Z'(z)\beta(r)y \\ 0 & 0 & \lambda_u \end{pmatrix}$$

for $\xi \in B(p,\rho)$.

Set $T_\xi = dg_{\xi/E^s_B \oplus E^c_B}$.

Lemma 4.5. *With the above notations we have*

$$l_\xi = l_\xi(F) \le \|T_\xi\|/\lambda_u.$$

Moreover the following estimates hold:

(i) *For $\xi \notin B(p,\rho)$ we have $l_\xi \le \lambda_c/\lambda_u$.*
(ii) *For $\xi \in B(p,\rho)$ we have $l_\xi < (\lambda_c + Z(z)\beta(r) + k)/\lambda_u$.*

In particular $l_\xi(F) < 1$ for all $\xi \in \mathbb{T}^3$ (if k is small).

Proof. If we write

$$dg_\xi = \begin{pmatrix} T_\xi & A_\xi \\ 0 & \lambda_u \end{pmatrix}$$

then it is not difficult to see that

$$F(\xi,t)(v) = \frac{1}{\lambda_u}(T_\xi(t(v)) + A_\xi v)$$

and therefore

$$\|F(\xi,t_1) - F(\xi,t_2)\| \le \frac{\|T_\xi\|}{\lambda_u}\|t_1 - t_2\|,$$

which implies $l_\xi \le \|T_\xi\|/\lambda_u$. Since for $\xi \notin B(p,\rho)$ we have $\|T_\xi\| = \lambda_c$, we obtain (i).

In order to prove (ii), set $T_\xi = D + S_\xi$ where $D = \left(\begin{smallmatrix} \lambda_s & 0 \\ 0 & \lambda_c \end{smallmatrix}\right)$ and

$$S_\xi = \begin{pmatrix} Z(z)(\beta(r)+\beta'(r)2x^2) & Z(z)\beta'(r)2xy \\ Z(z)\beta'(r)2xy & Z(z)(\beta(r)+\beta'(r)2y^2) \end{pmatrix}.$$

Observe that S_ξ is selfadjoint and has eigenvectors (when $\xi \neq p$) (x,y), $(-y,x)$ and eigenvalues

$$\lambda_1 = Z(z)(\beta(r) + 2\beta'(r)r), \qquad \lambda_2 = Z(z)\beta(r). \tag{4.3}$$

If $\xi = p$ then $S_\xi = Z(0)\beta(0)\,\mathrm{id}$. From the definition of g (recall Lemma 2.1 and (2.2)) we have $-k < \lambda_1 < \lambda_2 < \beta(0)$ and $\lambda_2 > 0$, $\lambda_2 - \lambda_1 < k$. Then, $\|S_\xi\| \le \max\{|\lambda_1|, |\lambda_2|\} \le \lambda_2 + k = Z(z)\beta(r) + k$ and so $\|T_\xi\| \le \lambda_c + \lambda_2 + k$. ∎

LEMMA 4.6. *Let $\lambda_1 = \lambda_{1,g} : \mathbb{T}^3 \to \mathbb{R}$ be defined by $\lambda_{1,g}(\xi) = 0$ if $\xi \notin B(p,\rho)$ and $\lambda_{1,g}(\xi) = Z(z)(\beta(r) + 2\beta'(r)r)$ for $\xi \in B(p,\rho)$. Then*

$$\|(dg_\xi)^{-1}\| = \tau_\xi = \tau_\xi(g) \le \frac{1}{\lambda_s + \lambda_{1,g}(\xi)}.$$

Proof. Write

$$dg_\xi = \begin{pmatrix} T_\xi & A_\xi \\ 0 & \lambda_u \end{pmatrix}.$$

Then

$$(dg_\xi)^{-1} = \begin{pmatrix} T_\xi^{-1} & -\lambda_u^{-1} T_\xi^{-1} A_\xi \\ 0 & \lambda_u^{-1} \end{pmatrix}.$$

Since $\|A_\xi\|$ is small, $\lambda_u^{-1} < 1$ and $\|T_\xi^{-1}\| \ge 1$ it follows that

$$\tau_\xi \le \|T_\xi^{-1}\|.$$

So we want to estimate $\|T_\xi^{-1}\|$. If $\xi \notin B(p,\rho)$ then

$$\|T_\xi^{-1}\| = \frac{1}{\lambda_s} = \frac{1}{\lambda_s + \lambda_1(\xi)}.$$

If $\xi = p$ then

$$T_p = \begin{pmatrix} \lambda_s + Z(0)\beta(0) & 0 \\ 0 & \lambda_c + Z(0)\beta(0) \end{pmatrix}$$

and so

$$\|T_p^{-1}\| = \frac{1}{\lambda_s + Z(0)\beta(0)} = \frac{1}{\lambda_s + \lambda_1(p)}.$$

For $\xi \in B(p,\rho)$, $\xi \neq p$, write $T_\xi = C_\xi + \widetilde{S}_\xi$ where

$$C_\xi = \begin{pmatrix} \lambda_s - \lambda_c & 0 \\ 0 & 0 \end{pmatrix},$$

$$\widetilde{S}_\xi = \begin{pmatrix} Z(z)(\beta(r) + \beta'(r)2x^2) + \lambda_c & Z(z)\beta'(r)2xy \\ Z(z)\beta'(r)2xy & Z(z)(\beta(r) + \beta'(r)2y^2) + \lambda_c \end{pmatrix}.$$

The selfadjoint map $\widetilde{S}_\xi$ has eigenvectors $(x,y), (-y,x)$ associated to the eigenvalues $\lambda_1 + \lambda_c$ and $\lambda_2 + \lambda_c$ where λ_1, λ_2 are as in (4.3).

Let $\mathcal{E}$ be the ellipse with axes in the (x, y) direction and $(-y, x)$ direction, with vertices of norm $1/(\lambda_2 + \lambda_c)$ and $1/(\lambda_1 + \lambda_c)$ respectively. We have $S_\xi(\mathcal{E}) = S^1$ (the unit circle). Thus

$$T_\xi(\mathcal{E}) \subset \left\{ v : 1 - \frac{\lambda_c - \lambda_s}{\lambda_c + \lambda_1} \le \|v\| \le 1 + \frac{\lambda_c - \lambda_s}{\lambda_c + \lambda_1} \right\}.$$

Setting $R = 1 - \frac{\lambda_c - \lambda_s}{\lambda_c + \lambda_1} = \frac{\lambda_s + \lambda_1}{\lambda_c + \lambda_1}$, we have

$$T_\xi^{-1}(\{v : \|v\| = R\}) \subset \operatorname{int}(\mathcal{E}) \subset \left\{ v : \|v\| \le \frac{1}{\lambda_1 + \lambda_c} \right\}.$$

Then

$$\|(T_\xi)^{-1}\| \le \frac{1}{R}\,\frac{1}{\lambda_1 + \lambda_c} = \frac{1}{\lambda_s + \lambda_1}. \ \blacksquare$$

4.1. A special family of linear Anosov diffeomorphisms on $\mathbb{T}^3$. In order to construct elements with E^u bundle of class C^r with r close to 3 we have seen that we need $B \in \mathrm{SL}(3, \mathbb{Z})$ with eigenvalues λ_s and λ_c arbitrary close. For this we will find a special family of matrices in $\mathrm{SL}(3, \mathbb{Z})$.

Let us begin with the following family $\mathcal{J} = \{M_a\}_{a \in \mathbb{N} \setminus \{0,1,2\}}$ of matrices in $\mathrm{SL}(3, \mathbb{Z})$ (inspired from the one in [McS]):

$$M_a = \begin{pmatrix} 0 & -1 & 0 \\ 1 & a^2 - 1 & a \\ 0 & a^3 + a & 1 \end{pmatrix}. \tag{4.4}$$

Lemma 4.7. *For every $a \in \mathbb{N} \setminus \{0, 1, 2\}$, M_a has eigenvalues $\alpha_a, \beta_a, \gamma_a$ such that*

$$\alpha_a < -a^2/3 < -1 < \beta_a < 0 < a^2 < \gamma_a.$$

Furthermore,

$$-\,2a^2/3 < \alpha_a < -a^2/3 \quad \textit{and} \quad a^2 < \gamma_a < 2a^2. \tag{4.5}$$

Proof. The characteristic polynomial of M_a is $P_a(\lambda) = -\lambda^3 + a^2\lambda^2 + a^4\lambda + 1$. Its derivative $P_a'(\lambda) = -3\lambda^2 + 2a^2\lambda + a^4$ has one negative root $\lambda = -a^2/3$ and a positive one $\lambda = a^2$. At the negative root of P_a' the polynomial P_a has a relative minimum, and at the positive root there is a relative maximum of P_a. The values of P_a at the roots are

$$P_a(-a^2/3) = -5a^6/27 + 1 < 0 \quad \text{and} \quad P_a(a^2) = a^6 + 1 > 0.$$

Thus, $P_a(\lambda)$ is as in Figure 10 and the eigenvalues of M_a (i.e. the roots of $P_a(\lambda)$) satisfy

$$\alpha_a < -a^2/3 < \beta_a < 0 < a^2 < \gamma_a.$$

For the proof of the other inequalities in (4.5) we just compute

$$P_a\left(-\frac{2a^2}{3}\right) = \frac{2}{3^3}a^6 + 1 > 0 \quad \text{and} \quad P_a(2a^2) = -2a^6 + 1 < 0. \ \blacksquare$$

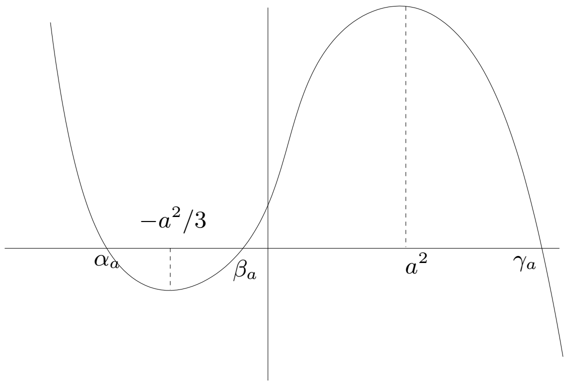

Fig. 10. The graph of $P_a(\lambda)$

We are ready to define our special family of linear Anosov maps:

$$\mathcal{I} = \{B_a = (M_a^2)^{-1} : M_a \in \mathcal{J},\ a \in \mathbb{N} \setminus \{0,1,2\}\} \tag{4.6}$$

Notice that $B_a \in \mathrm{SL}(3,\mathbb{Z})$ and the eigenvalues of B_a are the inverses of the squares of the eigenvalues of M_a and we have

$$\frac{1}{4a^4} < \frac{1}{\gamma_a^2} < \frac{1}{a^4} < \frac{9}{4a^4} < \frac{1}{\alpha_a^2} < \frac{9}{a^4} < 1 < \frac{1}{\beta_a^2}.$$

We summarize this in the following

Corollary 4.8. *For $B_a \in \mathcal{I}$ the following holds:*

(i) $B_a \in \mathrm{SL}(3,\mathbb{Z})$ *and has eigenvalues* $0 < \lambda_s(a) < \lambda_c(a) < 1 < \lambda_u(a)$.
(ii) *For every* $a \in \mathbb{N} \setminus \{0,1,2\}$ *we may write*

$$\lambda_s(a) = K_a/a^4 \quad \text{and} \quad \lambda_c(a) = K'_a/a^4 \tag{4.7}$$

where $1/10 < K_a < K'_a < 10$. *In particular* $\lambda_u(a) = a^8/K_aK'_a$.

With the next result we will conclude the proof of our Main Theorem:

Proposition 4.9. *For each* $r \in [1,3)$ *there exists* $B_a \in \mathcal{I}$ *such that for* $g_a = g_{B_a,k}$ *as defined in* (2.1) *and* (2.2) *with* k *sufficiently small the following holds: for the map* $F = F_{B_a,g_a} : L \to L$ *as defined in* (4.1) *and* (4.2) *and* $l_\xi(F), \tau_\xi(g_a)$ *as defined in Theorem* 4.1 *we have*

$$l_\xi(F)\tau_\xi(g_a)^r < 1 \quad \text{for all } \xi \in \mathbb{T}^3.$$

Proof. For simplicity, for $\xi \in \mathbb{T}^3$ set $l_{\xi,a} = l_\xi(F_{B_a,g_a})$ and $\tau_{\xi,a} = \tau_\xi(g_a)$.

Fix r with $1 \le r < 3$. It is enough to show that

$$\lim_{a\to\infty} l_{\xi,a}\tau_{\xi,a}^r = 0$$

uniformly in $\xi \in \mathbb{T}^3$. To do so, from Lemmas 4.5 and 4.6, we have, for $\xi \notin B(p,\rho)$,

$$(4.8)\qquad l_{\xi,a}\tau^r_{\xi,a} = \frac{\lambda_c(a)}{\lambda_u(a)}\,\frac{1}{\lambda_s(a)^r} = \frac{\lambda_c(a)^2}{\lambda_s(a)^{r-1}} = \frac{K'_a a^{4(r-1)}}{K_a a^8} \le 100\frac{a^{4(r-1)}}{a^8},$$

and for $\xi \in B(p,\rho)$,

$$\begin{aligned} l_{\xi,a}\tau^r_{\xi,a} &= \frac{\lambda_c(a)+Z(z)\beta(r)+k}{\lambda_u(a)}\left[\frac{1}{\lambda_s(a)+\lambda_{1,g_a}(\xi)}\right]^r \\ &= \frac{1}{\lambda_u(a)}\left[\frac{\lambda_c(a)+\lambda_{1,g_a}(\xi)}{(\lambda_s(a)+\lambda_{1,g_a}(\xi))^r} + \frac{k+Z(z)\beta(r)-\lambda_{1,g_a}(\xi)}{(\lambda_s(a)+\lambda_{1,g_a}(\xi))^r}\right]. \end{aligned}$$

Since $Z(z)\beta(r) - \lambda_{1,g_a}(\xi) \le 2k$ we have

$$\begin{aligned} l_{\xi,a}\tau^r_{\xi,a} &\le \frac{1}{\lambda_u(a)}\left[\frac{\lambda_c(a)+\lambda_{1,g_a}(\xi)}{(\lambda_s(a)+\lambda_{1,g_a}(\xi))^r} + \frac{3k}{(\lambda_s(a)+\lambda_{1,g_a}(\xi))^r}\right] \\ &\le \frac{1}{\lambda_u(a)}\left[\frac{\lambda_s(a)+\lambda_{1,g_a}(\xi)+(\lambda_c(a)-\lambda_s(a)+3k)}{(\lambda_s(a)+\lambda_{1,g_a}(\xi))^r}\right]. \end{aligned}$$

We may assume, for fixed a, that $3k < \lambda_s(a) < 10/a^4$. From the fact that $0 < \lambda_c(a) - \lambda_s(a) < 10/a^4$ and also that $\lambda_{1,g_a}(\xi) \ge -k$ we have

$$\begin{aligned} l_{\xi,a}\tau^r_{\xi,a} &\le \frac{1}{\lambda_u(a)}\left[\frac{\lambda_s(a)+\lambda_{1,g_a}(\xi)+20\frac{1}{a^4}}{(\lambda_s(a)+\lambda_{1,g_a}(\xi))^r}\right] \\ &\le \frac{1}{\lambda_u(a)}\left[\frac{1}{(\lambda_s(a)+\lambda_{1,g_a}(\xi))^{r-1}} + \frac{20}{a^4(\lambda_s(a)+\lambda_{1,g_a}(\xi))^r}\right] \\ &\le \frac{1}{\lambda_u(a)}\left[\frac{1}{(\lambda_s(a)-k)^{r-1}} + \frac{20}{a^4(\lambda_s(a)-k)^r}\right] \\ &\le \frac{100}{a^8}\left[\frac{2}{\lambda_s(a)^{r-1}} + \frac{40}{a^4\lambda_s(a)^r}\right] \\ &\le \frac{100}{a^8}[8a^{4(r-1)} + 40a^{4(r-1)}] \le 10^4\frac{a^{4(r-1)}}{a^8}. \end{aligned}$$

From this and (4.8) and taking into account that $1 \le r < 3$ we deduce for $a \in \mathbb{N}$ large enough that $l_{\xi,a}\tau^r_{\xi,a} < 1$ for any $\xi \in \mathbb{T}^2$. This completes the proof of the proposition. ∎

We can now conclude the proof of our Main Theorem: Let $1 \le r < 3$ and choose $B_a \in \mathcal{I}$ and $g_{B_a,k}$ from the above proposition. From Corollary 4.2 we find $\mathcal{U}(g_{B_a,k})$ and we choose $g \in \mathcal{U}(g_{B_a,k})$ of class C^∞ and having a homoclinic intersection associated to the fixed point p of unstable index 2. From Theorem 4.1, Corollary 4.2 and Remark 4.3 the unstable foliation $\mathcal{F}^u_g$ is of class C^r, and so, by Remark 3.2, the induced map $f = f_g : \mathbb{T}^2 \to \mathbb{T}^2$ is of class C^r. Finally, Theorem 3.3 implies our Main Theorem.

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